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# Well-posedness of Common Fixed Point Theorems for Three and Four Mappings Under Strict Contractive Conditions in Fuzzy Metric Spaces

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**Abstract.** None has studied the well-posedness of common fixed points in fuzzy metric spaces. In this paper, our target is to develop the well-posedness of common fixed points in fuzzy metric spaces. Also using weakly compatibility, implicit relation, property (E.A.) and strict contractive conditions, we have established the unique common fixed point for three self mappings and also for four self mappings in fuzzy metric spaces.

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*Key words:* Fuzzy Metric Space, Fixed Point, Common Fixed Point, Weakly Compatible Mappings, Implicit Relation, Property (E.A.), Well-Posedness.

## 1. Introduction

Motivated by a work due to Popa [13], different authors [2, 5, 12, 10] have tried to prove fixed point theorems using an implicit relation, which is a good idea since it covers several contractive conditions rather than one contractive condition in an ordinary metric space. In fact, it is seen that commuting implies weakly commuting which also implies compatible and there are examples in the literature verifying that the inclusions are proper, see [5]. In the paper [6], Jungck defined the weakly compatible maps and established that two maps are weakly compatible if they commute at their coincidence points. Using the concept of weakly compatible maps, implicit relation, property (E.A.), the authors of

[10] established the unique common fixed point for three self mappings under strict contractive conditions [1] in an ordinary metric space. Also the authors of [10] proved that such type fixed point problem is well posed. Aliouche [2], also established the unique common fixed point for four self mappings using the concept of weakly compatible maps, implicit relation, property (E.A.) and strict contractive condition in an ordinary metric space, but he did not establish the well-posedness of such type common fixed point.

So far our knowledge, a little bit of such type results have been developed in fuzzy metric spaces, a lot of common fixed point theorems for three and four self mappings and their well-posedness yet remain to develop in fuzzy metric spaces.

Fuzzy set theory was first introduced by Zadeh [18] in 1965 to describe the situation in which data are imprecise or vague or uncertain. It has a wide range of application in the field of population dynamics, chaos control, computer programming, medicine, etc.

The concept of fuzzy metric was first introduced by Kramosil and Michalek [8] and later on it is modified and a few concepts of mathematical analysis have been developed by George and Veeramani [3, 4] and also they have developed the fixed point theory in fuzzy metric spaces [17]. In fuzzy metric spaces, the notion of compatible maps under the name of asymptotically commuting maps was introduced in the paper [9] and then in the paper [16], the notion of weak compatibility has been studied in fuzzy metric spaces. However, the study of common fixed points of non-compatible maps is of great interest, which has been initiated by Pant. With the help of the property (E.A.), which was introduced in the paper [1], Pant and Pant [11] studied the common fixed points of a pair of non-compatible maps in fuzzy metric spaces.

None has studied the well-posedness of common fixed points in fuzzy metric spaces. In this paper, our target is to develop the well-posedness of common fixed points in fuzzy metric spaces. Also using weakly compatibility, implicit relation, property (E.A.) and strict contractive condition, we have established the unique common fixed point for three self mappings and also for four self mappings in fuzzy metric spaces.

## 2. Preliminaries

We quote some definitions and statements of a few theorems which will be needed in the sequel.

**Definition 2.1.** ([15]) A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous $t$-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative;

(ii) $*$ is continuous;

(iii) $a * 1 = a \quad \forall a \in [0,1]$;

(iv) $a * b \leq c * d$ whenever $a \leq c,\ b \leq d$ and $a, b, c, d \in [0,1]$.

**Result 2.2** ([7]) (a) *For any* $r_1, r_2 \in (0,1)$ *with* $r_1 > r_2$, *there exists* $r_3 \in (0,1)$ *such that* $r_1 * r_3 > r_2$,

(b) *For any* $r_5 \in (0,1)$, *there exists* $r_6 \in (0,1)$ *such that* $r_6 * r_6 \geq r_5$.

**Definition 2.3.** ([3]) The 3-tuple $(X, \mu, *)$ is called a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous $t$-norm and $\mu$ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

(i) $\mu(x, y, t) > 0$;

(ii) $\mu(x, y, t) = 1$ if and only if $x = y$;

(iii) $\mu(x, y, t) = \mu(y, x, t)$;

(iv) $\mu(x, y, s) * \mu(y, z, t) \leq \mu(x, z, s + t)$;

(v) $\mu(x, y, \cdot) : (0, \infty) \to (0, 1]$ is continuous;

for all $x, y, z \in X$ and $t, s > 0$.

