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Weighted Sharp Boundedness for Multilinear Commutators of Singular Integral on Spaces of Homogeneous Type

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Abstract. In this paper, we obtain sharp estimates for the multilinear commutators related to the singular integral operator on the spaces of homogeneous type. As an application, we obtain a weighted $L^p(p > 1)$ inequality and an $L \log L$-type estimate for the commutator.

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1. Introduction

Let $b \in BMO(\mathbb{R}^n)$, and T be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberg and Weiss (see [7]) proved that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. However, it was observed that $[b, T]$ is not bounded, in general, from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. In [19, 20], the sharp inequalities were obtained for some multilinear commutators of the Calderón-Zygmund singular integral operators. In [5, 11-15], the boundedness were obtained for some multilinear commutators of the singular integral operators on $\mathbb{R}^n$. In [2-4, 6, 9, 17, 21], the boundedness were obtained for some multilinear commutators of the singular integral operators on the spaces of homogeneous type. Our works are motivated by these papers. The main purpose of this paper is to prove sharp estimates for the multilinear commutator related to the singular integral operator on the spaces of homogeneous type. As an application, we obtain a weighted $L^p(p>1)$ inequality and an $L\log L$-type estimate for the commutator.

2. Preliminaries and theorems

Given a set X, a function $d : X \times X \to \mathbb{R}^+$ is called a quasi-distance on X if the following conditions are satisfied:

(i) for every x and y in X, $d(x,y) \geq 0$ and $d(x,y) = 0$ if and only if $x = y$;

(ii) for every x and y in X, $d(x,y) = d(y,x)$;

(iii) there exists a constant $k \geq 1$ such that

$$d(x,y) \leq k(d(x,z) + d(z,y)) \tag{1}$$

for every x, y and z in X.

Let μ be a positive measure on the σ-algebra of subsets of X which contains the r-balls $B(x,r) = \{y : d(x,y) < r\}$. We assume that μ satisfies a doubling condition, that is, there exists a constant A such that

$$0 < \mu(B(x,2r)) \leq A\mu(B(x,r)) < \infty \tag{2}$$

holds for all $x \in X$ and $r > 0$.

A structure (X,d,μ), with d and μ as above, is called a space of homogeneous type. The constants k and A in (1) and (2) will be called the constants of the space.

Then let us introduce some notations (see [1, 3, 8, 9, 16, 22]). In this paper, B will denote a ball of X, and for a ball B, let $f_B = \mu(B)^{-1}\int_B f(x)d\mu(x)$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y) - f_B| d\mu(y).$$

It is well-known that (see [9])

$$f^{\#}(x) \approx \sup_{B \ni x} \inf_{c \in C} \frac{1}{\mu(B)} \int_B |f(y) - c| d\mu(y).$$

For $0 < r < \infty$ we denote

$$f_r^{\#} = [(|f|^r)^{\#}]^{1/r}.$$

We say that b belongs to $BMO(X)$ if $b^\#$ belongs to $L^\infty(X)$ and define $\|b\|_{BMO} = \|b^\#\|_{L^\infty}$.

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

we write $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$. For $k \in N$ we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \geq 2$.

Let Φ be a Young function and Ψ be the complementary associated to Φ. For a function f, we define the Φ-average by

$$\|f\|_{\Phi,B} = \inf \left\{ \lambda > 0 : \frac{1}{\mu(B)} \int_B \Phi\left(\frac{|f(y)|}{\lambda}\right) d\mu(y) \leq 1 \right\}$$

and the maximal function by

$$M_\Phi f(x) = \sup_{x \in B} \|f\|_{\Phi,B}.$$

The main Young function to be used in this paper is $\Phi(t) = t\log^r(t+e)$ and its complementary $\Psi(t) = \exp\,(t^r) - 1$, and the corresponding Φ-averages and maximal functions are denoted by $\|\cdot\|_{L(\log L)^r,B}$, $M_{L(\log L)^r}$ and $\|\cdot\|_{\exp L^r,B}$, $M_{\exp L^r}$. We have the generalized Hölder's inequality

$$\frac{1}{\mu(B)} \int_B |f(y)g(y)| d\mu(y) \leq \|f\|_{\Phi,B} \|g\|_{\Psi,B}$$

and the following inequality for any $r > 0$ and $m \in N$:

$$M(f) \leq M_{L(\log L)^r}(f), \qquad M_{L(\log L)^m}(f) \approx M^{m+1}(f).$$

For $r \geq 1$ we denote

$$\|b\|_{osc_{\exp L^r}} = \sup_B \|b - b_B\|_{\exp L^r,B}.$$

The space $Osc_{\exp L^r}(X)$ is defined by

$$Osc_{\exp L^r}(X) = \{b \in L^1_{loc}(X) : \|b\|_{Osc_{\exp L^r}} < \infty\}.$$

