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Vector-Pseudodifferential Operators Related to Orthogonal Expansions of Generalized Functions and Applications to Systems of Dual Series Equations

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Abstract. The aim of the present work is to introduce some functional spaces for investigating vector- pseudodifferential operators involving orthogonal expansions of generalized functions and their applications to systems of dual series equations.

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1. Introduction

In the paper [3], the pseudo-differential operator of the form

$$(Au)(x) := S^{-1}[a(k)\widehat{u}(k)](x) := \sum_{k=0}^{\infty} a(k)\widehat{u}(k)\psi_k(x), \tag{1}$$

was investigated and it was applied for solving dual series equations. Here $\{\psi_k(x)\}_{k=0}^{\infty}$ is an orthonormal sequence of functions in L^2, $\widehat{u}(k) = (u, \psi_k)$ denotes the value of the generalized function u on the function $\overline{\psi_k(x)}$, $a(k)$ is a known function and is called the symbol of the operator Au.

The purpose of the present work is to see the feasibility of extending the approach in [3] for vector-pseudodifferetial oprators involving orthogonal expansions of generalized functions and their applications to systems of dual series equations.

The paper is divided into five sections and organized as follows. In Sections 2 we recall some definitions and results from the theory of orthonormal series expansions for generalized functions [10]. In Sections 3 and 4 we construct Sobolev spaces for the investigation of the pseudo-differential operators of the form

$$(A\mathbf{u})(x) = S^{-1}[\mathbf{A}(k)\widehat{\mathbf{u}}(k)](x), \tag{2}$$

where $\mathbf{u} = \mathbf{u}(x)$ and $\widehat{\mathbf{u}}(k) = S[\mathbf{u}](k)$ are n-vectors, $\mathbf{A}(k)$ is a n-square matrix. In Section 5 we present these results for investigation on the sovability of systems of dual series equations. Section 6 deals with the solvability of some mixed boundary value problems for the harmonic and biharmonic equations.

2. Integral transform of generalized functions

Following the Zemanian's idea, we consider the Hilbert space $L^2(J)$ with the scalar product and norm

$$(u,v) = \int_J u(x)\overline{v(x)}dx, \quad \|u\| = \sqrt{(u,u)},$$

where J is an open interval in $\mathbb{R}$. Consider the operator $\mathcal{N}$ introduced by Zemanian in [10, Chapter 9], with a complete orthonormal system of smooth eigenfunctions $\{\psi_k\} \subset L^2(J)$, where $D(\mathcal{N})$ is the domain of $\mathcal{N}, D(\mathcal{N}) \subset L^2(J)$. Each $\varphi \in L^2(J)$ can be expressed as

$$\varphi = \sum_k (\varphi, \psi_k)\psi_k,$$

where the series converges in $L^2(J)$. Throughout the paper $\mathcal{N}$ will be a differential expression of the form

$$\mathcal{N} = \overline{\theta_m(x)}(-D)_{n_m}...e(-D)^{n_2}\overline{\theta_1(x)}(-D)^{n_1}\overline{\theta_0(x)},$$

where $D = d/dx, n_k$ are positives integers and θ_k are smooth functions on J. Also,

$$\mathcal{N} = \overline{\theta_m(x)}(-D)_{n_m}...(-D)^{n_2}\overline{\theta_1(x)}(-D)^{n_1}\overline{\theta_0(x)},$$

where $\overline{\theta_k(x)}$ denotes the complex-conjugate of the function $\theta_k(x)$. Let $\{\lambda\}_{k=0}^{\infty}$ be the real eigenvalues corresponding to the eigenfunctions $\{\psi_k\}_{k=0}^{\infty} \subset D(\mathcal{N})$ such that $\lambda_k = 0(k^q), k \to \infty, q > 0$ and

$$\mathcal{N}\psi_k = \lambda_k\psi_k, \quad k = 0, 1, \dots.$$

We shall take the domain $D(\mathcal{N})$ of the operator $\mathcal{N}$ to be

$$D(\mathcal{N}) := \{\varphi \in C^{\infty}(J) : \mathcal{N}^n\varphi \in L^2(J), (\mathcal{N}^n\varphi, \psi_k) = (\varphi, \mathcal{N}^n\psi_k) \forall k, n = 0, 1, \dots\},$$

where $\mathcal{N}^0$ is the identity operator on $L^2(J)$.

Definition 2.1. ([10]) Denote by $\mathcal{A}$ the space of test functions $\varphi(x)$ such that

1) $\varphi(x) \in C^{\infty}(J)$;
2) $\forall m = 0, 1, 2 \cdots :\ \alpha_m(\varphi) := \|\mathcal{N}^m\varphi\| < \infty$;
3) $(\mathcal{N}^m\varphi, \psi_k) = (\varphi, \mathcal{N}^m\psi_k)$.

The sequence$\{\varphi_k(x)\}_{k=0}^{\infty}$ of functions from $\mathcal{A}$ is called convergent in $\mathcal{A}$ to zero if $\alpha_m(\varphi_k) \to 0$ as $k \to \infty$ $\forall m = 0, 1, 2, \dots$

Obviously, $\mathcal{A}$ is a linear space and $\psi_k(x) \in \mathcal{A}$ $\forall k$. In [10] it was shown that $\mathcal{A}$ is a complete space and $C_o^{\infty}(J) \subset \mathcal{A} \subset L^2(J)$, where $C_o^{\infty}(J)$ is the set of infinitely differentiable functions with compact support in J.