**Example 2.4.** Let $X = [0, \infty)$, $a * b = ab$ for every $a, b \in [0, 1]$ and $d$ be the usual metric. Define $\mu(x, y, t) = e^{-\frac{d(x,y)}{t}}$ for all $x, y \in X$. Then clearly $(X, \mu, *)$ is a fuzzy metric space.

**Example 2.5.** Let $(X, d)$ be a metric space, and let $a*b = ab$ or $a*b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Let $\mu(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$ and $t > 0$. Then $(X, \mu, *)$ is a fuzzy metric space and this fuzzy metric $\mu$ induced by $d$ is called the standard fuzzy metric [3].

*Note.* George and Veeramani [3] proved that every fuzzy metric space is a metrizable topological space. In that paper, also they have proved, if $(X, d)$ is a metric space, then the topology generated by $d$ coincides with the topology generated by the fuzzy metric $\mu$ of example (2.5). As a result, we can say that an ordinary metric space is a special case of fuzzy metric spaces.

**Definition 2.6.** ([14]) Let $(X, \mu, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if and only if

$$\lim_{n\to\infty} \mu(x_n, x_{n+p}, t) = 1 \text{ for each } t > 0 \text{ and } p = 1, 2, 3, \cdots$$

A sequence $\{x_n\}$ in $X$ is said to converge to $x \in X$ if and only if

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1 \text{ for each } t > 0.$$

A fuzzy metric space $(X, \mu, *)$ is said to be complete if and only if every Cauchy sequence in $X$ is convergent in $X$.

**Definition 2.7.** ([9]) Let $A$ and $B$ be maps from a fuzzy metric space $(X, \mu, *)$ into itself. The maps $A$ and $B$ are said to be compatible (or asymptotically commuting), if for all $t > 0$,

$$\lim_{n\to\infty} \mu(ABx_n, BAx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

**Definition 2.8.** ([16]) Let $A$ and $B$ be maps from a fuzzy metric space $(X, \mu, *)$ into itself. The maps are said to be weakly compatible if they commute at their coincidence points, that is, $Az = Bz$ implies that $ABz = BAz$.

**Remark 2.9.** Note that compatible mappings are weakly compatible but the converse is not true in general.

**Definition 2.10.** ([11]) Let $A$ and $B$ be two self-maps of a fuzzy metric space $(X, \mu, *)$. We say that $A$ and $B$ satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$$

for some $z \in X$.

**Remark 2.11.** Note that weakly compatible and property (E.A.) are independent to each other (see [12, Ex.2.2]).

**Definition 2.12.** ([11]) The mappings $A, B, S, T :\longrightarrow X$ of a fuzzy metric space $(X, \mu, *)$ satisfy a common property (E.A.) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$$

for some $z \in X$. If $B = A$ and $T = S$, we obtain Definition 2.6.

**Definition 2.13.** ([14]) Let $(X, \mu, *)$ be a fuzzy metric space. A subset $P$ of $X$ is said to be closed if for any sequence $\{x_n\}$ in $P$ converging to $x$, we have $x \in P$, that is

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1 \Longrightarrow x \in P \quad \forall t > 0.$$

## 3. Implicit Relations

**Definition 3.1.** Let $I = [0, 1]$ and $F : I^6 \to I$ be a continuous function. We define the following property:

- ($F_1$) $F(u(t), 1, 1, u(t), u(t), 1) > 1 \quad \forall\, t > 0,\ 0 \le u(t) < 1;$
- ($F_2$) $F(u(t), 1, u(t), 1, 1, u(t)) > 1 \quad \forall\, t > 0,\ 0 \le u(t) < 1;$
- ($F_3$) $F(u(t), u(t), 1, 1, u(t), u(t)) > 1 \quad \forall\, t > 0,\ 0 \le u(t) < 1.$

**Example 3.2.** Let $F(t_1, \cdots, t_6) := \dfrac{t_1 + t_2 + t_3}{k \max\{t_4, t_5, t_6\}}$, where $k \in (0, 1)$.

($F_1$) $F(u(t), 1, 1, u(t), u(t), 1) = \frac{u(t)+1+1}{k \max\{u(t), u(t), 1\}} > 1 \quad \forall\ t > 0,\ 0 \le u(t) < 1;$

($F_2$) $F(u(t), 1, u(t), 1, 1, u(t)) = \frac{u(t)+1+u(t)}{k \max\{1,1,u(t)\}} > 1 \quad \forall t > 0,\ 0 \le u(t) < 1;$

($F_3$) $F(u(t), u(t), 1, 1, u(t), u(t)) = \frac{u(t)+u(t)+1}{k \max\{1, u(t), u(t)\}} > 1\ \forall t > 0,\ 0 \le u(t) < 1.$

