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A kind of Random Deviation Theorems for Stochastic Sequence on Random Selection Systems and Laplace Transforms *

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Abstract. In this paper, the notion of asymptotic logarithmic likelihood ratio, as a measure of dissimilarity between the joint distribution density function and the marginal product density function, is introduced. A kind of strong limit theorems represented by inequalities for the dependent nonnegative stochastic sequence on the random selection system are obtained by using the tools of Laplace transform and the differentiation on a net. The bounds given by the theorems depend on sample points. Some results obtained are generalized.

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1. Introduction

Let $\{X_n, n \geq 0\}$ be a sequence of nonnegative integrable random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ with the joint distribution density function:

$$P(X_0 = x_0, \cdots, X_n = x_n) = f(x_0, \cdots, x_n) > 0,\ x_i \in S,\ 0 \leq i \leq n. \quad (1)$$

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Let Q be another probability measure on $(\Omega, \mathcal{F})$, $\{X_n, n \geq 0\}$ be an independent sequence of random variables on the measure Q, with the marginal product density function as follows:

$$Q(X_0 = x_0, \cdots, X_n = x_n) = g(x_0, \cdots, x_n) = \prod_{k=0}^{n} f_k(x_k), \tag{2}$$

where $f_k(x_k)$ represents the density function of the random variable $X_k(k = 0, 1, 2, \cdots)$.

Definition 1.1. Let f and g be defined as (1) and (2), $\{a_n,\ n \geq 0\}$ be a sequence of nonnegative random variables and $a_n \uparrow \infty$. Set

$$h(P|Q) = \limsup_{n\to\infty} \frac{1}{a_n} \log \frac{f(X_0, \cdots, X_n)}{g(X_0, \cdots, X_n)}. \tag{3}$$

$h(P|Q)$ is called the likelihood ratio of $\{X_n, n \geq 0\}$ on P relative to the measure Q with regard to $\{a_n, n \geq 0\}$.

In fact, $h(P|Q)$ is also called the limit relative logarithmic likelihood ratio or asymptotic logarithmic likelihood ratio of $\{X_n, n \geq 0\}$ on the measure P relative to Q with regard to $\{a_n, n \geq 0\}$, where log is the natural logarithm. Although $h(P|Q)$ is not a proper metric between the probability measures, we nevertheless think of it as a measure of "dissimilarity" between the joint distribution and the marginal product distribution of $\{X_n, n \geq 0\}$. Obviously, $h(P|Q) = 0$ if and only if $P = Q$. It will be shown in (6) that $h(P|Q) \geq 0$ a.s. in any case. Hence, $h(P|Q)$ can be used as a random measure of the deviation between the true joint distribution density function $f(x_0, \cdots, x_n), (n \geq 0)$ and the reference marginal product density function $\prod_{k=0}^{n} f_k(x_k), (n \geq 0)$. Roughly speaking, this deviation may be regarded as the case between $\{X_n, n \geq 0\}$ under the measure P and the independent case under the measure Q. The smaller $h(P|Q)$ is, the smaller the deviation is.

Liu Wen (see [7]) discussed a class of strong deviations for arbitrary stochastic sequence with respect to the marginal distribution by using generating function methods, he also studied the above problem by means of Laplace transform (see [9]). Yang Weiguo and Liu Wen (see [8]) investigated the strong deviation theorems for arbitrary information source relative to Markov information source. Wang Zhongzhi (see [12]) discussed the strong deviation theorems for the random sum of arbitrary stochastic sequences. Li Gaorong (see [6]) have explored a class of strong deviation theorems for the continuous stochastic sequence with respect to the marginal distribution by using the approach of Laplace transform. Wang Kangkang (see [10, 11]) have recently studied some strong deviation theorems for the arbitrary information source with respect to the mth-order nonhomogeneous Markov information source and some limit properties for nonhomogeneous Markov chain on the random selection systems.

In this paper, we generalize the research of Li and Wang to the case of the random sum of dependent nonnegative stochastic sequences on the gambling systems, and establish a class of small deviation theorems for arbitrary stochastic dominated sequences with random bounds for the dependent random variables, that is, to study the expressions

$$\liminf_{n} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [X_k - E_Q(X_k)], \ \limsup_{n} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [X_k - E_Q(X_k)].$$

We provide a lower bound for the liminf and an upper bound for the limsup in terms of some functions of Laplace transforms of the tail of $\{X_n, n \geq 0\}$, the differentiation on a net and the so-called asymptotic logarithmic likelihood ratio defined by (3). As corollaries, some strong limit theorems for the independent stochastic sequence are obtained and the results of Li (see [6]) are generalized.

Lemma 1.2. *Let f and g be two arbitrary probability measures, denote $\alpha > 0$, if*

$$\liminf_{n\to\infty} \left(\frac{a_n}{n^\alpha}\right) > 0 \quad P - a.s. \tag{4}$$

Then

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \frac{g(X_0, \cdots, X_n)}{f(X_0, \cdots, X_n)} \leq 0 \quad P - a.s. \tag{5}$$

