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An Algorithm for Solving Quadratic Fractional Program with Linear Homogeneous Constraints

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Abstract. This paper presents an algorithm for solving a quadratic fractional program when some of its constraints are homogeneous. Using these homogeneous constraints a transformation matrix T is constructed. With the help of this matrix T, the given problem is transformed into another quadratic fractional program with fewer constraints. A relationship between the original problem and the transformed problem is also established which ensures that the solution of the original problem can be obtained from the transformed problem. The algorithm finds out the global optimal basic feasible solution of the quadratic fractional program and is illustrated with the help of a numerical example.

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1. Introduction

Quadratic Fractional Programming Problem is a special type of Mathematical optimization problem, which involves maximization/minimization of the sum of a quadratic fractional and a linear function subject to linear constraints. The general quadratic fractional programming problem with homogeneous constraints can be written as

$$\text{maximize } f(x) = cx + \frac{x^t P x}{x^t Q x} \qquad (P_0)$$

$$\text{subject to } Ax = b,$$

$$x \geq 0,$$

with its i^{th} constraint as $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0$. In Problem (P_0), c is an n-dimensional row vector describing the coefficients of the linear terms in the objective function, P and Q are $(n \times n)$ symmetric matrices describing the coefficients of the quadratic term, x^t is the n-dimensional column vector of the decision variables, constraints coefficients are defined by an $(m \times n)$ matrix A, and b is an m-dimensional column vector of the right-hand-side values. We assume that a feasible solution exists and that the constraint region is bounded.

Research literature is full of variety of applications of quadratic programming such as portfolio optimization [9], structural analysis [1]. Various researchers have given algorithms to solve quadratic programs in literature [4, 5]. Also quadratic programming problem with homogeneous constraints occurs in many real life situations, for example, in portfolio optimization when investment in securities in one sector is dependent on the investment in securities in another sector. In the area of non-linear programming [10] the fractional programming has great significance on account of economic development that has necessitated optimizing the productivity per unit of inputs invested. Aggarwal S. P. [2] in 1973 solved a quadratic fractional program. Later in 1995 Muu and Tam [12] gave efficient algorithms for solving non-convex programs dealing with the product of two affine fractional functions. Basu et al. [3] in 2002 studied the transportation problem quadratic fractional functions. In 2007, Yamamoto et al. [13] gave an efficient algorithm for solving convex-convex quadratic fractional programs. Gupta et al. [8] in 1994 also studied extreme point quadratic fractional programming. In 2004, Cambini et al. [6] discussed on generalized linearity of Quadratic Fractional Functions. Murty K. G. [11] in 1968 gave a novel technique of ranking the extreme point solutions for a fixed charge problem which was further extended by Cobot and Francis [5] for convex quadratic programming problem.

In this paper an algorithm to maximize a pseudoconvex quadratic fractional function over a convex compact polytope is presented. The algorithm is based upon the systematic ranking of the extreme point solutions for finding a solution of the problem.

Also we have considered a quadratic fractional program with some of the constraints as homogeneous. Chadha [7] in 1999 solved a linear fractional program with homogeneous constraints. In this paper, we have developed an alternative approach to solve quadratic programming problem with homogeneous constraints. Our approach is an alternative to the existing variable elimination method to remove homogeneous constraints.

2. Theoretical development

Consider a Quadratic Fractional Program

$$\text{maximize} \quad Z(X) = \frac{(CX+\alpha)(\lambda_1 CX+\beta)}{(DX+\gamma)(\lambda_2 DX+\delta)} \tag{P_1}$$
$$\text{subject to} \quad AX = b,$$
$$X \geq 0, \lambda_1 > 0, \lambda_2 < 0,$$

where C, D are row vectors with n components, $\alpha, \beta, \gamma, \delta$ are scalars, X is a column vector with n components, b is a column vector with m components, $A = \{A_1, A_2, \cdots, A_n\}$ is an $m \times n$ matrix.

It is assumed that $(DX + \gamma) > 0$ and $(\lambda_2 DX + \delta) > 0$ for all $X \in S_1 = \{X : AX = b, X \geq 0\}$.

Now we define a special type of program in which some of the components of the right hand side vector b of the constraint equation are zero.