**Definition 3.3.** Let $I = [0, 1]$ and $F : I^6 \to I$ be a continuous function. We define the following property:

($F_4$) There exists $k \in (0, 1)$ such that $0 \le u(t), v(t), w(t) < 1$,

$$\begin{aligned} &F(u(t), v(t), 1, w(t), u(t), v(t)) \le 1 \quad \forall t > 0,\\ \Rightarrow\ & u(t) \ge \frac{1}{k} w(t). \end{aligned}$$

**Example 3.4.** Let $F(t_1, \cdots, t_6) := \max\{t_2, t_3, t_5, t_6\} - kt_1 + t_4$, where $k \in (0, 1)$

$$\begin{aligned} &F(u(t), v(t), 1, w(t), u(t), v(t)) \le 1 \quad \forall t > 0,\\ \Rightarrow\ & \max\{v(t), 1, u(t), v(t)\} - ku(t) + w(t) \le 1\\ \Rightarrow\ & 1 - ku(t) + w(t) \le 1\\ \Rightarrow\ & u(t) \ge \frac{1}{k} w(t). \end{aligned}$$

**Example 3.5.** Let F $(t_1, \cdots, t_6) := t_1 + t_2 + t_3 + \frac{1}{k} t_4 - 2t_5 - t_6$, where $k \in (0, 1)$

$$\begin{aligned} &F(u(t), v(t), 1, w(t), u(t), v(t)) \le 1 \quad \forall t > 0\\ \Rightarrow\ & u(t) + v(t) + 1 + \frac{1}{k} w(t) - 2u(t) - v(t) \le 1\\ \Rightarrow\ & u(t) \ge \frac{1}{k} w(t). \end{aligned}$$

## 4. Common Fixed Point

In the paper [1], to establish the common fixed point for a pair of non-compatible self mappings in an ordinary metric space, the authors used the property (E.A.) and strict contractive condition which is given by the metric. In that paper, also they have established the common fixed point for four non-compatible self mappings in an ordinary metric space by considering the property (E.A.) and strict contractive condition which is defined by a nondecreasing function of single variable. Later on, the authors of [11] established the common fixed point for a pair of non-compatible self mappings in a fuzzy metric space by considering the property (E.A.), $R$-weakly commuting map and strict contractive condition which is defined by the fuzzy metric. But here we will establish the common fixed point for three non-compatible self mappings and four non-compatible self mappings in a fuzzy metric space by considering the property (E.A.) and strict contractive condition which is defined by implicit function relation.

**Theorem 4.1.** *Let $A, B$ and $I$ be three self mappings of a fuzzy metric space $(X, \mu, *)$ such that:*

(i) *The pairs $\{A, I\}$ and $\{B, I\}$ are weakly compatible;*

(ii) *The mappings $A, B$ and $I$ satisfy the property (E.A.);*

(iii) $F(\mu(Ax, By, t), \mu(Ix, Iy, t), \mu(Ix, Ax, t), \mu(Iy, By, t), \mu(Ix, By, t), \mu(Iy, Ax, t)) \leq 1$ $\forall x \neq y \in X$, *where $F$ satisfies properties* $(\mathrm{F}_1)$, $(\mathrm{F}_2)$ *and* $(\mathrm{F}_3)$*;*

(iv) $I(X)$ *is closed.*

*Then the mappings $A, B$ and $I$ have a unique common fixed point.*

*Proof.* Since the set of mappings $\{A, B, I\}$ satisfies the property (E.A.), there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Ix_n = u \quad \text{for some } u \in X. \tag{1}$$

Since $I(X)$ is closed there exists a point $a \in X$ such that $u = Ia$. If the sequence $\{x_n\}$ satisfies

$$x_n = a, \quad \forall n \geq n_0$$

for some positive integer $n_0$, then from (1), we have

$$u = Aa = Ia = Ba.$$

So, we may suppose that $x_n \neq a$ for all integers $n$, (otherwise, we consider a subsequence satisfying this property). By putting $x = x_n$ and $y = a$ in (iii) we obtain

$$F(\mu(Ax_n, Ba, t), \mu(Ix_n, Ia, t), \mu(Ix_n, Ax_n, t), \mu(Ia, Ba, t), \mu(Ix_n, Ba, t), \mu(Ia, Ax_n, t)) \leq 1 \quad \forall\, t > 0.$$