Theorem 2.2. ([10]) *If* $\varphi \in \mathcal{A}$ *then*

$$\varphi(x) = \sum_{k=0}^{\infty} (\varphi, \psi)\psi_k(x),$$

where the series converges in $\mathcal{A}$.

Definition 2.3. ([10]) A generalized function is called any continuous linear functional on the space $\mathcal{A}$. We denote by $\mathcal{A}'$ the set of all generalized functions on $\mathcal{A}$ and by $(f, \varphi) = \langle f, \overline{\varphi} \rangle$ the value of the generalized function $f \in \mathcal{A}'$ on the test function $\overline{\varphi} \in \mathcal{A}$.

The space $\mathcal{A}'$ is complete and $L^2(J) \subset \mathcal{A}' \subset \mathcal{D}'(\mathcal{J})$, where $\mathcal{D}'(\mathcal{J})$ is the space of distributions on J (see [10]). Hence every function $f(x) \in L^2(J)$ determines a regular functional f by the formula

$$(f, \varphi) = \int_J f(x)\overline{\varphi(x)}dx, \quad \varphi \in \mathcal{A} \subset L^2(J). \tag{3}$$

Theorem 2.4. ([10]) *Each* $f \in \mathcal{A}'$ *can be expressed as*

$$f = \sum_{k=0}^{\infty} (f, \psi_k)\psi_k(x), \tag{4}$$

where the series converges in $\mathcal{A}'$.

Definition 2.5. ([10]) We consider the orthonormal expansion (4) as the inverse formula, defining a certain integral transformation of genertalized functions, which is given by the formula

$$\widehat{f}(k) := S[f](k) := (f, \psi_k), \quad f \in \mathcal{A}', \quad k = 0, 1, 2, \dots \tag{5}$$

The inverse mapping S^{-1} is given by formula (4) and it may be represented in the form

$$S^{-1}[\widehat{f}(k)](x) := \sum_{k=0}^{\infty} \widehat{f}(k)\psi_k(x) = f. \tag{6}$$

Note that for $f \in L^2(J)$ formula (5) is of the form

$$\widehat{f}(k) = \int_J f(x)\overline{\psi_k(x)}dx.$$

3. Sobolev spaces

First, we recall some defitions and propositions on Sobolev spaces related to orthogonal expansions of generalized functions in the space $\mathcal{A}'$ from [3].

Definition 3.1. ([3]) Let s be a real number. Denote by $h^s = h^s(J)$ the set of generalized functions $f \in \mathcal{A}'$ such that

$$\|f\|_s^2 := \sum_{k=0}^{\infty}(1+|k|)^{2s}|\widehat{f}(k)|^2 < \infty, \tag{7}$$

where $\hat{f}(k) = S[f](k)$. The scalar product in h^s is defined by the formula

$$(f,g)_s := \sum_{k=0}^{\infty}(1+|k|)^{2s}\widehat{f}(k)\overline{\widehat{g}(k)}. \tag{8}$$

Theorem 3.2. ([3]) *Let $h^{s,*}$ be the conjugate space of the space h^s. Then $h^{s,*}$ is isomorphic to the space h^{-s}. Besides, the value of a functional $f \in h^{-s}$ on an element $u \in h^s$ can be given by the formula*

$$(u,f)_0 = \sum_{n=0}^{\infty} \widehat{u}(n)\overline{\widehat{f}(n)}, \tag{9}$$

where $\widehat{f}(n) = S[f](n) = (f,\psi_n), \widehat{u}(n) = S[u](n) = (u,\psi_n)$.

In virtue of Theorem 3.2, we put $h^{s,*} \simeq h^{-s}$.

Definition 3.3. Let α be a real number. Denote by σ^α the class of functions $a(k)$ satisfying the condition

$$|a(k)| \leqslant C(1+k)^\alpha \quad \forall k = 0,1,2,... \tag{10}$$

where C is a positive constant. We shall say that the function $a(k)$ belongs to the class σ_o^α if $a(k) \in \sigma^\alpha$ and $a(k) \geq 0$. Finally, the function $a(k)$ belongs to the class σ_+^α if $C_1(1+k)^\alpha \leqslant a(k) \leqslant C_2(1+k)^\alpha \quad \forall k = 0,1,\dots$, where C_1 and C_2 are positive constants.

Theorem 3.4. ([3]) *Assume that* $a(k) \in \sigma^{\alpha}, u \in h^s, \widehat{u}(k) = S[u](k)$. *Then the pseudo-differential operator*

$$A[u](x) := S^{-1}[a(k)\widehat{u}(k)](x) := \sum_{k=0}^{\infty} a(k)\widehat{u}(k)\psi_k(x) \tag{11}$$

is bounded from h^s *into* $h^{s-\alpha}$. *If* $a(k) \in \sigma^{-\beta}$, *where* $\beta > 1/2$, *then the operator* A *is completely continuous in* h^s.

Let Ω be a certain interval of J. Let us introduce the following definitions.