Consider a Quadratic Fractional Program with Homogeneous Constraints

$$\text{maximize} \quad Z(X) = \frac{(CX+\alpha)(\lambda_1 CX+\beta)}{(DX+\gamma)(\lambda_2 DX+\delta)} \tag{P_2}$$
$$\text{subject to} \quad AX = b,$$
$$X \geq 0, \lambda_1 > 0, \lambda_2 < 0,$$
$$\text{with } a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = 0 \quad \text{for some } i,$$

where all the notations are same as in problem (P_1).

3. Development of transformation matrix T

Let $S = \{X : AX = b, X \geq 0\}$ denote the feasible region for (P_2). Let $X = \{x_1, x_2, \cdots, x_n\}$ be a solution of (P_2). If $x_k > 0, a_{ik} > 0$ then there exists at least one $x_l > 0$ with $a_{il} < 0$.

Now, corresponding to the first homogeneous constraint, let R be a $1 \times n$ row vector and $R = (R_1, R_2, \cdots, R_n)$, where $R_1, R_2, \ldots, R_n$ are the columns of R. So we partition $R = (R^0, R^+, R^-)$, where R^0 is the set of all columns of R for which $a_{ij} = 0 \ \forall\ i$, let the number of such columns be r; R^+ is the set of all columns of R for which $a_{ij} > 0 \ \forall\ i$, let the number of such columns be p; R^- is the set of all columns of R for which $a_{ij} < 0 \ \forall i$, let the number of such columns be q. Thus $p + q + r = n$.

We define a transformation matrix T of order $n \times (pq + r)$ such that the ith equation of $ATW = b$ will be identically zero, where W is a column vector with $(pq + r)$ components. Partition T as $T = (T_1, T_2)$ where T_1 consists of unit column vectors e_j corresponding to $a_{ij} = 0$ and T_2 consists of column vectors corresponding to w_{kl} $\forall k \in A^+$ and $l \in A^-$, where w_{kl} is defined as the (k, l)th element of W such that $R^k \in R^+$ $(k = 1, 2, \ldots, p)$ and $R^l \in R^-$ $(l = 1, 2, \ldots, q)$.

Thus the matrix T possesses n rows and $r + pq$ columns and is represented as

$$T = (T_1, T_2) = \{(e_j) \text{ such that } a_{ij} = 0 \ \forall i; (t_{kl}) \ \forall k \in A^+ \text{ and } l \in A^-\},$$

i.e., e_j is the jth column vector of the identity matrix I_n and

$$t_{kl} = -a_{il}e_k + a_{ik}e_l. \tag{1}$$

Remark 3.1. We have constructed the transformation matrix T in such a way that the ith equation of $ATW = b$ is identically zero. Therefore it is sufficient to show that the ith row of AT will have all its elements as zeros. Let A^i be the ith row of A, then for any $j \in A^0$, it is clear that $A^i e_j = 0$ which implies that $A^i T_1 = 0$. Also, $A^i t_{kl} = A^i(-a_{il}e_k + a_{ik}e_l) = -a_{il}a_{ik} + a_{ik}a_{il} = 0$ for any $(k,l), k \in A^+$ and $l \in A^-$. Hence all the elements of the ith row of ATW will have all the zeros. Also for the ith equation $b_i = 0$. Thus the ith equation of $ATW = b$ will be identically zero.

4. Equivalence relationship

Applying the transformation $X = TW$ in the original problem (P_2), we have the following transformed problem (P_3).

$$\text{Maximize } Z(W) = \frac{(\overline{C}W + \alpha)(\lambda_1 \overline{C}W + \beta)}{(\overline{D}W + \gamma)(\lambda_2 \overline{D}W + \delta)} \tag{P_3}$$
$$\text{subject to } \overline{A}W = b,$$
$$W \geq 0, \lambda_1 > 0, \lambda_2 < 0,$$

where $\overline{C} = CT, \overline{D} = DT, \overline{A} = AT$. Define $S_2 = \{W : \ \overline{A}W = b, W \geq 0\}$.

Theorem 4.1. *If X solves $AX = b$ then there exists a W such that $X = TW$, which solves $\overline{A}W = b$.*

To prove this theorem, we need to prove the following lemma.