Letting $n \to \infty$, we obtain

$$F(\mu(Ia, Ba, t), 1, 1, \mu(Ia, Ba, t), \mu(Ia, Ba, t), 1, ) \leq 1$$

which, by virtue of $(\mathrm{F}_1)$, implies that

$$\mu(Ia, Ba, t) = 1 \quad \forall t > 0 \Rightarrow Ia = Ba.$$

Since $x_n \neq a$ for all integers $n$ and putting $x = a, y = x_n$ in (iii), then we get

$$F(\mu(Aa, Bx_n, t), \mu(Ia, Ix_n, t), \mu(Ia, Aa, t), \mu(Ix_n, Bx_n, t), \mu(Ia, Bx_n, t), \mu(Ix_n, Aa, t)) \leq 1.$$

Letting $n \to \infty$, we obtain

$$F(\mu(Aa, Ia, t), 1, \mu(Ia, Aa, t), 1, 1, \mu(Ia, Aa, t)) \leq 1 \quad \forall t > 0,$$

which by virtue of ($F_2$), implies that $\mu(Aa, Ia, t) = 1 \quad \forall t > 0$.
Hence $Aa = Ia$. Therefore, we obtain

$$Aa = Ia = Ba.$$

We set $x = Aa = Ia = Ba$. We shall prove that $x$ is a common fixed point of the mappings $A, B$ and $I$.
Since the pairs $\{A, I\}$ and $\{B, I\}$ are weakly compatible, then we have

$$AIa = IAa \text{ and } BIa = IBa.$$

Therefore,

$$IIa = IAa = AIa = AAa,$$

$$\text{i.e., } IIa = AAa \Rightarrow Ix = Ax, \tag{2}$$

and

$$IIa = IBa = BIa = BBa = Ix = Bx. \tag{3}$$

If $x = a$, then we have $x = Ax = Ix = Bx$. Therefore $x$ is a common fixed point of the mappings $A, B$ and $I$. So, we may suppose that $x \neq a$. In this case, by using the equalities (2) and (3) and the inequalities (iii) we put $x = a$ and $y = x$ then

$$\begin{aligned} &F(\mu(Aa, Bx, t), \mu(Ia, Ix, t), \mu(Ia, Aa, t), \mu(Ix, Bx, t), \mu(Ia, Bx, t), \\ &\qquad \mu(Ix, Aa, t)) \leq 1 \quad \forall t > 0 \\ &\Rightarrow \mu(x, Ix, t), \mu(x, Ix, t), 1, 1, \mu(x, Ix, t), \mu(x, Ix, t)) \leq 1, \end{aligned}$$

a contradiction to ($F_3$), so we have $\mu(x, Ix, t) = 1$.
Hence $Ix = x$. Then we get from (2) and (3) $x = Ix = Ax = Bx$. Therefore $x$ is a common fixed point of $A, B$ and $I$.

Now, we show that the point $x$ is a unique common fixed point of $A, B$ and $I$.

Suppose that $A, B$ and $I$ have another common fixed point $x_1$. Then, we put $y = x_1$ in (iii)

$$\begin{aligned} &F(\mu(Ax, Bx_1, t), \mu(Ix, Ix_1, t), \mu(Ix, Ax, t), \mu(Ix_1, Bx_1, t), \\ &\qquad \mu(Ix, Bx_1, t), \mu(Ix_1, Ax, t)) \leq 1 \\ &\Rightarrow F(\mu(x, x_1, t), \mu(x, x_1, t), \mu(x, x, t), \mu(x_1, x_1, t), \mu(x, x_1, t), \\ &\qquad \mu(x_1, x, t)) \leq 1 \\ &\Rightarrow F(\mu(x, x_1, t), \mu(x, x_1, t), 1, 1, \mu(x, x_1, t), \mu(x_1, x, t)) \leq 1, \end{aligned}$$

a contradiction to ($F_3$), we get $x = x_1$.

This completes the proof. ■

**Theorem 4.2.** *Let $A, B, S$ and $T$ be self-mappings of a fuzzy metric space $(X, \mu, *)$ satisfying the following conditions:*

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X),$$

$$F(\mu(Ax, By, t), \mu(Sx, Ty, t), \mu(Ax, Sx, t), \mu(By, Ty, t), \mu(Sx, By, t), \mu(Ax, Ty, t)) \leq 1, \tag{4}$$

*for all $x, y$ in $X$ and where $F$ satisfies properties* $(\mathrm{F}_1)$, $(\mathrm{F}_2)$ *and* $(\mathrm{F}_3)$.
*Suppose that $(A, S)$ or $(B, T)$ satisfies property $(E.A.)$ and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. If the range of one of $A, B, S$ and $T$ is a closed subset of $X$, then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

*Proof.* Suppose that $(B, T)$ satisfies property (E.A.), then there exists a sequence $\{x_n\}$ in $x$ such that