Proof. It is easy to see that $Z_n = g(X_0, \cdots, X_n)/f(X_0, \cdots, X_n)$ is a nonnegative superior martingale and $E_P(Z_n) \leq 1$ (see [1]), where E_P denotes the expectation under the measure P. Hence for all $\varepsilon > 0$, we have by Markov's inequality

$$\begin{aligned} & \sum_{n=1}^{\infty} P\left\{\frac{1}{n^\alpha} \log \frac{g(X_0, \cdots, X_n)}{f(X_0, \cdots, X_n)} \geq \varepsilon\right\} \\ = & \sum_{n=1}^{\infty} P\left\{\frac{1}{n^\alpha} \log Z_n \geq \varepsilon\right\} = \sum_{n=1}^{\infty} P\left(\log Z_n \geq n^\alpha \varepsilon\right) \\ = & \sum_{n=1}^{\infty} P\left(Z_n \geq \exp\{n^\alpha \varepsilon\}\right) \leq \sum_{n=1}^{\infty} \frac{E_P(Z_n)}{\exp(\varepsilon n^\alpha)} \\ \leq & \sum_{n=1}^{\infty} \exp(-\varepsilon n^\alpha) < \infty. \end{aligned} \tag{6}$$

Since $\varepsilon > 0$ is arbitrary, by the Borel-Cantelli lemma it follows from (6) that

$$\limsup_{n\to\infty} \frac{1}{n^\alpha} \log \frac{g(X_0, \cdots, X_n)}{f(X_0, \cdots, X_n)} \leq 0 \quad P - \text{a.s.} \tag{7}$$

Obviously (4) and (7) imply (5). ∎

We have by (3), (4) and (5) that

$$h(P|Q) \geq \liminf_{n\to\infty} \frac{1}{a_n} \log \frac{f(X_0, \cdots, X_n)}{g(X_0, \cdots, X_n)} \geq 0 \quad P - \text{a.s.} \tag{8}$$

By (2) and (3), we have

$$h(P|Q) = \limsup_{n\to\infty} \frac{1}{a_n(\omega)} \log[f(X_0, \cdots, X_n) \Big/ \prod_{k=0}^{n} f_k(X_k)]. \tag{9}$$

Let $\{X_n, n \geq 0\}$ be a sequence of random variables, and $f_k(X_k)$, $k = 1, 2, \cdots, n$ be the marginal density functions of $f(X_0, \cdots, X_n)$. We define the Laplace transform and the tailed-probability Laplace transform as follows:

$$W_k(s) = \int_0^{+\infty} e^{-sx_k} f_k(x_k) dx_k, \tag{10}$$

$$Q_k(s) = \int_0^{+\infty} e^{-sx} \int_x^{+\infty} f_k(x_k) dx_k dx. \tag{11}$$

In order to explain the real meaning of the notion of the random selection, we consider the following gambling model. Let $\{X_n, n \geq 0\}$ be a stochastic sequence with the joint distribution (1), and $g(x)$ be a real-valued function defined on S. Interpret X_n as the result of the nth trial, the type of which may change at each step. Let $\mu_n = Y_n g(X_n)$ denote the gain of the bettor at the nth trial, where Y_n represents the bet size, $g(X_n)$ is determined by the gambling rules, and $\{Y_n, n \geq 0\}$ is called a gambling system or a random selection system. The bettor's strategy is to determine $\{Y_n, n \geq 1\}$ by the results of the former $n-1$ trials. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n is independent of X_{n-1}, X_{n-2}, $\cdots$, X_1 as $n \geq 1$. Thus $\sum_{k=1}^{n} Y_k g(X_k)$ represents the total gain in the first n trials, $\sum_{k=1}^{n} b_k$ the accumulated entrance fees, and $\sum_{k=1}^{n} [Y_k g(X_k) - b_k]$ the accumulated net gain. Motivated by the classical definition of "fairness" of game of chance (see [3]), we introduce the following definition:

Definition 1.3. The game is said to be fair, if for almost all $\omega \in \{\omega : \sum_{k=1}^{\infty} Y_k = \infty\}$, the accumulated net gain in the first n trials is of smaller order of magnitude than the accumulated stake $\sum_{k=1}^{n} Y_k$ as n tends to infinity, that is

$$\lim_{n\to\infty} \frac{1}{\sum_{k=1}^{n} Y_k} \sum_{k=1}^{n} [Y_k g(X_k) - b_k] = 0 \tag{12}$$

$$\text{a.s. on } \{\omega : \sum_{k=1}^{\infty} Y_k = \infty\}. \tag{13}$$

We will establish some strong limit theorems represented by inequalities to generalize the above equation in the second paragraph.

Definition 1.4. Let $\{X_n, n \geq 0\}$ be a sequence of independent nonnegative random variables on the measure Q, then $\{X_n, n \geq 0\}$ is said to be stochastically dominated by a random variable X if there exists a constant $D > 0$, such that $\forall x > 0,\ n \geq 0$,

$$Q(X_n > x) \le D \cdot Q(X > x). \tag{14}$$

and denoted by $\{X_n, n \ge 0\} \prec X$.

Lemma 1.5. ([6]) *Let $W_k(s)$ and $Q_k(s)$ be defined by* (10) *and* (11), $s \in [-s_0, s_0]$. *Then*

$$Q_k(s) = \frac{1 - W_k(s)}{s}, \tag{15}$$

and

$$Q_k(0) = \int_0^{+\infty} \int_x^{+\infty} f_k(x_k) dx_k dx = E_Q(X_k) = \int_0^{+\infty} x_k f_k(x_k) dx_k,\ k = 0, 1, 2, \cdots, n. \tag{16}$$