It is clear that (see [22])

$$\|b - b_{2^k B}\|_{\exp L^r, 2^k B} \leq Ck \|b\|_{Osc_{\exp L^r}}.$$

It is obvious that $Osc_{\exp L^r}(X)$ coincides with the $BMO(X)$ space if $r = 1$. For $r_j > 0$ and $b_j \in Osc_{\exp L^{r_j}}(X)$ for $j = 1 \cdots m$, we denote

$$\frac{1}{r}=\frac{1}{r_1}+\cdots+\frac{1}{r_m} \quad \text{and} \quad \|\vec{b}\|=\prod_{j=1}^{m}\|b_j\|_{Osc_{\exp L^{r_j}}}.$$

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements. For $\sigma \in C_j^m$ set $\sigma^c=\{1,\cdots,m\}\setminus\sigma$. For $\vec{b}=(b_1,\cdots,b_m)$ and $\sigma=\{\sigma(1),\cdots,\sigma(j)\}\in C_j^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\cdots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(j)}$ and $\|\vec{b}_\sigma\|_{Osc_{\exp L^{r_\sigma}}}=\|b_{\sigma(1)}\|_{Osc_{\exp L^{r_{\sigma(1)}}}}\cdots\|b_{\sigma(j)}\|_{Osc_{\exp L^{r_{\sigma(j)}}}}$.

Denote the Muckenhoupt weights by A_p for $1 \le p < \infty$ (see [8]).

We are going to consider the multilinear commutators related to the singular integral operator defined below.

Definition 2.1. Suppose b_j $(j=1,\cdots,m)$ are the fixed locally integrable functions on X. Let T be the singular integral operator

$$T(f)(x)=\int_X K(x,y)f(y)d\mu(y),$$

where K is a locally integrable function on $X\times X\setminus\{(x,y):x=y\}$ and satisfies the following properties:

(1) $|K(x,y)| \le \dfrac{C}{\mu(B(x,d(x,y)))}$,

(2) $|K(x,y)-K(x,y')|+|K(y,x)-K(y',x)| \le C\dfrac{d(y,y')^\delta}{\mu(B(y,d(x,y)))d(x,y)^\delta}$,

when $d(x,y)\ge 2d(y,y')$ with some $\delta\in(0,1]$.

The multilinear commutator of the singular integral operator is defined by

$$T_{\vec{b}}(f)(x)=\int_X \prod_{j=1}^{m}(b_j(x)-b_j(y))K(x,y)f(y)d\mu(y).$$

Note that if $b_1=\cdots=b_m$, then $T_{\vec{b}}$ is just the mth-order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 2, 4, 6, 10, 18-21]). Our main purpose is to establish sharp inequality for the multilinear commutator.

The following theorems are our main results:

Theorem 2.2. *Let $r_j \ge 1$ and $b_j \in Osc_{\exp\ L^{r_j}}(X)$ for $j=1,\cdots,m$. Denote*

$$\frac{1}{r}=\frac{1}{r_1}+\cdots+\frac{1}{r_m}.$$

For any $0<p<q<1$ there exists a constant $C>0$ such that, for any $f\in C_0^\infty(X)$ and any $\tilde{x}\in X$,

$$(T_{\vec{b}}(f))_p^{\#}(\tilde{x}) \leq C\left(\|\vec{b}\|M_{L(logL)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^{m}\sum_{\sigma\in C_j^m} M_q(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x})\right).$$

Theorem 2.3. *If $1 < p < \infty$ and $\omega \in A_p$ then*

$$\|T_{\vec{b}}(f)\|_{L^p(\omega)} \leq C\|\vec{b}\|\|f\|_{L^p(\omega)}.$$

Theorem 2.4. *If $\omega \in A_1$ then there exists a constant $C > 0$ such that, for all $\lambda > 0$,*

$$\omega(\{x \in X : T_{\vec{b}}(f)(x) > \lambda\}) \leq C\int_X \Phi\left(\frac{\|\vec{b}\||f(x)|}{\lambda}\right)\omega(x)d\mu(x).$$

3. Proofs of theorems

To prove the theorems, we need the following lemmas.