Lemma 4.2. *If $\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = V,\ \alpha_i \geq 0,\ \beta_j \geq 0$, then there exists a matrix $Y = (y_{ij} \geq 0)$ such that $\sum_{j=1}^{q} y_{ij} = \alpha_i$ and $\sum_{i=1}^{p} y_{ij} = \beta_j$.*

Proof of the lemma. Define $y_{ij} = \dfrac{\alpha_i \times \beta_j}{V}$. Now

$$\sum_{i=1}^{p} y_{ij} = \sum_{i=1}^{p} \frac{\alpha_i \times \beta_j}{V} = \frac{\beta_j \sum_{i=1}^{p} \alpha_i}{V} = \frac{\beta_j \times V}{V} = \beta_j,$$

and

$$\sum_{j=1}^{q} y_{ij} = \sum_{j=1}^{q} \frac{\alpha_i \times \beta_j}{V} = \frac{\alpha_i \sum_{j=1}^{q} \beta_j}{V} = \frac{\alpha_i \times V}{V} = \alpha_i.$$

Also $y_{ij} \geq 0$ (because $\alpha_i \geq 0, \beta_j \geq 0$). Thus the existence of the matrix $Y = (y_{ij} \geq 0)$ is ensured. ■

Proof of Theorem 4.1. As X is a solution of $AX = b$, then its ith constraint equation will be $\sum_{k\in A^+} a_{ik}x_k + \sum_{i\in A^-} a_{il}x_l = 0$. Putting

$$a_{ik}x_k = \alpha_k \quad \text{and} \quad -a_{il}x_l = \beta_l, \tag{2}$$

we have $\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j$. Thus by Lemma 4.2, $\exists$ a matrix $Y = (y_{ij} \geq 0)$ of order $p \times q$ such that

$$\sum_{l=1}^{q} y_{kl} = \alpha_k \quad \text{and} \quad \sum_{k=1}^{p} y_{kl} = \beta_l. \tag{3}$$

Now we define a vector $W = \begin{bmatrix} W^1 \\ W^2 \end{bmatrix}$, where W^1 is a column vector with r components and W^2 is a column vector with pq components. Also,

$$w_j^i = x_j \quad \forall j \in A^0 \quad \text{and} \quad w_{kl}^2 = \frac{-y_{kl}}{a_{ik}a_{il}} \quad \forall k \in A^+ \text{ and } l \in A^-. \tag{4}$$

Clearly, $W \geq 0$ (because $a_{ik} > 0,\ a_{il} < 0$ and $y_{kl} \geq 0$).

Next, we wish to prove that $ATW = b$ which is equivalent to proving that $TW = X$. Now,

$$\begin{aligned}
TW &= T^1W^1 + T^2W^2 \\
&= \sum_j e_j w_j^1 + \sum_k \sum_l t_{kl} w_{kl}^2 \\
&= \sum_j e_j x_j + \sum_k \sum_l (-a_{il}e_k + a_{ik}e_l)\left(\frac{-y_{kl}}{a_{ik} \times a_{il}}\right) \quad \text{(using (1) and (4))} \\
&= \sum_j e_j x_j + \sum_k \sum_l \frac{e_k a_{il} y_{kl}}{a_{ik}a_{il}} - \sum_k \sum_l \frac{a_{ik} e_l y_{kl}}{a_{ik}a_{il}} \\
&= \sum_j e_j x_j + \sum_k \frac{e_k}{a_{ik}}\alpha_k - \sum_l \frac{e_l}{a_{il}}\beta_l \quad \text{(using (3))} \\
&= \sum_j e_j x_j + \sum_k \frac{e_k}{a_{ik}} a_{ik}x_k - \sum_l \frac{e_l}{a_{il}}(-a_{il x_l}) \quad \text{(using (2))} \\
&= \sum_{j\in A^0} e_j x_j + \sum_{k\in A^+} e_k x_k - \sum_{l\in A^-} e_l x_l = X \\
\Rightarrow TW &= X.
\end{aligned}$$

Since $AX = b \therefore ATW = b \Rightarrow \overline{A}W = b$, hence W solves $\overline{A}W = b$. ■

Theorem 4.3. *If X^* solves the problem (P_2) then $W^*(X^* = TW^*)$ solves the problem (P_3).*