2. Main results and their proofs

Theorem 2.1. *Let $\{X_n, n \ge 0\}$ be a sequence of arbitrary random variables with the joint distribution* (1), *and $\{X_n, n \ge 0\} \prec X$ on the measure Q. Let $h(P|Q)$, $W_k(s)$ and $Q_k(s)$ be defined as before, $\{\sigma_n, n \ge 0\}$ be a nondecreasing nonnegative stochastic sequence,*

$$E_Q(X) = \int_0^{+\infty} x f(x) dx < \infty. \tag{17}$$

Let

$$D(\omega) = \{\omega : \liminf_{n\to\infty} \left(\sum_{k=0}^{[\sigma_n]} Y_k \Big/ n^\alpha \right) > 0,\ h(P|Q) < \infty\}, \tag{18}$$

then

$$\liminf_n \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [X_k - E_Q(X_k)] \ge \alpha(h(P|Q)) \quad P - a.s.\ \omega \in D(\omega), \tag{19}$$

where

$$\alpha(x) = \sup\{\varphi(s, x),\ 0 < s \le s_0\},\ 0 \le x < +\infty. \tag{20}$$

$$\varphi(s, x) = \liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [Q_k(s) - Q_k(0)] - \frac{x}{s},\ 0 < s \le s_0,\ 0 \le x < +\infty. \tag{21}$$

$$\alpha(x) \le 0,\ \lim_{x\to 0^+} \alpha(x) = \alpha(0) = 0. \tag{22}$$

Here $[c]$ represents the integral part of c, E_Q the expectation with respect to the measure Q.

Remark 2.2. In Theorem 2.1, in this case we have

$$h(P|Q) = \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log[f(X_0, \cdots, X_{[\sigma_n]}) \Big/ \prod_{k=0}^{[\sigma_n]} f_k(X_k)].$$

Proof of Theorem 2.1. For arbitrary $s \in [-s_0, s_0]$, denote

$$D_{x_0\cdots x_n} = \{\omega : X_k = x_k,\ 0 \le k \le n\},\ x_k \in S.$$

Then

$$P(D_{x_0\cdots x_n}) = P(X_0 = x_0, \cdots, X_n = x_n) = f(x_0, \cdots, x_n). \tag{23}$$

$D_{x_0\cdots x_n}$ is called an nth-order elementary cylinder. Let N_n be the collection of nth-order elementary cylinders, N be the collection consisting of ϕ, Ω, and all cylinder sets N_n. Define a set function μ on N as follows:

$$\begin{aligned}\mu(D_{x_0\cdots x_n}) &= \prod_{k=0}^{n} \left(\frac{1}{W_k(s)}\right)^{y_k} e^{-sy_kx_k} f_k(x_k) \\ &= \frac{1}{\prod_{k=0}^{n} W_k(s)^{y_k}} \exp(-s\sum_{k=0}^{n} x_k y_k) \prod_{k=0}^{n} f_k(x_k).\end{aligned} \tag{24}$$

$$\mu(I_{x_0}) = \sum_{x_1\in S_1} \mu(D_{x_0x_1}),\ \mu(\Omega) = \sum_{x_0\in S_0} \mu(I_{x_0}), \tag{25}$$

where

$$y_k = f_{k-1}(x_0, \cdots, x_{k-1}),\ k \ge 1.$$

We have by (24) that

$$\begin{aligned}&\int_0^{+\infty} \mu(D_{x_0,\cdots,x_n}) dx_n \\ &= \int_0^{+\infty} \prod_{k=0}^{n} \left(\frac{1}{W_k(s)}\right)^{y_k} e^{-sy_kx_k} f_k(x_k) dx_n \\ &= \prod_{k=0}^{n-1} \frac{e^{-sy_kx_k} f_k(x_k)}{W_k(s)^{y_k}} \int_0^{+\infty} \frac{e^{-sy_nx_n} f_n(x_n)}{W_n(s)^{y_n}} dx_n \\ &= \mu(D_{x_0,\cdots,x_{n-1}}) \int_0^{+\infty} \frac{e^{-sy_nx_n} f_n(x_n)}{W_n(s)^{y_n}} dx_n \\ &= \mu(D_{x_0,\cdots,x_{n-1}}).\end{aligned} \tag{26}$$

When $y_n = 0$, we get

$$\int_0^{+\infty} \frac{e^{-sy_nx_n} f_n(x_n)}{W_n(s)^{y_n}} dx_n = \int_0^{+\infty} f_n(x_n) dx_n = 1. \tag{27}$$

When $y_n = 1$, we obtain

$$\int_0^{+\infty} \frac{e^{-sy_nx_n} f_n(x_n)}{W_n(s)^{y_n}} dx_n = \int_0^{+\infty} \frac{e^{-sx_n} f_n(x_n)}{W_n(s)} dx_n = \frac{W_n(s)}{W_n(s)} = 1. \tag{28}$$

It follows from (24)-(26) that μ is a measure on N. Since N is a semi-algebra, μ has a unique extension to the σ-field $\sigma(N)$. Let

$$T_n(s,\omega) = \sum_{D \in N_n} \frac{\mu(D_{x_0 \cdots x_n})}{P(D_{x_0 \cdots x_n})} I_{D_{x_0 \cdots x_n}},$$

where $I_{D_{x_0 \cdots x_n}}$ denotes the indicate function of $D_{x_0 \cdots x_n}$, that is

$$T_n(s,\omega) = \frac{\mu(D_{X_0(\omega) \cdots X_n(\omega)})}{P(D_{X_0(\omega) \cdots X_n(\omega)})}. \tag{29}$$

It is easy to see that $\{N_n, n \geq 0\}$ is a net relative to (Ω, A, P), where A denotes the σ-algebra of events on which P is defined. By the differentiation on a net of Hewitt and Stromberg (see [4], p. 373), there exists $A(s) \in \sigma(N)$ with $P(A(s)) = 1$ such that

$$\lim_n T_n(s,\omega) = T_\infty(s,\omega) < \infty \quad P - \text{a.s. } \omega \in A(s),$$

that is

$$\lim_n T_n(s,\omega) = T_\infty(s,\omega) < \infty \quad P - \text{a.s.} \tag{30}$$