$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = z \quad \text{for some } z \in X.$$

Therefore, we have

$$\lim_{n\to\infty} \mu(Bx_n, Tx_n, t) = 1.$$

Since $B(x) \subset S(x)$, there exists in $X$ a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Putting $x = y_n$ and $y = x_n$ in (4),

$$\begin{aligned}
& F(\mu(Ay_n, Bx_n, t), \mu(Sy_n, Tx_n, t), \mu(Ay_n, Sy_n, t), \\
& \qquad \mu(Bx_n, Tx_n, t), 1, \mu(Ay_n, Tx_n, t)) \leq 1 \quad \forall t > 0 \\
\Rightarrow\ & F(\mu(Ay_n, Bx_n, t), \mu(Bx_n, Tx_n, t), \mu(Ay_n, Bx_n, t), \\
& \qquad \mu(Bx_n, Tx_n, t), 1, \mu(Ay_n, Tx_n, t)) \leq 1 \\
\Rightarrow\ & F(\mu(Ay_n, Bx_n, t), \mu(Bx_n, Bx_n, t), \mu(Ay_n, Bx_n, t) \\
& \qquad \mu(Bx_n, Bx_n, t), 1, \mu(Ay_n, Bx_n, t)) \leq 1.
\end{aligned}$$

Taking the limit as $n \longrightarrow \infty$, we obtain

$$F(\mu(Ay_n, Bx_n, t), 1, \mu(Ay_n, Bx_n, t), 1, 1, \mu(Ay_n, Bx_n, t)) \leq 1,$$

which is a contradiction to $(\mathrm{F}_2)$, then we have

$$\lim_{n\to\infty} \mu(Ay_n, Bx_n, t) = 1 \Rightarrow \lim_{n\to\infty} Ay_n = z.$$

Suppose that $S(X)$ is a closed subspace of $X$. Then $z = Su$ for some $u \in X$. Putting $x = u$ and $y = x_n$ in (4) we obtain

$$\begin{aligned}
& F(\mu(Au, Bx_n, t), \mu(Su, Tx_n, t), \mu(Au, Su, t), \mu(Bx_n, Tx_n, t), \\
& \qquad \mu(Su, Bx_n, t), \mu(Au, Tx_n, t)) \leq 1.
\end{aligned}$$

Letting $n \longrightarrow \infty$ we have

$$F(\mu(Au, z, t), 1, \mu(Au, z, t), 1, 1, \mu(Au, z, t)) \leq 1,$$

which is a contradiction to ($F_2$). Hence,

$$\begin{aligned} &\mu(Au, z, t) = 1 \\ \Rightarrow\ & Au = z \\ \Rightarrow\ & z = Au = Su. \end{aligned}$$

Since $A(X) \subset T(X)$, there exists $v \in x$ such that $z = Au = Tv$. If $Az \neq z$ and putting $x = u$ and $y = v$ in (4), then we get

$$\begin{aligned} &F(\mu(Au, Bv, t), \mu(Su, Tv, t), \mu(Au, Su, t), \mu(Bv, Tv, t), \\ &\qquad\qquad\qquad\qquad \mu(Su, Bv, t), \mu(Au, Tv, t)) \leq 1, \\ &F(\mu(z, Bv, t), 1, 1, \mu(z, Bv, t), \mu(z, Bv, t), 1) \leq 1, \end{aligned}$$

a contradiction to ($F_1$). Then

$$\begin{aligned} &Bv = z \\ \Rightarrow\ & Tv = Bv = z \\ \Rightarrow\ & Au = Su = z = Tv = Bv. \end{aligned}$$

Since the pair $(A, S)$ is weakly compatible, we have

$$ASu = SAu \Longrightarrow Az = Sz.$$

If $Az \neq z$ and putting $x = z = y$ in (4)

$$\begin{aligned} &F(\mu(Az, Bv, t), \mu(Sz, Tv, t), \mu(Az, Sz, t), \mu(Bv, Tv, t), \\ &\qquad\qquad\qquad\qquad \mu(Sz, Bv, t), \mu(Az, Tv, t)) \leq 1, \\ &F(\mu(Az, z, t), \mu(Az, z, t), 1, 1, \mu(Az, z, t), \mu(Az, z, t)) \leq 1, \end{aligned}$$

which is a contradiction to ($F_3$). Then $Az = Sz = z$.
Since the pair $(B, T)$ is weakly compatible, we have

$$BTv = TBv \quad \text{i.e.,} \ \ Bz = Tz.$$

If $Bz \neq z$ and putting $x = z = y$ in (4)

$$\begin{aligned} &F(\mu(Az, Bz, t), \mu(Sz, Tz, t), \mu(Az, Sz, t), \mu(Bz, Tz, t), \\ &\qquad\qquad\qquad\qquad \mu(Sz, Bz, t), \mu(Az, Tz, t)) \leq 1, \\ &F(\mu(z, Bz, t), \mu(z, Bz, t), 1, 1, \mu(z, Bz, t), \mu(z, Bz, t)) \leq 1, \end{aligned}$$

which is a contradiction to ($F_3$ ).