Corollary 4.4. *If W^* solves the problem (P_3) then there exists $X^* = TW^*$ which solves the problem (P_2) and the optimal values of the two objective functions are equal.*

Note 1. In order to solve problem (P_3) which is the same as problem (P_1), as all the homogeneous constraints are removed, we consider two problems (P_1') and (P_1'') related to problem (P_1) as

$$\max_{x\in S_1} U(X) = (CX+\alpha)(\lambda_1 CX+\beta) = \left(\sum_j c_j x_j + \alpha\right)\left(\lambda_1 \sum_j c_j x_j + \beta\right), \qquad (P_1')$$

$$\min_{x\in S_1} W(X) = (DX+\gamma)(\lambda_2 DX+\delta) = \left(\sum_j d_j x_j + \gamma\right)\left(\lambda_2 \sum_j d_j x_j + \delta\right). \qquad (P_1'')$$

Corollary 4.5. *Let $U(X)$ be a non-negative convex function and $W(X)$ be a positive concave function, then the function*

$$Z(X) = \frac{U(X)}{W(X)} = \frac{(CX+\alpha)(\lambda_1 CX+\beta)}{(DX+\gamma)(\lambda_2 DX+\delta)},$$

$\lambda_1 > 0, \lambda_2 < 0, (DX+\gamma) > 0$ and $(\lambda_2 DX+\delta) > 0$ is pseudoconvex and this function $Z(X)$ defined on the polytope S_1 attains its maximum at one of its vertices.

Since it is known that if $\nabla Z(X) = 0$ at an interior point, then X is a minimum point. So unless $Z(X)$ is constant, a maximum must happens on the boundary. Applying the same idea to the faces and the edges, the maximum is located at a vertex of the polytope S_1. This motivates the systematic scanning of the extreme points of S_1 for obtaining the optimal solution of the problem (P_3).

Result. Let $W = \min_{x\in S_1} W(X) = \min_{x\in S_1} (DX+\gamma)(\lambda_2 DX+\delta)$ and $U = \max_{x\in S_1} U(X) = \max_{x\in S_1} (CX+\alpha)(\lambda_1 CX+\beta)$. Then $Z(X) \le \frac{U}{W} \quad \forall X \in S_1$.

Corollary 4.6. *If $\frac{U_{k+1}}{W} \le \max\{Z(X)/X \in \bigcup_{i=1}^{k} S_1^i\} < \frac{U_k}{W}$, then the optimal value of $Z(X)$ in the problem (P_1) is $\max\{Z(X)/X \in \bigcup_{i=1}^{k} S_1^i\}$, where S_1^i is the set of the ith best extreme point solutions of (P_1') and U_{k+1} is the value of $U(X)$ for $X \in S_1^{k+1}$.*

Proof. For $s > k+1, \frac{U_s}{W} < \frac{U_{k+1}}{W} \le \max\{Z(X)/X \in \bigcup_{i=1}^{k} S_1^i\}$ (by hypothesis). Also for all $X \in S_1^s, Z(X) \le \frac{U_s}{W} < \max\{Z(X)/X \in \bigcup_{i=1}^{k} S_1^i\}$. Therefore no extreme point of S_1 after the $(k+1)$th best extreme point solution of (P_1') can yield objective function value of the problem (P_1) better than $\max\{Z(X)/X \in \bigcup_{i=1}^{k} S_1^i\}$. Thus $\max\{Z(X)/X \in \bigcup_{i=1}^{k} S_1^i\}$ is the optimal value of the objective function in the problem (P_1).

If $\max\{Z(X)/X \in \bigcup_{i=1}^{k} S_1^i\} = Z(X_\nu)$, then X_ν is an optimal solution of the problem (P_1). ■

Next, we find out the basic feasible solution of the problem (P_1') in Theorem 4.7.