By (18), (29), (30) and Lemma 1.2, we have

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log T_{[\sigma_n]}(s,\omega) \leq 0 \quad P - \text{a.s. } \omega \in D(\omega). \tag{31}$$

By (23), (24) and (29), (31) can be rewritten as

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log \left[\frac{1}{\prod_{k=0}^{[\sigma_n]} W_k(s)^{Y_k}} \frac{\exp(-s \sum\limits_{k=0}^{[\sigma_n]} X_k Y_k) \prod\limits_{k=0}^{[\sigma_n]} f_k(X_k)}{f(X_0, \cdots, X_{[\sigma_n]})} \right] \leq 0$$

$$P - a.s. \ \omega \in D(\omega), \tag{32}$$

that is

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \left\{ -\sum_{k=0}^{[\sigma_n]} \log W_k(s)^{Y_k} - s\sum_{k=0}^{[\sigma_n]} X_k Y_k - \log[\frac{f(X_0, \cdots, X_{[\sigma_n]})}{\prod\limits_{k=0}^{[\sigma_n]} f_k(X_k)}] \right\} \leq 0$$

$$P - \text{a.s. } \omega \in D(\omega). \tag{33}$$

By (18), (31) and (33), we have

$$\limsup_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}(-s)X_kY_k\le\limsup_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}Y_k\log W_k(s)+h(P|Q)$$
$$P-\text{a.s. }\omega\in D(\omega). \tag{34}$$

Let $0<s\le s_0$, dividing two sides of (34) by $-s$, we have

$$\liminf_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}X_kY_k\ge\liminf_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}\frac{Y_k\log W_k(s)}{-s}+\frac{h(P|Q)}{-s}$$
$$P-\text{a.s. }\omega\in D(\omega). \tag{35}$$

By (35) and Lemma 1.5, the inequality $1-1/x\le\log x\le x-1(x>0)$ and the property of the inferior limit

$$\liminf_{n\to\infty}(a_n-b_n)\ge d\Rightarrow\liminf_{n\to\infty}(a_n-c_n)\ge\liminf_{n\to\infty}(b_n-c_n)+d,$$

we obtain

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}Y_k[X_k-E_Q(X_k)]\\
\ge\ &\liminf_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}Y_k[\frac{\log W_k(s)}{-s}-E_Q(X_k)]-\frac{h(P|Q)}{s}\\
\ge\ &\liminf_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}Y_k[\frac{W_k(s)-1}{-s}-E_Q(X_k)]-\frac{h(P|Q)}{s}\\
=\ &\liminf_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}Y_k[Q_k(s)-Q_k(0)]-\frac{h(P|Q)}{s}\quad P-\text{a.s. }\ \omega\in D(\omega).
\end{aligned} \tag{36}$$

Letting $s=0$ in (34), we arrive at

$$h(P|Q)\ge0\qquad P-\text{a.s. }\ \omega\in D(\omega). \tag{37}$$

Let

$$g(s)=\liminf_{n\to\infty}\frac{1}{\sum_{k=0}^{[\sigma_n]}Y_k}\sum_{k=0}^{[\sigma_n]}Y_k[Q_k(s)-Q_k(0)],\quad 0<s\le s_0, \tag{38}$$

then by (21) and (38), we have

$$\varphi(s,x)=g(s)-\frac{x}{s},\quad 0<s\le s_0,\quad x\ge0, \tag{39}$$

$$\alpha(x) = \sup\{g(s) - \frac{x}{s}, \quad 0 < s \le s_0\}, \quad x \ge 0. \tag{40}$$

By L'Hospital rule, we have

$$\lim_{s\to 0} Q_k(s) = -\lim_{s\to 0} W_k^{'}(s) = E_Q(X_k).$$

Obviously $g(s) \le 0$, $\varphi(s,x) \le 0$, hence $\alpha(x) \le 0$.

Let $Q(s)$ denote the Laplace transform of $\int_x^{+\infty} f(u)du$, $(u > 0)$

$$Q(s) = \int_0^{+\infty} e^{-sx} \int_x^{+\infty} f(u)dudx. \tag{41}$$

If $0 \le s - t < s \le s_0$, by (11), (14) and (38), we have

$$\begin{aligned}
0 &< g(s-t) - g(s) \\
&= \liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [Q_k(s-t) - Q_k(0)] \\
&\quad - \liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [Q_k(s) - Q_k(0)] \\
&= \liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [Q_k(s-t) - Q_k(0)] \\
&\quad + \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [Q_k(0) - Q_k(s)] \\
&\le \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [Q_k(s-t) - Q_k(s)] \\
&= \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k \Big[\int_0^{+\infty} e^{-(s-t)x} \int_x^{+\infty} f_k(x_k)dx_kdx \\
&\qquad\qquad - \int_0^{+\infty} e^{-sx} \int_x^{+\infty} f_k(x_k)dx_kdx\Big] \\
&= \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k \Big[\int_0^{+\infty} (e^{-(s-t)x} - e^{-sx}) \int_x^{+\infty} f_k(x_k)dx_kdx\Big] \\
&\le \limsup_{n\to\infty} \frac{D}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k \Big[\int_0^{+\infty} (e^{-(s-t)x} - e^{-sx}) \int_x^{+\infty} f(u)dudx\Big], \\
&\qquad\qquad (e^{-(s-t)x} - e^{-sx} > 0)
\end{aligned}$$

$$= \limsup_{n\to\infty} \frac{D}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k \Big[\int_0^{+\infty} e^{-(s-t)x} \int_x^{+\infty} f(u)dudx - \int_0^{+\infty} e^{-sx} \int_x^{+\infty} f(u)dudx \Big]$$
$$\le D[Q(s-t) - Q(s)]. \tag{42}$$