Definition 3.5. ([3]) Denote by $h^s_o(\Omega)$ the space defined as the closure of the set $C^{\infty}_o(\Omega)$ of infinitely differentiable functions with compact supports in $\overline{\Omega}$ with respect to the norm (7). The norm in $h^s_\circ(\Omega)$ is defined by the same (7). The symbol $h^s_{oo}(\Omega)$ denotes the set of functions belonging to $h^s(J)$ with support contained in $\overline{\Omega}$.

It is clear that $h^s_o(\Omega) \subset h^s_{oo}(\Omega)$.

Definition 3.6. ([3]) The space $h^s(\Omega)$ is defined as the set of generalized functions f from $\mathcal{D}'(\Omega)$ having extensions $lf \in h^s$. The norm in $h^s(\Omega)$ is defined by the formula

$$\|f\|_{h^s(\Omega)} := \inf_l \|lf\|_s, \tag{12}$$

where the infimum is taken over all possible extentions $lf \in h^s$.

Theorem 3.7. ([3]) *Let* $u \in h^s_\circ(\Omega), f \in h^{-s}(\Omega)$ *and* lf *be an extension of the function* f *belonging to* $h^{-s}(\Omega)$. *Then the series*

$$[u, f] := (u, lf)_0 := \sum_{n=0}^{\infty} S[u](n)\overline{S[lf]}(n) \tag{13}$$

does not depends on the choice of the extension lf. *Therefore this series defines a linear continuous functional on* $h^s_\circ(\Omega)$. *Conversely, for every linear continuous functional* $\phi(u)$ *on* $h^s_\circ(\Omega)$ *there exists an element* $f \in h^{-s}(\Omega)$ *such that* $\Phi(u) = [u, f]$ *and* $\|\phi\| = \|f\|_{h^{-s}(\Omega)}$.

Let $h^{s,*}_\circ(\Omega)$ be the conjugate space of the space $h^s_\circ(\Omega)$. In virtue of Theorem 3.7 we put $h^{s,*}_\circ(\Omega) \simeq h^{-s}(\Omega)$. We now construct similar spaces for vector-functions. We shall use bold letters for denoting vector-values and matrices.

Definition 3.8. Let X be a linear topological space. We denote the direct product of n elements X by X^n. A topology in X^n is given by the usual topology of the direct product. Denote by $\mathbf{u}$ a vector of the form $\mathbf{u} = (u_1, u_2, ..., u_n)$, and

$$\mathcal{A}^n = \mathcal{A} \times \mathcal{A} ... \times \mathcal{A}, \quad \mathcal{A}'^n = \mathcal{A}' \times \mathcal{A}' ... \times \mathcal{A}'.$$

For the vectors $\mathbf{u} \in \mathcal{A}'^n$, $\mathbf{w} \in \mathcal{A}^n$ we put

$$(\mathbf{u},\mathbf{w}) = \sum_{j=1}^{n}(u_j, w_j). \tag{14}$$

For a vector $\mathbf{u} \in \mathcal{A}'^n$ we write

$$\widehat{\mathbf{u}}(k) = S[\mathbf{u}](k) := ((u_1,\psi_k),(u_2,\psi_k),\ldots,(u_n,\psi_k))^T \tag{15}$$

and

$$\mathbf{u}(x) = S^{-1}[\widehat{\mathbf{u}}](x) := \Big(\sum_{k=0}^{\infty}\widehat{u}_1(k)\psi_k(x),\ldots,\sum_{k=0}^{\infty}\widehat{u}_n(k)\psi_k(x)\Big)^T. \tag{16}$$

Definition 3.9. Let $h^{s_j},\ h_o^{s_j}(\Omega),\ h_{oo}^{s_j}(\Omega),\ h^{s_j}(\Omega)$ be the Sobolev spaces, where $j=1,2,\ldots,n;\ \Omega$ is a certain set of the interval in J. We put

$$\vec{s} = (s_1,s_2,\ldots,s_n), \quad \mathbf{h}^{\tilde{\mathbf{s}}} = h^{s_1}\times h^{s_2}\times\cdots\times h^{s_n},$$

$$\mathbf{h}^{\tilde{\mathbf{s}}}{}_o(\Omega) = h_o^{s_1}(\Omega)\times h_o^{s_2}(\Omega)\times\cdots\times h_o^{s_n}(\Omega),$$

$$\mathbf{h}^{\tilde{\mathbf{s}}}{}_{oo}(\Omega) = h_{oo}^{s_1}(\Omega)\times h_{oo}^{s_2}(\Omega)\times\cdots\times h_{oo}^{s_n}(\Omega),$$

$$\mathbf{h}^{\tilde{\mathbf{s}}}(\Omega) = h^{s_1}(\Omega)\times h^{s_2}(\Omega)\times\cdots\times h^{s_n}(\Omega).$$

The scalar product and the norm in $\mathbf{h}^{\tilde{\mathbf{s}}},\ \mathbf{h}^{\tilde{\mathbf{s}}}{}_o(\Omega)$ and $\mathbf{h}^{\tilde{\mathbf{s}}}{}_{oo}(\Omega)$ are given by the formulas

$$(\mathbf{u},\mathbf{v})_{\vec{s}} = \sum_{j=1}^{n}(u_j,v_j)_{s_j}, \quad ||\mathbf{u}||_{\vec{s}} = \Big(\sum_{j=1}^{n}||u_j||_{s_j}^2\Big)^{1/2}, \tag{17}$$

where $\|u_j\|_{s_j}$ and $(u_j,v_j)_{s_j}$ are given by the formulas (7) and (8) respectively. The norm in $\mathbf{h}^{\tilde{\mathbf{s}}}(\Omega)$ is defined by the equality

$$\|\mathbf{u}\|_{\mathbf{h}^{\tilde{\mathbf{s}}}(\mathbf{\Omega})} := \Big(\sum_{j=1}^{n}\|u_j\|_{h^{s_j}(\Omega)}^2\Big)^{1/2}, \tag{18}$$

where $\vec{s} = (s_1,s_2,\ldots,s_n), s_j \in \mathbb{R}\ (j=1,2,\ldots,n)$.