Hence $z = Bz = Tz = Az = Sz$ and $z$ is a common fixed point of $A, B, S$ and $T$.

Suppose that $A, B, S$ and $T$ have another fixed point $z_1$. Then, we put $x = z$ and $y = z_1$ in (4)

$$\begin{aligned} &F(\mu(Az, Bz_1, t), \mu(Sz, Tz_1, t), \mu(Az, Sz, t), \mu(Bz_1, Tz_1, t), \\ &\qquad\qquad\qquad \mu(Sz, Bz_1, t), \mu(Az, Tz_1, t)) \leq 1, \\ &F(\mu(z, z_1, t), \mu(z, z_1, t), 1, 1, \mu(z, z_1, t), \mu(z, z_1, t)) \leq 1, \end{aligned}$$

which is a contradiction to ($F_3$ ), then we get $z = z_1$. ■

**Corollary 4.3.** *Let $A, B, S$ and $T$ be self-mappings of a fuzzy metric space $(X, \mu, *)$ satisfying the following conditions*

$$\begin{aligned} &F(\mu(Ax, By, t), \mu(Sx, Ty, t), \mu(Ax, Sx, t), \mu(By, Ty, t), \\ &\qquad\qquad\qquad \mu(Sx, By, t), \mu(Ax, Ty, t) \leq 1, \end{aligned}$$

*for all $x, y \in X$ and where $F$ satisfies properties* ($F_1$)*,* ($F_2$) *and* ($F_3$)*.*

*Suppose that $(A, S)$ or $(B, T)$ satisfies property (E.A.) and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. If $S(X)$ and $T(X)$ are closed subsets of $X$, then $A, B, S$ and $T$ have a unique common fixed point in $X$.*

## 5. Well-posedness of Common Fixed Point Theorems

**Definition 5.1.** Let $(X, \mu, *)$ be a fuzzy metric space and $\mathcal{P}$ a set of self-mappings of $X$. The common fixed point problem of the set $\mathcal{P}$ is said to be well-posed if

(1) $\mathcal{P}$ has a unique common fixed point $x$ in $X$ (that is, $x$ is the unique point in $X$ such that $Ax = x \quad \forall A \in \mathcal{P}$;

(2) For every sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} \mu(x_n, Ax_n, t) = 1 \quad \forall A \in \mathcal{P},$$

we have

$$\lim_{n\to\infty} \mu(x_n, x, t) = 1.$$

**Theorem 5.2.** *Let $A, B$ and $I$ be three self mappings of a fuzzy metric space $(X, \mu, *)$ such that:*

(i) *The pairs $\{A, I\}$ and $\{B, I\}$ are weakly compatible;*

(ii) *The mappings $A, B$ and $I$ satisfy the property (E.A.);*

(iii) $F(\mu(Ax, By, \frac{t}{2}), \mu(Ix, Iy, \frac{t}{2}), \mu(Ix, Ax, \frac{t}{2}), \mu(Iy, By, \frac{t}{2}), \mu(Ix, By, \frac{t}{2}),$ $\mu(Iy, Ax, \frac{t}{2})) \leq 1 \; \forall x \neq y \in X$*, where $F$ satisfies properties* ($F_1$)*,* ($F_2$)*,* ($F_3$) *and* ($F_4$)*;*

(iv) $I(X)$ *is closed.*
*Then the common fixed point problem of* $A, B$ *and* $I$ *is well posed.*

*Proof.* Let $\{x_n\}$ be a sequence in $X$ such that

$$\lim_{n\to\infty} \mu(x_n, Ax_n, \frac{t}{2}) = \lim_{n\to\infty} \mu(x_n, Bx_n, \frac{t}{2}) = \lim_{n\to\infty} \mu(x_n, Ix_n, \frac{t}{2}) = 1.$$