Theorem 4.7. *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a basic feasible solution of (P_1') with basis matrix B. Then it will be an optimal basic feasible solution if*

$$\begin{cases} R_j \leq 0 & \forall j \notin B, \\ R_j = 0 & \forall j \in B, \end{cases}$$

where $R_j = (z_j - c_j)[\theta_j \lambda_1 (z_j - c_j) - \lambda_1 V_1 - V_2]$, $z_j = c_B B^{-1} a_j \quad \forall j \notin B$ and $z_j' = \lambda_1 c_B B^{-1} a_j = \lambda_1 z_j \quad \forall j \notin B$, V_1 and V_2 are the values of $\sum_j c_j x_j + \alpha$ and $\lambda_1 \sum_j c_j x_j + \beta$ at the current basic feasible solution corresponding to the basis B, and θ_j is the level at which a non basic vector a_j enters the basis replacing some basic vector of B.

Proof. Let U^0 be the objective function value of the problem (P_1'). Let $U^0 = V_1 V_2$. Let $\widehat{U}$ be the value of the objective function at the current basic feasible solution $\widehat{X} = \{x_{ij}\}$ corresponding to the basis B obtained on entering the vector a_j into the basis. Then

$$\widehat{U} = [V_1 + \theta_j (c_j - z_j)][V_2 + \theta_j \lambda_1 (c_j - z_j)].$$

Now

$$\begin{aligned} \widehat{U} - U^0 &= [V_1 + \theta_j (c_j - z_j)][V_2 + \theta_j \lambda_1 (c_j - z_j)] - V_1 V_2 \\ &= V_1 V_2 + V_1 \theta_j \lambda_1 (c_j - z_j) + V_2 \theta_j (c_j - z_j) + \theta_j^2 (c_j - z_j) \lambda_1 (c_j - z_j) - V_1 V_2 \\ &= V_1 \theta_j \lambda_1 (c_j - z_j) + V_2 \theta_j (c_j - z_j) + \theta_j^2 (c_j - z_j) \lambda_1 (c_j - z_j) \\ &= \theta_j (c_j - z_j)[\lambda_1 V_1 + V_2 + \theta_j \lambda_1 (c_j - z_j)]. \end{aligned}$$

This basic feasible solution will give an improved value of U if $\widehat{U} > U^0$, i. e., $\widehat{U} - U^0 > 0$, i. e., $\theta_j (c_j - z_j)[\lambda_1 V_1 + V_2 + \theta_j \lambda_1 (c_j - z_j)] > 0$.

Since $\theta_j \geq 0$

$$\therefore (c_j - z_j)[\lambda_1 V_1 + V_2 + \lambda_1 \theta_j (c_j - z_j)] > 0. \tag{5}$$

$\Rightarrow$ One can move from one basic feasible solution to another basic feasible solution on entering the vector a_j into the basis for which condition (5) is satisfied.

It will be an optimal basic feasible solution if

$$\begin{aligned} & (z_j - c_j)[\theta_j \lambda_1 (z_j - c_j) - \lambda_1 V_1 - V_2] \leq 0 \\ \text{or} \quad & R_j \leq 0 \quad \forall j \notin B, \end{aligned}$$

where $R_j = (z_j - c_j)[\theta_j \lambda_1 (z_j - c_j) - \lambda_1 V_1 - V_2]$. Also, it can easily be verified that $R_j = 0\ \forall j \in B$. ■

5. Ranking the extreme points of the problem (P_1')

Step 1. Find an optimal basic feasible solution of problem (P_1'). Let $X_1 = \{x_j\}$ be a basic feasible solution of (P_1') with basis matrix B_1. Then it will be an optimal basic feasible solution if

$$\begin{cases} R_j^1 \leq 0 & \forall j \notin B_1, \\ R_j^1 = 0 & \forall j \in B_1, \end{cases}$$

where $R_j^1 = (z_j - c_j)[\theta_j \lambda_1 (z_j - c_j) - \lambda_1 V_1 - V_2]$, V_1 and V_2 are the values of $\sum_j c_j x_j + \alpha$ and $\lambda_1 \sum_j c_j x_j + \beta$ at the current basic feasible solution corresponding to basis B_1 and θ_j is the level at which a non basic vector a_j enters the basis replacing some basic vector of B_1.