Theorem 3.10. *Let* $\mathbf{h}^{\vec{s},*}(J)$ *be the dual space of the space* $\mathbf{h}^{\vec{s}}(J)$*. Then* $\mathbf{h}^{\vec{s},*}(J)$ *is isomorphic to the space* $\mathbf{h}^{-\vec{s}}(J)$*. Moreover, the value of a functional* $\mathbf{f} \in \mathbf{h}^{-\vec{s}}(J)$ *on an element* $\mathbf{u} \in \mathbf{h}^{\vec{s}}(J)$ *is given by*

$$(\mathbf{f},\mathbf{u})_o = \sum_{j=1}^{n}\sum_{k=0}^{\infty}\overline{\widehat{f_j(k)}}\widehat{u_j}(k), \tag{19}$$

where $\widehat{u_j}(k) = S[u_j](k), \widehat{f_j}(k) = S[f_j](k)$.

Proof. Due to Riesz theorem, for any functional $\Phi(\mathbf{u}), \mathbf{u} \in \mathbf{h}^{\vec{s}}(J)$, there exists an element $\mathbf{v} \in \mathbf{h}^{\vec{s}}(J)$, such that $\Phi(\mathbf{u}) = (\mathbf{v},\mathbf{u})_{\vec{s}}$ and its norm $\|\Phi\| = \sup_{\|\mathbf{u}\|_{\vec{s}}=1}|\Phi(\mathbf{u})|$ equals $\|\mathbf{v}\|_{\vec{s}}$. Denote

$$\widehat{\mathbf{f}}(k) = [\widehat{f}_1(k), \ldots, \widehat{f}_n(k)] := [(1+|k|)^{2s_1}\widehat{v}_1(k), \ldots, (1+|k|)^{2s_n}\widehat{v}_n(k)]. \qquad (20)$$

We have

$$\sum_{k=0}^{\infty}(1+|k|)^{-2s_j}|\widehat{f}_j(k)|^2 = \sum_{k=0}^{\infty}(1+|k|)^{2s_j}|\widehat{v}_j(k)|^2 (j=1,2,\ldots,n).$$

Therefore $\mathbf{f} := S^{-1}[\widehat{\mathbf{f}}] \in \mathbf{h}^{-\vec{s}}(J)$, $\|\mathbf{f}\|_{-\vec{s}} = \|\mathbf{v}\|_{\vec{s}} = \|\Phi\|$ and $(\mathbf{v}, \mathbf{u})_{\vec{s}} = (\mathbf{f}, \mathbf{u})_o$, where

$$(\mathbf{f}, \mathbf{u})_o = \sum_{j=1}^{n}\sum_{k=0}^{\infty}\overline{\widehat{f}_j(k)}\widehat{u}_j(k).$$

By this, (20) establishes an isomorphism between $\mathbf{h}^{\vec{s},*}(J)$ and $\mathbf{h}^{-\vec{s}}(J)$. Besides, the value of a functional $\mathbf{f} \in \mathbf{h}^{-\vec{s}}(J)$ on an element $\mathbf{u} \in \mathbf{h}^{\vec{s}}(J)$ is given by the formula (19). The proof of Theorem 3.10 is complete. ■

Theorem 3.11. *Let $\Omega \subset J$, $\mathbf{u} \in \mathbf{h}^{\vec{s}}(\Omega)$, $\mathbf{f} \in \mathbf{h}^{-\vec{s}}(\Omega)$ and $\mathbf{lf} \in \mathbf{h}^{-\vec{s}}(J)$ be an extension of* $\mathbf{f}$ *from Ω to J. Then*

$$[\mathbf{f}, \mathbf{u}] := (\mathbf{lf}, \mathbf{u})_o := \sum_{j=1}^{n}\sum_{k=0}^{\infty}\overline{\widehat{l_j f_j}(k)}\widehat{u}_j(k) \qquad (21)$$

does not depend on the choice of the extension $\mathbf{lf}$. *Therefore this formula defines a linear continuous functional on $\mathbf{h}_o^{\vec{s}}(\Omega)$. Conversely, for every linear continuous functional $\Phi(\mathbf{u})$ on $\mathbf{h}_o^{\vec{s}}(\Omega)$, there exists an element $\mathbf{f} \in \mathbf{h}^{-\vec{s}}(\Omega)$ such that $\Phi(\mathbf{u}) = [\mathbf{u}, \mathbf{f}]$ and $\|\Phi\| = \|\mathbf{f}\|_{\mathbf{h}^{-\vec{s}}(\Omega)}$.*