By (42) we know that $g(s)$ is a continuous function with respect to s on the interval $[0, s_0]$, hence it is easy to see $\varphi(s, x)$ is also a continuous function on the interval $[0, s_0]$ with respect to s. Let Q_* be a set of rational numbers which is dense in the interval $[0, s_0]$. By (39) and (40), we have for every $\omega \in D(\omega)$, $\exists \lambda_n(\omega) \in Q_*$ $(n = 1, 2, \cdots)$ such that

$$\lim_n \varphi(\lambda_n(\omega), h(P|Q)) = \alpha(h(P|Q)). \tag{43}$$

By (36), (38), (39) and (40), we have

$$\liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [X_k - E_Q(X_k)] \ge \varphi(\lambda_n(\omega), h(P|Q)), \ n = 1, 2, \cdots$$
$$P-\text{a.s.} \ \ \omega \in D(\omega). \tag{44}$$

By (43) and (44) we have

$$\liminf_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [X_k - E_Q(X_k)] \ge \alpha(h(P|Q)) \quad P-\text{a.s.}\ \omega \in D(\omega). \tag{45}$$

Hence (19) follows from (45).

When $0 < s \le s_0$, by (14) and (41), we have

$$\frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [Q_k(s) - Q_k(0)]$$
$$= \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [\int_0^{+\infty} e^{-sx} \int_x^{+\infty} f_k(x_k)dx_k dx - \int_0^{+\infty} \int_x^{+\infty} f_k(x_k)dx_k dx]$$
$$= \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [\int_0^{+\infty} (e^{-sx} - 1) \int_x^{+\infty} f_k(x_k)dx_k dx]$$
$$\ge \frac{D}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [\int_0^{+\infty} (e^{-sx} - 1) \int_x^{+\infty} f(u)dudx] \quad (e^{-sx} - 1 \le 0)$$
$$= \frac{D}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k [\int_0^{+\infty} e^{-sx} \int_x^{+\infty} f(u)dudx - \int_0^{+\infty} \int_x^{+\infty} f(u)dudx]$$
$$= D[Q(s) - Q(0)]. \tag{46}$$

For $x > 0$, we have that

$$\alpha(x) \geq \varphi(\sqrt{x}, x) = g(\sqrt{x}) - \frac{x}{\sqrt{x}} \geq D[Q(\sqrt{x}) - Q(0)] - \sqrt{x}, \quad n \geq 1. \tag{47}$$

While for $x = 0$, we have

$$\alpha(0) \geq g(\sqrt{1/n}) \geq D[Q(\sqrt{1/n}) - Q(0)], \quad n \geq 1. \tag{48}$$

Since $\alpha(x) \leq 0$ $(x \geq 0)$, (22) follows from (47) and (48). ■

Theorem 2.3. *Under the assumption of Theorem 2.1, we denote*

$$E_Q(X) = \int_0^{+\infty} x f(x) dx < \infty, \tag{49}$$

we get

$$\limsup_{n} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n(\omega)]} Y_k\{X_k - E_Q(X_k)\} \leq \beta(h(P|Q)) \quad P - a.s. \ \ \omega \in D(\omega), \tag{50}$$

where

$$\beta(x) = \inf\{\psi(s, x), \quad -s_0 \leq s < 0\}, \quad 0 \leq x < +\infty, \tag{51}$$

$$\psi(s, x) = \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[Q_k(s) - Q_k(0)] - \frac{x}{s}, \ -s_0 \leq s < 0, \ 0 \leq x < +\infty, \tag{52}$$

$$\beta(x) \geq 0; \ \lim_{x\to 0^+} \beta(x) = \beta(0) = 0. \tag{53}$$

Proof. Let $-s_0 \leq s < 0$, dividing the two sides of (34) by $-s$, we have

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} X_k Y_k \leq \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} \frac{Y_k \log W_k(s)}{-s} + \frac{h(P|Q)}{-s}$$

$$P - \text{a.s.} \ \ \omega \in D(\omega). \tag{54}$$

By (54), Lemma 1.5 and the inequality $1 - 1/x \leq \log x \leq x - 1 (x > 0)$, in virtue of the property of the superior limit

$$\limsup_{n\to\infty}(a_n - b_n) \leq d \Rightarrow \limsup_{n\to\infty}(a_n - c_n) \leq \limsup_{n\to\infty}(b_n - c_n) + d,$$

we have

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[X_k - E_Q(X_k)]$$

$$\leq \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[\frac{\log W_k(s)}{-s} - E_Q(X_k)] - \frac{h(P|Q)}{s}$$

$$\leq \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} [\frac{W_k(s)-1}{-s} - E_Q(X_k)] - \frac{h(P|Q)}{s}$$

$$= \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} [Q_k(s) - Q_k(0)] - \frac{h(P|Q)}{s} \quad P-\text{a.s.} \;\; \omega \in D(\omega). \quad (55)$$

Let Q^* be the set of rational numbers in the interval $[-s_0, 0)$, by (55), we have

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[X_k - E_Q(X_k)]$$

$$\leq \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} [Q_k(s) - Q_k(0)] - \frac{h(P|Q)}{s} \;\; P-\text{a.s.} \; \omega \in D(\omega) \; \forall s \in Q^*. \quad (56)$$