Lemma 3.1. ([20]) *Let $r_j \geq 1$ for $j = 1, \cdots, m$. Denote*

$$\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}.$$

Then

$$\frac{1}{\mu(B)}\int_B |f_1(x)\cdots f_m(x)g(x)|d\mu(x)$$
$$\leq C\|f\|_{\exp L^{r_1},B}\cdots\|f\|_{\exp L^{r_m},B}\|g\|_{L(\log L)^{1/r},B}.$$

Lemma 3.2. ([17]) *Let $\omega \in A_1$ and $1 < p < \infty$. Then T is bounded on $L^p(\omega)$ and weak type of $(L^1(\omega), L^1(\omega))$.*

Lemma 3.3. *Let (X, d, μ) be a space of homogeneous type. Then there exists a positive constant C such that the inequality*

$$\int_B T(f)(x)d\mu(x) \leq C\mu(B)^{-1}\int_X f(x)d\mu(x)$$

holds for any ball B.

Proof. By Tonelli's theorem and the fact that $\int_B K(x,y)d\mu(x) \leq C\mu(B)^{-1}$ we have

$$\int_B T(f)(x)d\mu(x) = \int_B\int_X K(x,y)f(y)d\mu(y)d\mu(x)$$

$$= \int_X \int_B K(x,y) f(y) d\mu(x) d\mu(y)$$
$$\leq C\mu(B)^{-1} \int_X f(y) d\mu(y).$$

■

Proof of Theorem 2.2. It suffices to prove that for $f \in C_0^\infty(X)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{\mu(B)} \int_B |T_{\vec{b}}(f)(x) - C_0|^p d\mu(x) \right)^{1/p}$$
$$\leq C \left(\|\vec{b}\| M_{L(\log L)^{1/r}}(f)(\tilde{x}) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_q(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}) \right).$$

Fix a ball $B = B(x_0, r)$ and $\tilde{x} \in B$. We first consider the *case m=1* . Write, for $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{(2B)^c}$,

$$T_{b_1}(f)(x) = (b_1(x) - (b_1)_{2B}) T(f)(x) - T((b_1 - (b_1)_{2B}) f_1)(x) - T((b_1 - (b_1)_{2B}) f_2)(x).$$

Let $C_0 = T(((b_1)_{2B} - b_1) f_2)(x_0)$, then

$$\begin{aligned} |T_{b_1}(f)(x) - C_0| &\leq |(b_1(x) - (b_1)_{2B}) T(f)(x)| + |T((b_1)_{2B} - (b_1) f_1)(x)| \\ &\quad + |T(((b_1)_{2B} - b_1) f_2)(x) - T(((b_1)_{2B} - b_1) f_2)(x_0)| \\ &= I(x) + II(x) + III(x). \end{aligned}$$

For $I(x)$, by Hölder's inequality with exponent $1/l + 1/l' = 1$ with $1 < l < q/p$ and $pl = q$, we get

$$\begin{aligned} &\left(\frac{1}{\mu(B)} \int_B |I(x)|^p d\mu(x) \right)^{1/p} \\ &= \left(\frac{1}{\mu(B)} \int_B |b_1(x) - (b_1)_{2B}|^p |T(f)(x)|^p d\mu(x) \right)^{1/p} \\ &\leq C \left(\frac{1}{\mu(2B)} \int_{2B} |b_1(x) - (b_1)_{2B}|^{pl'} d\mu(x) \right)^{1/pl'} \left(\frac{1}{\mu(B)} \int_B |T(f)(x)|^{pl} d\mu(x) \right)^{1/pl} \\ &\leq C \|b_1\|_{Osc_{\exp L^r}} M_{pl}(T(f))(\tilde{x}) \\ &\leq C \|b_1\|_{Osc_{\exp L^r}} M_q(T(f))(\tilde{x}). \end{aligned}$$