Proof. Let $\mathbf{l'f}$ be another extension of $\mathbf{f}$. Then we have $\mathbf{lf} - \mathbf{l'f} = \mathbf{0}$ on Ω, i.e.

$$(\mathbf{lf} - \mathbf{l'f}, \mathbf{w})_o = \mathbf{0} \quad \forall \mathbf{w} \in (C_o^{\infty}(\Omega))^n. \qquad (22)$$

Since $(C_o^{\infty}(\Omega))^n$ is dense in $\mathbf{h}_o^{\vec{s}}(\Omega)$, from (22) it follows that

$$(\mathbf{lf} - \mathbf{l'f}, \mathbf{u})_o = 0 \quad \forall \mathbf{u} \in \mathbf{h}_o^{\vec{s}}(\Omega).$$

This implies $(\mathbf{l'f}, \mathbf{u})_o = (\mathbf{lf}, \mathbf{u})_o$. Therefore the sum in (21) does not depend on the choice of the extension $\mathbf{lf}$. By (21),

$$|(\mathbf{lf}, \mathbf{u})_o| \leqslant \|\mathbf{u}\|_{\vec{s}}.\|\mathbf{lf}\|_{-\vec{s}}.$$

Since $(\mathbf{lf}, \mathbf{u})_o$ does not depend on the choice of $\mathbf{lf}$

$$|(\mathbf{lf}, \mathbf{u})_o| \leqslant \|\mathbf{u}\|_{\vec{s}} \inf_{\mathbf{l}} \|\mathbf{lf}\|_{-\vec{s}} = \|\mathbf{u}\|_{\vec{s}}.\|\mathbf{f}\|_{\mathbf{h}^{-\vec{s}}(\Omega)}. \qquad (23)$$

Therefore, each $\mathbf{f} \in \mathbf{h}^{-\vec{s}}(\Omega)$ gives a continuous functional on $\mathbf{h}_o^{\vec{s}}(\Omega)$ by the formula (21). Let $\Phi(\mathbf{u})$ be a linear continuous functional on $\mathbf{h}_o^{\vec{s}}(\Omega)$. The space $\mathbf{h}_o^{\vec{s}}(\Omega) \subset \mathbf{h}^{\vec{s}}(J)$ is a Hilbert space with respect to the scalar product (17). Therefore due to Riesz theorem there exists a function vector $\mathbf{v} \in \mathbf{h}_o^{\vec{s}}(\Omega)$ such that

$\Phi(\mathbf{u}) = (\mathbf{v}, \mathbf{u})_s$. We put $\hat{\mathbf{f}}_o(k) = [(1+|k|)^{2s_1}\widehat{v}_1(k), ..., (1+|k|)^{2s_n}\widehat{v}_n(k)]$, $\mathbf{f}_o = S^{-1}[\hat{\mathbf{f}}_o]$. Then $\mathbf{f}_o \in \mathbf{h}^{-\vec{s}}(J)$, $p\mathbf{f}_o = \mathbf{f} \in \mathbf{h}^{-\vec{s}}(\Omega)$, where p denotes the restriction operator to Ω. We have $\Phi(\mathbf{u}) = (\mathbf{v}, \mathbf{u})_{\vec{s}} = (\mathbf{f}_o, \mathbf{u})_o$ and $||\Phi|| = ||\mathbf{v}||_{\vec{s}} = ||\mathbf{f}_o||_{-\vec{s}} \geq ||\mathbf{f}||_{\mathbf{h}^{-\vec{s}}(\Omega)}$. On the other hand, in virtue of (23) we have $||\Phi|| = \sup_{||\mathbf{u}||_{\vec{s}} \leqslant 1} |\Phi(\mathbf{u})| \leqslant ||\mathbf{f}||_{\mathbf{h}^{-\vec{s}}(\Omega)}$. Thus, $||\Phi|| = ||f||_{\mathbf{h}^{-\vec{s}}(\Omega)}$. The proof is complete. ∎

4. Vector-pseudodifferential operators

Consider the vector-pseudodifferential operator of the form

$$(A\mathbf{u})(x) := S^{-1}[\mathbf{A}(k)\widehat{\mathbf{u}}(k)](x), \; k = 0, 1, \ldots; x \in J \subset \mathbb{R}, \tag{24}$$

where $\mathbf{A}(k) = \|a_{ij}(k)\|_{n\times n}$ is a square matrix of order n, $\mathbf{u} = (u_1, u_2, \ldots, u_n)^T$ is a vector, transposed to the line vector $(u_1, u_2, ..., u_n)$, and $\widehat{\mathbf{u}}(k) := S[\mathbf{u}] = (S[u_1], S[u_2], \ldots, S[u_n])^T$. We introduce the following classes.