Let

$$g(s) = \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[Q_k(s) - Q_k(0)], \quad -s_0 \leq s < 0. \quad (57)$$

By (52) and (57), we have

$$\psi(s,x) = g(s) - \frac{x}{s}, \quad -s_0 \leq s < 0, \; x \geq 0, \quad (58)$$

$$\beta(x) = \inf\{g(s) - \frac{x}{s}, \quad -s_0 \leq s < 0\}. \quad (59)$$

Obviously $g(s) \geq 0, \psi(s,x) \geq 0$, hence $\beta(x) \geq 0$. By imitating (41) and (42), we can also know that $g(s)$ is a continuous function with respect to s on the interval $[-s_0, 0]$, then it is easy to see that $\psi(s,x)$ is also a continuous function with respect to s on the interval $[-s_0, 0]$. By (51) for each $\omega \in D(\omega)$, take $\lambda_n(\omega) \in Q^*$, $(n = 1, 2, \cdots)$, such that

$$\lim_n \psi(\lambda_n(\omega), h(P|Q)) = \beta(h(P|Q)). \quad (60)$$

By (56)-(58), it can be obtained that

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[X_k - E_Q(X_k)] \leq \psi(\lambda_n(\omega), h(P|Q))$$

$$P-\text{a.s.} \;\; \omega \in D(\omega). \;\; n = 1, 2, \cdots \quad (61)$$

By (60) and (61), we obtain

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[X_k - E_Q(X_k)] \le \beta(h(P|Q)) \quad P-\text{a.s. } \omega \in D(\omega). \tag{62}$$

Hence (50) holds from (62). When $-s_0 \le s < 0$, similar to (46) we have

$$\frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[Q_k(s) - Q_k(0)] \le D[Q(s) - Q(0)], \ (e^{-sx} - 1 \ge 0)$$

$$P-\text{a.s. } \omega \in D(\omega). \tag{63}$$

For $x > 0$, we have

$$\beta(x) \le \psi(\sqrt{x}, x) = g(\sqrt{x}) - \frac{x}{\sqrt{x}} \le D[Q(\sqrt{x}) - Q(0)] - \sqrt{x}. \tag{64}$$

While for $x = 0$, we have

$$\beta(0) \le g(\sqrt{1/n}) \le D[Q(\sqrt{1/n}) - Q(0)], \ n \ge 1. \tag{65}$$

Noticing that $\beta(x) \ge 0$ $(x \ge 0)$, (53) follows from (64) and (65). ■

Definition 2.4. Let $\{X_n, n \ge 0\}$ be a sequence of independent nonnegative random variables on the measure Q, then $\{X_n, n \ge 0\}$ is said to be stochastically dominated in Cesaro sense by a random variable X if there exists a constant $D > 0$ such that $\forall x > 0,\ n \ge 1$,

$$\sum_{k=1}^{n} Q(X_k > x) \le nD \cdot Q(X > x)$$

and denoted by $\{X_n, n \ge 0\} \prec X(c)$.

Theorem 2.5. *Let $\{X_n, n \ge 0\}$ be a sequence of arbitrary nonnegative random variables, $\{\sigma_n, n \ge 0\}$, $h(P|Q)$, $W_k(s)$ and $Q_k(s)$ be defined as above. If $\{X_n, n \ge 0\} \prec X(c)$, $Q(s) < \infty$, then*

$$\limsup_{n} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n(\omega)]} Y_k\{X_k - E_Q(X_k)\} \le \beta(h(P|Q)) \quad P-a.s. \ \omega \in D(\omega), \tag{66}$$

$$\liminf_{n} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n(\omega)]} Y_k\{X_k - E_Q(X_k)\} \ge \alpha(h(P|Q)) \quad P-a.s. \ \omega \in D(\omega), \tag{67}$$

where

$$\alpha(x) = \sup\{D[Q(s) - Q(0)] - \frac{x}{s}, \quad 0 < s \le s_0\}, \quad 0 \le x < +\infty, \tag{68}$$

$$\alpha(x) \le 0, \quad \lim_{x\to 0^+} \alpha(x) = \alpha(0) = 0. \tag{69}$$

$$\beta(x) = \inf\{D[Q(s) - Q(0)] - \frac{x}{s}, \quad -s_0 \leq s < 0\}, \quad 0 \leq x < +\infty. \tag{70}$$

$$\beta(x) \geq 0, \quad \lim_{x \to 0^+} \beta(x) = \beta(0) = 0. \tag{71}$$

Remark 2.6. The proof of Theorem 2.5 is similar to that of Theorem 2.1 and Theorem 2.3.

Corollary 2.7. *Under the assumption of Theorem 2.5, we have*

$$\liminf_n \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k\{X_k - E_Q[X_k]\} \geq D[Q(1) - Q(0)] - h(P|Q)$$

$$P - a.s. \quad \omega \in D(\omega), \tag{72}$$

$$\limsup_n \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k\{X_k - E_Q[X_k]\} \leq D[Q(-1) - Q(0)] + h(P|Q)$$

$$P - a.s. \quad \omega \in D(\omega). \tag{73}$$

Proof. Let $s = 1$, $x = h(P|Q)$ in Theorem 2.5, we have by (68)

$$\alpha(x) \geq D[Q(1) - Q(0)] - h(P|Q).$$

Therefore (72) follows from (67) and (68). Let $s = -1$, $x = h(P|Q)$ in Theorem 2.5, we have by (70) that

$$\beta(x) \leq D[Q(-1) - Q(0)] + h(P|Q).$$

Similarly (73) follows from (66) and (70). ■

Corollary 2.8. *Let $\{X_n, n \geq 0\}$ be a sequence of independent nonnegative random variables with the product density function* (2)*, let*

$$H(\omega) = \{\omega : \liminf_{n \to \infty} \left(\sum_{k=0}^{[\sigma_n]} Y_k \Big/ n^{\alpha} \right) > 0\}. \tag{74}$$