For $II(x)$, by Hölder's inequality, we have

$$\left(\frac{1}{\mu(B)}\int_B |II(x)|^p d\mu(x)\right)^{1/p} \le \frac{1}{\mu(B)}\int_B |II(x)|d\mu(x)$$
$$\le \frac{1}{\mu(B)}\int_B |T((b_1-(b_1)_{2B})f_1)(x)|d\mu(x)$$
$$\le C\mu(2B)^{-2}\int_{2B} |b_1(x)-(b_1)_{2B}||f(x)|d\mu(x)$$
$$\le C\|b_1-(b_1)_{2B}\|_{\exp L^r,2B}\|f\|_{L(\log L)^{1/r},2B}$$
$$\le C\|b_1\|_{Osc_{\exp L^r}} M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For $III(x)$, by the properties of K, the doubling condition of μ, Minkowski's inequality and Hölder's inequality we obtain, for $x \in B$,

$$C(x) = \left|\int_X (K(x,y)-K(x_0,y))((b_1)_{2B}-b_1(y))f_2(y)d\mu(y)\right|$$
$$\le C\int_{(2B)^c}\left[\frac{|d(x_0,x)|}{|d(x,y)|}\right]^\delta \frac{1}{|\mu(B(x_0,d(x_0,y)))|}|(b_1)_{2B}-b_1(y)||f(y)|d\mu(y)$$
$$\le C\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB}\left[\frac{|d(x_0,x)|}{|d(x,y)|}\right]^\delta \frac{1}{|\mu(B(x_0,d(x_0,y)))|}|(b_1)_{2B}-b_1(y)||f(y)|d\mu(y)$$
$$\le C\sum_{k=1}^\infty \frac{\mu(B)^\delta}{\mu(2^kB)^{\delta+1}}\int_{2^{k+1}B}|b_1(y)-(b_1)_{2B}||f(y)|d\mu(y)$$
$$\le C\sum_{k=1}^\infty 2^{-k(\delta+1)}k\|b_1\|_{Osc_{\exp L^r}} M_{L(\log L)^{1/r}}(f)(\tilde{x})$$
$$\le C\|b_1\|_{Osc_{\exp L^r}} M_{L(\log L)^{1/r}}(f)(\tilde{x}),$$

thus
$$\left(\frac{1}{\mu(B)}\int_B |III(x)|^p d\mu(x)\right)^{1/p} \le CM_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

Now, we consider the *case* $m \ge 2$. We have known that, for $\vec{b} = (b_1, \cdots, b_m)$,

$$T_{\vec{b}}(f)(x)$$
$$= \int_X \Big[\prod_{j=1}^m (b_j(x)-b_j(y))\Big]K(x,y)f(y)d\mu(y)$$
$$= \int_X \prod_{j=1}^m \left[(b_j(x)-(b_j)_{2B})-(b_j(y)-(b_j)_{2B})\right]K(x,y)f(y)d\mu(y)$$

$$
\begin{aligned}
&= \sum_{j=0}^{m} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - (\vec{b})_{2B})_\sigma \int_X (\vec{b}(y) - (\vec{b})_{2B})_{\sigma^c} K(x,y) f(y) d\mu(y) \\
&= (b_1(x) - (b_1)_{2B}) \cdots (b_m(x) - (b_m)_{2B}) T(f)(x) \\
&\quad + (-1)^m T((b_1 - (b_1)_{2B}) \cdots (b_m - (b_m)_{2B}) f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - (\vec{b})_{2B})_\sigma \int_X (\vec{b}(y) - (\vec{b})_{2B})_{\sigma^c} K(x,y) f(y) d\mu(y) \\
&= (b_1(x) - (b_1)_{2B}) \cdots (b_m(x) - (b_m)_{2B}) T(f)(x) \\
&\quad + (-1)^m T((b_1 - (b_1)_{2B}) \cdots (b_m - (b_m)_{2B}) f)(x) \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - (\vec{b})_{2B})_\sigma T_{\vec{b}_{\sigma^c}}(f)(x).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\left| T_{\vec{b}}(f)(x) - T((b_1 - (b_1)_{2B}) \ldots (b_m - (b_m)_{2B}) f_2)(x_0) \right| \\
\le &\left| (b_1(x) - (b_1)_{2B}) \ldots (b_m(x) - (b_m)_{2B}) T(f)(x) \right| \\
&+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left| (\vec{b}(x) - (\vec{b})_{2B})_\sigma T_{\vec{b}_{\sigma^c}}(f)(x) \right| \\
&+ \left| T((b_1 - (b_1)_{2B}) \ldots (b_m - (b_m)_{2B}) f_1)(x) \right| \\
&+ \left| T((b_1 - (b_1)_{2B}) \ldots (b_m - (b_m)_{2B}) f_2)(x) \right. \\
&\left. - T((b_1 - (b_1)_{2B}) \ldots (b_m - (b_m)_{2B}) f_2)(x_0) \right| \\
= &\, I_1(x) + I_2(x) + I_3(x) + I_4(x).
\end{aligned}
$$