Step 2. To find the 2^{nd} best basic feasible solution of problem (P_1'), construct the set H_1 defined as follows

$$H_1 = \bigcup \{j / R_j^1 < 0 \quad \forall j \notin B_1\}.$$

Find

$$\max_{j \in H_1} \{\theta_j R_j^1\} = \theta_s R_s^1 \quad \text{(say)}.$$

Then on entering a non-basic vector $a_s \notin B_1$ at the level θ_s and on replacing some basic vector in B_1 yields a 2^{nd} best basic feasible solution of (P_1').Then the corresponding 2^{nd} best value of $U(X)$ is $U_2 = (U_1 + \theta_s R_s^1)$. If $\max_j \{\theta_j R_j^1\}$ is obtained for only one index $j \in H_1$ then the 2^{nd} best basic feasible solution is unique. Otherwise there may be more than one 2^{nd} best basic feasible solution.

Step 3. To find the $(k+1)$th best basic feasible solution of the problem (P_1') $(k \geq 2)$.Suppose that the kth best basic feasible solutions of problem (P_1') have been obtained and B_k is the corresponding basis. The set H_k is defined as

$$H_k = \bigcup \{j / R_j^k < 0, j \notin B_k\}.$$

Find

$$\max \left\{ \max_{j \in \overline{H}_1} (U_1 + \theta_j R_j^1), \cdots, \max_{j \in \overline{H}_k} (U_k + \theta_j R_j^k) \right\},$$

where $\overline{H}_q = H_q \backslash \{j \backslash j \in H_q; j \in \bigcup_{i=q+1}^{k} B_i\} \quad q = 1, 2, \ldots, k-1, \overline{H}_k = H_k$.

If

$$\max \left\{ \max_{j \in \overline{H}_1} (U_1 + \theta_j R_j^1), \cdots, \max_{j \in \overline{H}_k} (U_k + \theta_j R_j^k) \right\}$$
$$= \max_{j \in \overline{H}_p} (U_p + \theta_j R_j^p) = U_p + \theta_s R_s^p \text{ (say)},$$

then entry of $a_s \notin B_p$ into the set of the basic vectors B_p will yield a $(k+1)$th best basic feasible solution with $(k+1)$th best value in (P_1') as $(U_p + \theta_s R_s^p)$.

6. Algorithm for finding a global optimum solution of problem (P_1)

Global optimal solution of problem (P_1) can be obtained by successively tightening the bounds on the objective function value. This is achieved by ranking the extreme point solutions of problem (P_1') in descending order of values of $U(X)$. The steps involved in the algorithm are as follows.

Step 1. Find $\min_{x \in S_1} (DX + \gamma)(\lambda_2 DX + \delta)$. Let its minimum value be W.

Step 2. Find S_1^1, the set of the optimal solutions of problem (P_1'). If

$$\frac{U_1}{W} = \max\{Z(X)/X \in S_1^1\} = Z(X_1) \quad \text{(say)}$$

stop. $X \in S_1^1$ is an optimal solution of (P_1) and optimal value is given by $Z(X_1) = \dfrac{U_1}{W \text{ at } X_1}$. Otherwise, go to Step 3.

Step 3. $(k \geq 2)$ Find the set S_1^k of the kth best extreme point solutions of (P_1'). If either (a) $S_1^k \neq \emptyset$ and $\dfrac{U_k}{W} \leq \max\{Z(X)/X \in \bigcup_{i=1}^{k-1} S_1^i\} = Z(X_\nu)$ or (b) $S_1^k = \emptyset, U_{i+1} < U_i,\ i = 1, \ldots, k-2$ and $\dfrac{U_{k-1}}{W} > \max\{Z(X)/X \in \bigcup_{i=1}^{k-1} S_1^i\} = Z(X_\nu)$, then go to Step 4. Otherwise, the lower bound on $Z(X)$ is updated as $\max\{Z(X)/X \in \bigcup_{i=1}^{k-1} S_1^i\}$ and upper bound as $\dfrac{U_k}{W}$, and then return to Step 3, replacing k by $k+1$.

Step 4. Algorithm ends yielding an optimal solution X_ν.

Note 2. MATLAB code is developed for reducing the original problem (P_2) to transformed problem (P_3) without homogeneous constraints. We find out the transformation matrices through MATLAB and it takes less than 5 seconds to find all transformation matrices.

7. Convergence of the algorithm

The algorithm converges in a finite number of iterations as only extreme point solutions of the problem (P_1') are investigated in a systematic manner and these are finite in number. As we tighten the bounds of the objective function value, the upper bound decreases and the lower bound increases, thereby the algorithm ends when upper bound is less than or equal to the lower bound.