Then

$$\lim_n \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k[X_k - E_P(X_k)] = 0 \quad P - a.s. \quad \omega \in H(\omega). \tag{75}$$

Proof. Letting $P \equiv Q$ in Theorem 2.1, in this case, $f(x_0, \cdots, x_n) = \prod_{k=0}^{n} f_k(x_k)$ and $E_Q = E_P$. Therefore, we obtain

$$h(P|Q) = \limsup_{n \to \infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log[f(X_0, \cdots, X_{[\sigma_n]}) / g(X_0, \cdots, X_{[\sigma_n]})]$$

$$= \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log[f(X_0, \cdots, X_{[\sigma_n]}) \Big/ \prod_{k=0}^{[\sigma_n]} f_k(X_k)] \equiv 0 < \infty \ P - \text{a.s.} \tag{76}$$

Hence $H(\omega) = D(\omega)$ a.s. Corollary 2.8 follows from (19) and (22) of Theorem 2.1, (50) and (53) of Theorem 2.3. ■

If $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov information source, then as $n \geq m$,

$$\begin{aligned} &P(X_n = x_n | X_0 = x_0, \cdots, X_{n-1} = x_{n-1}) \\ &= P(X_n = x_n | X_{n-m} = x_{n-m}, \cdots, X_{n-1} = x_{n-1}). \end{aligned} \tag{77}$$

Denote

$$q(i_0, \cdots, i_{m-1}) = P(X_0 = i_0, \cdots, X_{m-1} = i_{m-1}), \tag{78}$$

$$p_n(j|i_1, \cdots, i_m) = P(X_n = j | X_{n-m} = i_1, \cdots, X_{n-1} = i_m). \tag{79}$$

$q(i_0, \cdots, i_{m-1})$ is called the m dimensional initial distribution, $p_n(j|i_1, \cdots, i_m)$, $n \geq m$ are called the mth-order transition probabilities, and

$$P_n = (p_n(j|i_1, \cdots, i_m)) \tag{80}$$

are called the mth-order transition matrices. In this case,

$$p(x_0, \cdots, x_n) = q(x_0, \cdots, x_{m-1}) \prod_{k=m}^{n} p_k(x_k | x_{k-m}, \cdots, x_{k-1}). \tag{81}$$

Theorem 2.9. *Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov information source taking values in the set $S = \{0, 1, 2, \cdots, N\}$ under the measure P defined as above, if $Q(s) < \infty$, $\{X_n, n \geq 0\} \prec X$, let*

$$\limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=m}^{[\sigma_n]} \left[\frac{p_k(i_m | i_0, \cdot\cdot\cdot, i_{m-1})}{f_k(i_m)} - 1 \right]^+ \leq 0 \quad \forall i_0, \cdot\cdot\cdot, i_m \in S. \tag{82}$$

Then

$$\lim_{n} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=0}^{[\sigma_n]} Y_k \{X_k - E_Q[X_k]\} = 0 \quad P - a.s. \ \omega \in H(\omega). \tag{83}$$

Proof. By (82), noticing $\log x \leq x - 1$ $(x > 0)$, $a \leq [a]^+$, we have

$$h(P|Q) = \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log[\frac{f(X_0, \cdot\cdot\cdot, X_{[\sigma_n]})}{g(X_0, \cdot\cdot\cdot, X_{[\sigma_n]})}]$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \frac{1}{\sum\limits_{k=0}^{[\sigma_n]} Y_k} \log[\frac{p(X_0,\cdots,X_{m-1}) \prod\limits_{k=m}^{[\sigma_n]} p_k(X_k|X_{k-m},\cdots,X_{k-1})}{\prod\limits_{k=0}^{m-1} f_k(X_k) \prod\limits_{k=m}^{[\sigma_n]} f_k(X_k)}] \\
&\le \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log[p(X_0,\cdots,X_{m-1}) - \prod_{k=0}^{m-1} f_k(X_k)] \\
&\quad + \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \log \prod_{k=m}^{[\sigma_n]} \frac{p_k(X_k|X_{k-m},\cdots,X_{k-1})}{f_k(X_k)} \\
&\le \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=m}^{[\sigma_n]} \log \frac{p_k(X_k|X_{k-m},\cdots,X_{k-1})}{f_k(X_k)} \\
&\le \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=m}^{[\sigma_n]} \left[\frac{p_k(X_k|X_{k-m},\cdots,X_{k-1})}{f_k(X_k)} - 1\right]^+ \\
&= \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=m}^{[\sigma_n]} \sum_{i_0,\cdots,i_m\in S} \delta_{i_0}(X_{k-m})\cdots\delta_{i_m}(X_k)\times \\
&\qquad\qquad \left[\frac{p_k(i_m|i_0,\cdots,i_{m-1})}{f_k(i_m)} - 1\right]^+ \\
&\le \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=m}^{[\sigma_n]} \sum_{i_0,\cdots,i_m\in S} \left[\frac{p_k(i_m|i_0,\cdots,i_{m-1})}{f_k(i_m)} - 1\right]^+ \\
&\le \sum_{i_0,\cdots,i_m\in S} \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{[\sigma_n]} Y_k} \sum_{k=m}^{[\sigma_n]} \left[\frac{p_k(i_m|i_0,\cdots,i_{m-1})}{f_k(i_m)} - 1\right]^+ \le 0 \\
&\qquad\qquad P-\text{a.s. } \ \omega\in H(\omega). \qquad (84)
\end{aligned}
$$