Definition 4.1. Let $\mathbf{A}(k) = \|a_{ij}(k)\|_{n\times n}$, $k = 0, 1, 2, \ldots$ be a square matrix of order n, $\alpha_j, \beta_{ij} \in \mathbb{R}$ $(i, j = 1, 2, \ldots, n)$. Denote by $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and by $\Sigma^{\vec{\alpha}}$ the class of matrices $\mathbf{A}(k) = \|a_{ij}(k)\|_{n\times n}$, such that

$$a_{ii}(k) \in \sigma^{\alpha_i}, \; a_{ij}(k) \in \sigma^{\alpha_{ij}}, \; \alpha_{ij} \leqslant \frac{1}{2}(\alpha_i + \alpha_j). \tag{25}$$

We say that the matrix $\mathbf{A}(k)$ belongs to the class $\boldsymbol{\Sigma}_+^{\vec{\alpha}}$ if $\mathbf{A}(k) \in \boldsymbol{\Sigma}^{\vec{\alpha}}$ and it is Hermitian, i.e. $(\overline{\mathbf{A}(k)})^T = \mathbf{A}(k)$, and it satisfies the condition

$$\overline{\mathbf{w}}^T\mathbf{A}\mathbf{w} \geq C_1 \sum_{j=1}^{n} (1+|k|)^{\alpha_j} |w_j|^2 \; \forall \mathbf{w} = (w_1, w_2, \ldots, w_n)^T \in \mathbb{C}^n, \tag{26}$$

where C_1 is some positive constant. Finally, we say that the matrix $\mathbf{A}(k) \in \boldsymbol{\Sigma}^{\vec{\alpha}}$ belongs to the class $\boldsymbol{\Sigma}_o^{\vec{\alpha}}$ if it is positive-definite for almost every $k = 0, 1, 2, \ldots$

Theorem 4.2. *Let $\mathbf{A}(k) = \mathbf{A}_+(k)$ belong to the class $\boldsymbol{\Sigma}_+^{\vec{\alpha}}$. Then the scalar product and the norm in $\mathbf{h}^{\vec{\alpha}/2}(J)$ may be defined by the formulas*

$$(\mathbf{u}, \mathbf{v})_{\mathbf{A}_+, \vec{\alpha}/2} = \sum_{k=0}^{\infty} \overline{S[\mathbf{v}^T](k)} \mathbf{A}_+(k) S[\mathbf{u}](k), \tag{27}$$

$$\|\mathbf{u}\|_{\mathbf{A}_+, \vec{\alpha}/2} = \Big(\sum_{k=0}^{\infty} \overline{S[\mathbf{u}^T](\xi)} \mathbf{A}_+(k) S[\mathbf{u}](k) \Big)^{1/2}. \tag{28}$$

Proof. Using Cauchy-Schwarz inequality we have

$$\overline{\mathbf{w}(k)}^T \mathbf{A}\mathbf{w}(k) \leqslant C_2 \sum_{j=1}^{n} (1+k)^{\alpha_j} |w_j(k)|^2 \quad k = 0, 1, 2, \ldots, \tag{29}$$

where C_2 is a positive constant. In (25) and (29) replacing $w_j(k)$ with $\hat{u}_j(k) = S[u_j](k)$ and $\mathbf{w}(k)$ with $S[\mathbf{u}](k)$, and summing over $k = 0, 1, 2, ...$, we have

$$C_1 \sum_{j=1}^{n} \sum_{k=0}^{\infty} (1+|k|)^{\alpha_j} |S[u_j](k)|^2 \leqslant \sum_{k=0}^{\infty} \overline{S[\mathbf{u}^T](k)} \mathbf{A}_+(k) S[\mathbf{u}](k)$$

$$\leqslant C_2 \sum_{j=1}^{n} \sum_{k=0}^{\infty} (1+|k|)^{\alpha_j} |S[u_j](k)|^2. \tag{30}$$

Due to (7) and (17), from (30) we derive (28). It is clear that the sum (27) defines a scalar product in $\mathbf{h}^{\vec{\alpha}/2}(J)$. ■

Theorem 4.3. *Let* $\mathbf{A}(k) \in \boldsymbol{\Sigma}^{\vec{\alpha}}$, $\mathbf{u} \in \mathbf{h}^{\vec{s}}(J)$, *where*

$$\vec{s} = \vec{\alpha}/2 \pm \vec{\varepsilon}, \;\; \vec{\varepsilon} = (\varepsilon, \varepsilon, \dots, \varepsilon), \varepsilon \geq 0. \tag{31}$$

Then the pseudo-differential operator $A\mathbf{u}$ *defined by the formula*

$$(A\mathbf{u})(x) := S^{-1}[\mathbf{A}(\mathbf{k})\widehat{\mathbf{u}}(k)](x), \;\; x \in J \tag{32}$$

is bounded from $\mathbf{h}^{\vec{s}}(J)$ *into* $\mathbf{h}^{\vec{s}-\vec{\alpha}}(J)$. *If* $\mathbf{A}(k) \in \boldsymbol{\Sigma}^{-\vec{\beta}}$ $(\vec{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, $\beta_i > \dfrac{1}{2}; i = 1, 2, \dots, n)$, *then the operator* A *is completely continuous in* $\mathbf{h}^{\vec{s}}(J)$.