8. Conclusion

The process described in this paper can be extended to define T if $AX = b$ has more than two homogeneous constraints. In case there are s homogeneous constraints, we define s transformation matrices $T(1), T(2), \ldots, T(s)$. Note that $T(2)$ is determined once $AT(1)$ has been computed. In general, $T(s)$ is determined only when $AT(1)T(2)\ldots T(s-1)$ has been computed. The alternative method developed in this paper transforms the original problem to a new formulation, which has less number of constraints but more number of variables. Here, it may be noted that as the complexity of simplex type algorithms depends more on the number of constraints than the number of variables. Therefore alternative method developed in this paper is useful when there are huge number of homogeneous constraints in a quadratic fractional programming problem.

9. Numerical illustration

$$\begin{aligned}
\max Z(X) =& \frac{(2x_1 + 2x_2 + 12)(x_1 + x_2 + 17)}{(-2x_1 + x_2 + 15)(2x_1 - x_2 + 11)} \qquad (A_1)\\
\text{subject to} \quad & x_1 + 2x_2 + x_3 = 2\\
& 3x_1 + x_2 + x_4 = 4\\
& -5x_1 + 3x_2 + x_5 = 0\\
& -x_1 + x_2 + x_6 = 0\\
& x_i \geq 0 \qquad \forall i = 1, 2, 3, 4, 5, 6.
\end{aligned}$$

Solution: Clearly the numerator of problem (A_1) is a non-negative convex function and denominator is a positive concave function. Therefore the objective function in problem (A_1) is a pseudoconvex function and attains its maximum at an extreme point of the feasible region.

Since there are two homogeneous constraints in problem (A_1), therefore there will be two transformation matrices. The problem (A_1) is of the form

$$\begin{aligned}
\text{maximize } Z(X) =& \frac{(CX + \alpha)(\lambda_1 CX + \beta)}{(DX + \gamma)(\lambda_2 DX + \delta)}\\
\text{subject to} \quad & AX = b\\
& X \geq 0, \lambda_1 = 1/2, \lambda_2 = -1 \quad \text{with two homogeneous constraints,}\\
\text{where} \quad & C = (2 \quad 2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) \quad \lambda_1 C = (1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)\\
& D = (-2 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0) \; \lambda_2 D = (2 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)
\end{aligned}$$

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 0 \\ -5 & 3 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first transformation matrix $T(1)$ corresponding to the first homogeneous constraint is given by

$$T(1) = \begin{pmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

On applying the transformation $X = T(1)W$ in problem (A_1), the problem (A_1) can be equivalently written in the form

$$\begin{aligned} &\text{maximize } Z(W) = \frac{(\overline{C}W + \alpha)(\lambda_1 \overline{C}W + \beta)}{(\overline{D}W + \gamma)(\lambda_2 \overline{D}W + \delta)} \\ &\text{subject to} \quad \overline{A}W = b \\ &\qquad\qquad\quad W \geq 0, \\ &\text{where} \qquad\quad \overline{C} = CT(1), \overline{D} = DT(1). \end{aligned}$$

Thus we have the following problem

$$\begin{aligned} \max\ Z(W) = &\frac{(16w_4 + 2w_5 + 12)(8w_4 + w_5 + 17)}{(-w_4 - 2w_5 + 15)(w_4 + 2w_5 + 11)} \qquad (A_{11}) \\ \text{subject to} \quad &w_1 + 13w_4 + w_5 = 2 \\ &w_2 + 14w_4 + 3w_5 = 4 \\ &w_3 + 2w_4 - w_5 = 0 \\ &w_i \geq 0 \quad \forall i = 1, 2, 3, 4, 5. \end{aligned}$$

Since in the transformed problem (A_{11}) there is another homogeneous constraint, therefore we find the second transformation matrix $T(2)$ given by

$$T(2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

On applying the transformation $W = T(2)Y$ in problem (A_{11}), we have

$$\max\ Z(Y) = \frac{(2y_3 + 20y_4 + 12)(y_3 + 10y_4 + 17)}{(-2y_3 - 5y_4 + 15)(2y_3 + 5y_4 + 11)} \qquad (B_1)$$

$$\begin{aligned} \text{subject to} \quad & y_1 + y_3 + 15y_4 = 2 \\ & y_2 + 3y_3 + 20y_4 = 4 \\ & y_i \geq 0 \quad \forall i = 1,2,3,4. \end{aligned}$$