Putting $y = x_n$ in (iii) then

$$\begin{aligned} &F(\mu(Ax, Bx_n, \frac{t}{2}), \mu(Ix, Ix_n, \frac{t}{2}), \mu(Ix, Ax, \frac{t}{2}), \mu(Ix_n, Bx_n, \frac{t}{2}),\\ &\qquad\qquad \mu(Ix, Bx_n, \frac{t}{2}), \mu(Ix_n, Ax, \frac{t}{2})) \leq 1\\ \Rightarrow\ &F(\mu(x, Bx_n, \frac{t}{2}), \mu(x, Ix_n, \frac{t}{2}), \mu(x, x, \frac{t}{2}), \mu(Ix_n, Bx_n, \frac{t}{2}),\\ &\qquad\qquad \mu(x, Bx_n, \frac{t}{2}), \mu(Ix_n, x, \frac{t}{2})) \leq 1\\ \Rightarrow\ &F(\mu(x, Bx_n, \frac{t}{2}), \mu(x, Ix_n, \frac{t}{2}), 1, \mu(Ix_n, Bx_n, \frac{t}{2}), \mu(x, Bx_n, \frac{t}{2}),\\ &\qquad\qquad \mu(Ix_n, x, \frac{t}{2})) \leq 1. \end{aligned}$$

By ($F_4$) we get

$$\mu(x, Bx_n, \frac{t}{2}) \geq \frac{1}{k}\mu(Ix_n, Bx_n, \frac{t}{2}).$$

Therefore,

$$\begin{aligned} \mu(x, x_n, t) &\geq \mu(x, Bx_n, \frac{t}{2}) * \mu(Bx_n, x_n, \frac{t}{2})\\ &\geq \frac{1}{k}\{\mu(Ix_n, Bx_n, \frac{t}{2})\} * \mu(Bx_n, x_n, \frac{t}{2})\\ &\geq \frac{1}{k}\{\mu(Ix_n, x_n, \frac{t}{4}) * \mu(Bx_n, x_n, \frac{t}{4})\} * \mu(Bx_n, x_n, \frac{t}{2}). \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \mu(x, x_n, t) \geq \frac{1}{k} > 1, \lim_{n\to\infty} \mu(x, x_n, t) = 1.$$

This completes the proof. ■

**Theorem 5.3.** *Let* $A, B, S$ *and* $T$ *be self-mappings of a fuzzy metric space* $(X, \mu, *)$ *satisfying the following conditions:*

$$A(X) \subset T(X) \quad and \quad B(X) \subset S(X),$$

$$\begin{aligned} F(\mu(Ax, By, \frac{t}{2}), &\mu(Sx, Ty, \frac{t}{2}), \mu(Ax, Sx, \frac{t}{2}), \mu(By, Ty, \frac{t}{2})\\ &\mu(Sx, By, \frac{t}{2}), \mu(Ax, Ty, \frac{t}{2})) \leq 1, \end{aligned} \tag{5}$$

*for all $x, y \in X$ and where $F$ satisfies properties* ($F_1$), ($F_2$), ($F_3$) *and* ($F_4$).

*Suppose that $(A, S)$ or $(B, T)$ satisfies property (E.A.) and the pairs $(A, S)$ and $(B, T)$ are weakly compatible. If the range of one of $A, B, S$ and $T$ is a closed subset of $X$, then the common fixed point problem of $A, B, S$ and $T$ is well posed.*

*Proof.* Suppose that $(B, T)$ satisfies property (E.A.), then there exists a sequence $\{x_n\}$ in $X$ such that

$$\lim_{n\to\infty} \mu(x_n, Bx_n, \frac{t}{2}) = \lim_{n\to\infty} \mu(x_n, Tx_n, \frac{t}{2}) = 1.$$

Putting $y = x_n$ in (5), we have

$$\begin{aligned} &F(\mu(Ax, Bx_n, t), \mu(Sx, Tx_n, \frac{t}{2}), \mu(Ax, Sx, \frac{t}{2}), \mu(Bx_n, Tx_n, \frac{t}{2}), \\ &\qquad \mu(Sx, Bx_n, \frac{t}{2}), \mu(Ax, Tx_n, \frac{t}{2})) \leq 1, \\ &F(\mu(x, Bx_n, \frac{t}{2}), \mu(x, Tx_n, \frac{t}{2}), 1, \mu(Bx_n, Tx_n, \frac{t}{2}), \mu(x, Bx_n, \frac{t}{2}), \\ &\qquad \mu(x, Tx_n, \frac{t}{2})) \leq 1. \end{aligned}$$