Proof. Let $\mathbf{v}(x) = (A\mathbf{u})(x) := S^{-1}[\mathbf{A}(\mathbf{k})\widehat{\mathbf{u}}(k)](x)$. Hence

$$\widehat{\mathbf{v}}(k) := \mathbf{A}(\mathbf{k})\hat{\mathbf{u}}(k). \tag{33}$$

We shall consider an m-component of the vector (33). We have

$$\widehat{v}_m(k) = \sum_{i=1}^{n} a_{mi}(k)\widehat{u}_i(k), \;\; (m = 1, 2, \dots, n), \;\; (k = 0, 1, 2, \dots). \tag{34}$$

Multiplying by $(1+k)^{s_m - \alpha_m}$ both parts of (34), we obtain

$$(1+k)^{s_m-\alpha_m}\widehat{v}_m(k) = \sum_{i=1}^{n} (1+k)^{s_m-\alpha_m} a_{mi}(k)\widehat{u}_i(k).$$

$$= \sum_{i=1}^{n} [a_{mi}(k)(1+k)^{s_m-\alpha_m}(1+k)^{-s_i}][(1+k)^{s_i}\widehat{u}_i(k)]. \tag{35}$$

Applying Cauchy-Schwarz inequlity to the equality (35), we receive

$$(1+k)^{2(s_m-\alpha_m)}|\widehat{v}_m(k)|^2 \leqslant$$

$$\sum_{i=1}^{n} |a_{mi}(k)|^2 (1+k)^{2(s_m-\alpha_m-s_i)} \sum_{j=1}^{n} (1+k)^{2s_i} |\widehat{u}_i(k)|^2. \tag{36}$$

We have

$$|a_{mi}(k)| \leqslant C(1+k)^{\alpha_m/2+\alpha_i/2} \ \ \forall k=0,1,\ldots.$$

Thanks to (31), we get

$$\sum_{i=1}^{n} |a_{mi}(k)|^2 (1+k)^{2(s_m-\alpha_m-s_i)} \leqslant C \ \ \forall k=0,1,2,\ldots \tag{37}$$

From (36), (37) we get

$$\|v_m\|^2_{s_m-\alpha_m} \leqslant C \sum_{i=1}^{n} \|u_i\|^2_{s_i}, (m=1,2,\ldots,n), \mathbf{v}(x)=(A\mathbf{u})(x) \in \mathbf{h}^{\vec{s}-\vec{\alpha}}(J).$$

Now we assume that $\alpha_i = -\beta_i,\ \beta_i > 1/2\ (i=1,2,\ldots,n)$. In this case we have $\mathbf{h}^{\vec{s}-\vec{\alpha}}(J) = \mathbf{h}^{\vec{s}+\vec{\beta}}(J) \subset \mathbf{h}^{\vec{s}}(J)$. Let δ_{ij} be the Kronecker symbol. We rewite (34) in the form

$$\widehat{v}_m(k) = \sum_{j=0}^{\infty} \sum_{i=1}^{n} a_{mi}(j) \widehat{u}_i(j) \delta_{kj} \ \ m=1,2,\ldots,n. \tag{38}$$

Multiply by $(1+k)^{s_m}$ both parts of (38) and introduce notations

$$f_m(k) = (1+k)^{s_m} \widehat{v}_m(k), \ \ g_m(k) = (1+k)^{s_m} \widehat{u}_m(k).$$

$$f_m = \{f_m(k)\}_{k=0}^{\infty}, \ \ g_m = \{g_m(k)\}_{k=0}^{\infty} \ \ m=1,2,\ldots,n.$$

Obviously, for each $m=1,2,\ldots,n,\ g_m \in l_2$. We show that $f_m \in l_2$ too. Indeed, we have

$$\|f_m\|^2_{l_2} = \sum_{k=0}^{\infty} (1+k)^{2s_m} |\widehat{v}_m(k)|^2 = \sum_{k=0}^{\infty} (1+k)^{2\alpha_m} |(1+k)^{s_m-\alpha_m} \widehat{v}_m(k)|^2.$$

Since $\alpha_m = -\beta_m, \beta_m > 1/2 > 0, (1+k)^{2\alpha_m} \leqslant 1$. Thus $\|f_m\|^2_{l_2} \leqslant \|v_m\|^2_{-\alpha_m/2} < +\infty$. We have

$$f_m(k) = \sum_{i=1}^{n} \sum_{j=0}^{\infty} a_{mi}(j) g_i(j) \delta_{kj} \frac{(1+k)^{s_m}}{(1+j)^{s_i}}. \tag{39}$$

Applying Cauchy- Schwarz inequality to (39) we have

$$\|f_m\|^2_{l_2} \leqslant n \sum_{i=1}^{n} \|g_i\|^2_{l_2} \sum_{k=0}^{\infty} |a_{mi}(k)|^2 (1+k)^{2(s_m-s_i)}. \tag{40}$$

We show that

$$\sum_{k=0}^{\infty} |a_{mi}(k)|^2 (1+k)^{2(s_m-s_i)} < +\infty.$$

Indeed, we have

$$\sum_{k=0}^{\infty}|a_{mi}(k)|^2(1+k)^{2(s_m-s_i)} \leqslant \sum_{k=0}^{\infty}C(1+k)^{\alpha_m+\alpha_i}(1+k)^{2(s_m-s_i)} =$$

$$C\sum_{k=0}^{\infty}(1+k)^{2\alpha_m} = C\sum_{k=0}^{\infty}(1+k)^{-2\beta_m} = C\sum_{k=0}^{\infty}\frac{1}{(1+k)^{2\beta_m}} < +\infty.$$