To solve the above problem, we first solve

$$\begin{aligned} \max\ U(Y) =& (2y_3 + 20y_4 + 12)(y_3 + 10y_4 + 17) \qquad (B_1') \\ \text{subject to} \quad & y_1 + y_3 + 15y_4 = 2 \\ & y_2 + 3y_3 + 20y_4 = 4 \\ & y_i \geq 0 \quad \forall i = 1,2,3,4. \end{aligned}$$

The optimal basic feasible solution of problem (B_1') is given by

$$Y_1 = \{y_1 = 0, y_2 = 0, y_3 = 4/5, y_4 = 2/25\}$$

and maximum value of $U(Y) = U_1 = 7068/25$.

Now, we solve

$$\begin{aligned} \max\ W(Y) =& (-2y_3 - 5y_4 + 15)(2y_3 + 5y_4 + 11) \qquad (B_1'') \\ \text{subject to} \quad & y_1 + y_3 + 15y_4 = 2 \\ & y_2 + 3y_3 + 20y_4 = 4 \\ & y_i \geq 0 \quad \forall i = 1,2,3,4. \end{aligned}$$

Its optimal basic feasible solution is given by $\{y_1 = 2, y_2 = 4, y_3 = 0, y_4 = 0\}$ and minimum value of $W(Y) = W = 165$. Thus

$$\begin{aligned} \text{upper bound} &= \frac{U_1}{W} = \frac{7068/25}{165} = \frac{7068}{4125} \\ \text{and lower bound} &= \frac{U_1}{W \text{ at } Y_1} = \frac{7068/25}{4221/25} = \frac{7068}{4221}. \end{aligned}$$

Since lower bound < upper bound, therefore we proceed further.

For the 2nd best basic feasible solution of the problem (B_1'), we find

$$\max\{\theta_1 R_1, \theta_2 R_2\} \quad \text{as} \quad R_1, R_2 < 0,$$

for non-basic variables y_1, y_2 in the optimal solution of problem (B_1')

$$= \max\{2/3(-1556/75), 4/3(-778/75)\} = \max\left\{\frac{-3112}{225}, \frac{-3112}{225}\right\} = \frac{-3112}{225}.$$

Since maximum is obtained for two indices, therefore there may be two second best basic feasible solutions. The 2nd best basic feasible solution is $Y_2 = \{y_1 = 2/3, y_2 = 0, y_3 = 4/3, y_4 = 0\}$ with $U_2 = (44/5)(55/3) = 2420/9$. Thus

$$\text{current upper bound} = \frac{U_2}{W} = \frac{2420/9}{165} = \frac{2420}{1485}$$
$$\text{and current lower bound} = \frac{U_2}{W \text{ at } Y_2} = \frac{2420/9}{2747/9} = \frac{2420}{2747}.$$

Update lower bound as $\max\{Z(Y)/Y \in S_1^1\} = \dfrac{7068}{4221}$ and update upper bound as $\dfrac{U_2}{W} = \dfrac{2420}{1485}$.

Since lower bound > upper bound, therefore we stop.

Thus 1st best basic feasible solution gives the optimal basic feasible solution. Hence optimal basic feasible solution of problem (B_1) is $\{y_1 = 0, y_2 = 0, y_3 = 4/5, y_4 = 2/25\}$ and maximum value of $Z(Y) = \dfrac{7068}{4221}$.

Now the solution to the original problem is given by $T(1)T(2)Y = X$

$$\Rightarrow \begin{pmatrix} 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 5 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 4/5 \\ 2/25 \end{pmatrix} = \begin{pmatrix} 6/5 \\ 2/5 \\ 0 \\ 0 \\ 24/5 \\ 4/5 \end{pmatrix}.$$

Hence the optimal solution to original problem (A_1) is

$$\{x_1 = 6/5, x_2 = 2/5, x_3 = 0, x_4 = 0, x_5 = 24/5, x_6 = 4/5\}$$

and the maximum value of $Z(X) = \dfrac{7068}{4221}$.

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