By ($F_4$) we get

$$\mu(x, Bx_n, \frac{t}{2}) \geq \frac{1}{k}\mu(Bx_n, Tx_n, \frac{t}{2}).$$

Therefore,

$$\begin{aligned} \mu(x, x_n, t) &\geq \mu(x, Bx_n, \frac{t}{2}) * \mu(Bx_n, x_n, \frac{t}{2}) \\ &\geq \frac{1}{k}\{\mu(Bx_n, Tx_n, \frac{t}{2})\} * \mu(Bx_n, x_n, \frac{t}{2}) \\ &\geq \frac{1}{k}\{\mu(Bx_n, x_n, \frac{t}{4}) * \mu(Tx_n, x_n, \frac{t}{4})\} * \mu(Bx_n, x_n, \frac{t}{2}). \end{aligned}$$

Taking limit as $n \longrightarrow \infty$,

$$\begin{aligned} &\lim_{n\to\infty} \mu(x, x_n, t) \geq \frac{1}{k} > 1, \\ &\lim_{n\to\infty} \mu(x, x_n, t) = 1. \end{aligned}$$

This completes the proof. ∎

**Corollary 5.4.** *Let $A, B, S$ and $T$ be self-mappings of a fuzzy metric space $(X, \mu, *)$ satisfying the following conditions:*

$$\begin{aligned} F(\mu(Ax, By, \frac{t}{2}), &\mu(Sx, Ty, \frac{t}{2}), \mu(Ax, Sx, \frac{t}{2}), \mu(By, Ty, \frac{t}{2}) \\ &\mu(Sx, By, \frac{t}{2}), \mu(Ax, Ty, \frac{t}{2})) \leq 1, \end{aligned}$$

*for all* $x, y \in X$ *and where* $F$ *satisfies properties* $(\mathrm{F}_1)$, $(\mathrm{F}_2)$, $(\mathrm{F}_3)$ *and* $(\mathrm{F}_4)$.

*Suppose that* $(A, S)$ *or* $(B, T)$ *satisfies property* $(E.A.)$ *and the pairs* $(A, S)$ *and* $(B, T)$ *are weakly compatible. If* $S(X)$ *and* $T(X)$ *are closed subsets of* $X$*, then the common fixed point problem of* $A, B, S$ *and* $T$ *is well posed .*

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## References

1. M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, *J. Math. Anal. Appl.*, **270** (2002), 181-188.
2. A. Aliouche, Common fixed point theorems via an implicit relation and new properties, *Soochow J. Math.*, **33(4)** (2007), 593-601.
3. A. George and P. Veeramani, On Some result in fuzzy metric spaces, *Fuzzy Sets and Systems*, **64** (1994), 395-399.
4. A. George and P. Veeramani, On some results of analysis for fuzzy metric spaces, *Fuzzy Sets and Systems*, **90**(1997), 365-368.
5. G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. and Math. Sci.*, **9** (1986), 771-779.
6. G. Jungck, Common fixed points for non-continuous non-self maps on non metric space, *Far East J. Math. Sci.*, **4(2)** (1996), 199-215.
7. E. P. Klement, R. Mesiar and E. Pap, *Triangular norms*, Kluwer, Dordrecht, 2000.
8. O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, **11** (1975), 326-334.
9. S. N. Mishra, N. Sharma and S. L. Singh, Common fixed points of maps on fuzzy metric spaces, *Int. J. Math. Sci.*, **17** (1994), 253-258.
10. Mohamed Akkouchi and Valeriu Popa, Well-posedness of a common fixed point problem for three mappings under strict contractive conditions, *Matematica-Informatica-Fizica*, **LXI(2)** (2009), 1689-1695.
11. V. Pant and R. P. Pant, Fixed points in fuzzy metric space for noncompatible maps, *Soochow J. Math.*, **33(4)** (2007), 647-655.
12. H. K. Pathak, R. R. Lopez and R. K. Verma, A common fixed point theorem using implicit relation and property $(E.A)$ in metric spaces, *Filomat*, **21(2)** (2007), 211-234.
13. V. Popa, A general fixed point theorem under strict implicit contractive condition, *Stud. Cercet. Stiint. Ser. Mat. Univ. Bacau*, **15** (2005), 129-133.
14. T. K. Samanta and Iqbal H. Jebril, Finite dimentional intuitionistic fuzzy normed linear space, *Int. J. Open Probl. Comput. Math.*, **2(4)** (2009), 574-591.
15. B. Schweizer and A. Sklar, Statistical metric spaces, *Pacific J. Math.*, **10** (1960), 314-334.
16. B. Singh and S. Jain, Weak compatibility and fixed point theorems in fuzzy metric spaces, *Ganita*, **56(2)** (2005), 167-176.
17. R. Vasuki and P. Veeramani, Fixed point theorems and Cauchy sequences in fuzzy metric spaces, *Fuzzy Sets and Systems*, **135** (2003), 415-417.
18. L. A. Zadeh, Fuzzy sets, *Information and control*, **8** (1965), 338-353.