For $I_1(x)$ and $I_2(x)$, similar to the proof of the case $m = 1$, we get

$$
\left(\frac{1}{\mu(B)} \int_B |I_1(x)|^p d\mu(x) \right)^{1/p} \le C M_q(T(f))(\tilde{x})
$$

and

$$
\left(\frac{1}{\mu(B)} \int_B |I_2(x)|^p d\mu(x) \right)^{1/p} \le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} M_{L(\log L)^{1/r}}(f)(\tilde{x}).
$$

For $I_3(x)$, by Lemma 3.1 and Lemma 3.3, we obtain

$$
\begin{aligned}
&\left(\frac{1}{\mu(B)} \int_B |I_3(x)|^p d\mu(x) \right)^{1/p} \\
&\le \frac{C}{\mu(2B)} \int_{2B} |b_1(x) - (b_1)_{2B} \cdots |b_m(x) - (b_m)_{2B}||f(x)| d\mu(x)
\end{aligned}
$$

$$\leq C\|b_1 - (b_1)_{2B}\|_{\exp L^{r_1},2B} \cdots \|b_m - (b_m)_{2B}\|_{\exp L^{r_m},2B}\|f\|_{L(\log L)^{1/r},2B}$$
$$\leq C\|\vec{b}\| M_{L(\log L)^{1/r}}(f)(\tilde{x}).$$

For $I_4(x)$, similar to $III(x)$, we have

$$\begin{aligned}
I_4(x) &= \left|\int_X \prod_{j=1}^m (b_j(y) - (b_j)_{2B})(K(x,y) - K(x_0,y))f_2(y)d\mu(y)\right| \\
&\leq C \int_{(2B)^c} \left|\prod_{j=1}^m (b_j(y) - (b_j)_{2B})\right| \left[\frac{|d(x_0,x)|}{|d(x,y)|}\right]^\delta \frac{1}{|\mu(B(x_0,d(x_0,y)))|}|f(y)|d\mu(y) \\
&\leq C\sum_{k=1}^\infty \int_{2^{k+1}B\setminus 2^kB} \left|\prod_{j=1}^m (b_j(y) - (b_j)_{2B})\right| \left[\frac{|d(x_0,x)|}{|d(x,y)|}\right]^\delta \frac{|f(y)|}{|\mu(B(x_0,d(x_0,y)))} d\mu(y) \\
&\leq C\sum_{k=1}^\infty \frac{\mu(B)^\delta}{\mu(2^kB)^{\delta+1}} \int_{2^{k+1}B} \left|\prod_{j=1}^m (b_j(y) - (b_j)_{2B})\right| |f(y)|d\mu(y) \\
&\leq C\sum_{k=1}^\infty 2^{-k(\delta+1)} \prod_{j=1}^m \|b_j - (b_j)_{2^{k+1}B}\|_{\exp L^{r_j},2^{k+1}B}\|f\|_{L(\log L)^{1/r},2^{k+1}B} \\
&\leq C\sum_{k=1}^\infty 2^{-k(\delta+1)} k^m \prod_{j=1}^m \|b_j\|_{Osc_{\exp L^{r_j}}} M_{L(\log L)^{1/r}}(f)(\tilde{x}) \\
&\leq C\|\vec{b}\| M_{L(\log L)^{1/r}}(f)(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 2.2. ∎

By using Theorem 2.2 and the boundedness of T, we obtain Theorem 2.3 and Theorem 2.4.

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