By (8) and (84), we know

$$h(P|Q) = 0 \quad P-\text{a.s. } \omega \in H(\omega).$$

Therefore, $H(\omega) = D(\omega)$ a.s. Theorem 2.9 follows from (19), (22), (50) and (53). ■

Corollary 2.10. ([6]) *Let* $\{X_n, n \ge 0\}$ *be a sequence of arbitrary random variables with the joint distribution* (1)*, and* $E_Q(X) < \infty$*, denote*

$$B(\omega) = \{\omega : h(P|Q) < \infty\}, \qquad (85)$$

$$\int_x^{+\infty} f_k(x_k)dx_k \le \int_x^{+\infty} f(u)du, \qquad (86)$$

then

$$\liminf_{n} \frac{1}{n} \sum_{k=0}^{n} \{X_k - E_Q(X_k)\} \geq \alpha(h(P|Q)) \quad P-a.s. \;\; \omega \in B(\omega), \tag{87}$$

$$\limsup_{n} \frac{1}{n} \sum_{k=0}^{n} \{X_k - E_Q(X_k)\} \leq \beta(h(P|Q)) \quad P-a.s. \;\; \omega \in B(\omega), \tag{88}$$

where

$$\alpha(x) = \sup\{\varphi(s,x), \quad 0 < s \leq s_0\}, \quad 0 \leq x < +\infty, \tag{89}$$

$$\varphi(s,x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} [Q_k(s) - Q_k(0)] - \frac{x}{s}, \quad 0 < s \leq s_0, \quad 0 \leq x < +\infty, \tag{90}$$

$$\alpha(x) \leq 0, \quad \lim_{x\to 0^+} \alpha(x) = \alpha(0) = 0, \tag{91}$$

$$\beta(x) = \inf\{\psi(s,x), \quad -s_0 \leq s < 0\}, \quad 0 \leq x < +\infty, \tag{92}$$

$$\psi(s,x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} [Q_k(s) - Q_k(0)] - \frac{x}{s}, \quad -s_0 \leq s < 0, \quad 0 \leq x < +\infty, \tag{93}$$

$$\beta(x) \geq 0, \quad \lim_{x\to 0^+} \beta(x) = \beta(0) = 0. \tag{94}$$

Proof. Letting $\sigma_n(\omega) = n,\ Y_k \equiv 1,\ k \geq 0,\ 0 < \alpha < 1$, we have

$$\liminf_{n\to\infty} \left(\sum_{k=0}^{[\sigma_n(\omega)]} Y_k \Big/ n^\alpha \right) = \liminf_{n\to\infty} \frac{n}{n^\alpha} = \liminf_{n\to\infty} n^{1-\alpha} = \infty. \tag{95}$$

Hence $B(\omega) = D(\omega)$ a.s. Letting $D = 1$, $\{X_n, n \geq 0\} \prec X$ implies that (86) holds. Corollary 2.10 follows from Theorem 2.1 and Theorem 2.3 immediately. ■

Theorem 2.11. *Under the assumption of Theorem 2.9, if*

$$\liminf_{n\to\infty} \frac{f_n(X_n)}{p_n(X_n|X_{n-m}, \cdots, X_{n-1})} \geq 1 \quad a.s. \tag{96}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} (X_k - E_Q(X_k)) = 0 \quad P-a.s. \;\; \omega \in B(\omega). \tag{97}$$

Proof. Denote

$$R_n(\omega) = \frac{g(X_0, \cdots, X_n)}{f(X_0, \cdots, X_n)} = \frac{\prod_{k=0}^{n} f_k(X_k)}{p(X_0, \cdots, X_{m-1}) \prod_{k=m}^{n} p_k(X_k|X_{k-m}, \cdots, X_{k-1})}, \tag{98}$$

we know that for an arbitrary sequence of positive numbers $\{a_n, n \geq 1\}$,

$$\liminf_{n\to\infty} \sqrt[n]{a_n} \geq \liminf_{n\to\infty}(a_n/a_{n-1}). \tag{99}$$

Therefore, by (96), (98) and (99),

$$\begin{aligned}\liminf_{n\to\infty}[R_n(\omega)]^{\frac{1}{n}} &\geq \liminf_{n\to\infty}[R_n(\omega)/R_{n-1}(\omega)] \\ &= \liminf_{n\to\infty} \frac{f_n(X_n)}{p_n(X_n|X_{n-m},\cdots,X_{n-1})} \geq 1. \end{aligned} \tag{100}$$

By (100) we get

$$\liminf_{n\to\infty} \frac{1}{n} \log R_n(\omega) \geq 0 \tag{101}$$

and

$$\begin{aligned} h(P|Q) = & \limsup_{n\to\infty} \frac{1}{n} \log \frac{f(X_0,\cdots,X_n)}{g(X_0,\cdots,X_n)} \\ = & -\liminf_{n\to\infty} \frac{1}{n} \log R_n(\omega) \leq 0 \quad P-\text{a.s.} \ \ \omega \in B(\omega). \end{aligned} \tag{102}$$

By (8) and (102), we obtain

$$h(P|Q) = 0 \quad P-\text{a.s.} \ \ \omega \in B(\omega). \tag{103}$$

Analogously, (97) follows from (19), (22), (50) and (53). ■

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