Thus, for each $m \in \{1,2,\ldots,n\}$, (39) defines a certain linear continuous operator $L_m : f_m = L_m g_m$ from l_2 into l_2. In virtue of Theorem 3.4 the operator L_m is completely continuous. We can rewrite (39) in the following vector-form. We introduce notations

$$\mathbf{f}(k) = (f_1(k), f_2(k), \ldots, f_n(k))^T, \ \ \mathbf{g}(k) = (g_1(k), g_2(k), \ldots, g_n(k))^T,$$
$$\mathbf{f} = \{\mathbf{f}(k)\}_{k=0}^{\infty} = (\{f_1(k)\}_{k=0}^{\infty}, \ \{f_2(k)\}_{k=0}^{\infty}, \ldots, \{f_n(k)\}_{k=0}^{\infty})^T,$$
$$\mathbf{g} = \{\mathbf{g}(k)\}_{k=0}^{\infty} = (\{g_1(k)\}_{k=0}^{\infty}, \ \{g_2(k)\}_{k=0}^{\infty}, \ldots, \{g_n(k)\}_{k=0}^{\infty})^T,$$
$$\mathbf{l}_2 = \mathbf{l}_2^n = l_2 \times l_2 \times \cdots \times l_2, \ \ b_{mi}(k,j) = a_{mi}(j)\frac{(1+k)^{s_m}}{(1+j)^{s_i}},$$

$\mathbf{B}(k,j) = \|b_{mi}(k,j)\|_{n\times n}$ is a square matrix of order n.

Then (39) has the vector - form

$$\mathbf{f}(k) = \sum_{j=0}^{\infty}\mathbf{B}(k,j)\mathbf{g}(j)\delta_{kj}. \tag{41}$$

Thus (41) defines a linear continuous operator $L : \mathbf{f} = L\mathbf{g}$ from $\mathbf{l}_2$ into $\mathbf{l}_2$. Now we show that the operator L is completely continuous in $\mathbf{l}_2$. Indeed, let $\{\mathbf{g}_p\} = [\{\{g_{1p}(k)\}_{k=0}^{\infty}\}_p, \{\{g_{2p}(k)\}_{k=0}^{\infty}\}_p, \ldots, \{\{g_{np}(k)\}_{k=0}^{\infty}\}_p]^T$ be a bounded sequence in $\mathbf{l}_2$. Then, for each $m = 1,2,\ldots,n$, $\{\{g_{mp}(k)\}_{k=0}^{\infty}\}_p$ is a bounded sequence in l_2. Then we have

$$\sum_{k=0}^{\infty}(1+k)^{2s_m}|\widehat{u}_{mp}(k)|^2 = \sum_{k=0}^{\infty}|g_{mp}(k)|^2 \leqslant C_m \ \ \forall p = 1,2,\ldots$$

Thus the sequence $\{u_{mp}(x)\}_p$ is bounded in $h^{s_m}(J)$. Then there exists a subsequence $\{\{f_{mp'}(k)\}_{k=0}^{\infty}\}_{p'}$ converging in l_2, therefore there exists a subsequence $\{\{\widehat{v}_{mp'}(k)\}_k\}_p' = \{\{(1+k)^{-s_m}f_{mp}(k)\}_k\}_{p'}$ converging in l_2. This means that one has found a subsequence $\{v_{mp'}(x)\}_{p'}$ converging in $h^{s_m}(J)$. Thus there exists the subsequence $\{\mathbf{v}_{p'}(x)\}_{p'} = \{(A\mathbf{u}_{p'})(x)\}_{p'}$ converging in $\mathbf{h}^{\vec{s}}(J)$. The proof of Theorem 4.3 is complete. ■

Theorem 4.4. *Assume that* $\mathbf{B}(k) \in \boldsymbol{\Sigma}^{2\vec{s}-\vec{\beta}}(\vec{\beta} = (\beta,\beta,...,\beta), \ \beta > 1/2; \ i = 1,2,\ldots,n)), \mathbf{u} \in \mathbf{h}_o^{\vec{s}}$ *and* p *is the restriction operator to* $\Omega \subset J$*. Then the pseudodifferential operator* B *defined by the formula*

$$(B\mathbf{u})(x) = pS^{-1}[\mathbf{B}(k)\hat{\mathbf{u}}(k)](x), \;\; \hat{\mathbf{u}}(k) = S[\mathbf{u}](k)$$

is completely continuous from $\mathbf{h}_o^{\vec{s}}(\Omega)$ *into* $\mathbf{h}^{-\vec{s}}(\Omega)$.

Proof. Applying Theorem 4.3 with $\vec{\alpha} = 2\vec{s} - \vec{\beta}, \; \vec{\varepsilon} = \vec{\beta}/2$ we can show that the operator B is continuous from $\mathbf{h}_o^{\vec{s}}(\Omega) \subset \mathbf{h}^{\vec{s}}(J)$ into $\mathbf{h}^{-\vec{s}+\vec{\beta}}(\Omega) = p\mathbf{h}^{-\vec{s}+\vec{\beta}}(J)$. We put

$$\begin{aligned} &\widetilde{B}[\mathbf{u}](x) = pS^{-1}[\mathbf{B}(k)\widehat{\mathbf{u}}(k)](x), \;\; x \in J, \\ &\mathbf{v} = pJ_{-\vec{s}}\widetilde{B}[\mathbf{u}], \;\; L\mathbf{v} = pl\mathbf{v}, \end{aligned}$$

where $J_{-\vec{s}}$ denotes the embedding operator to $\mathbf{h}^{-\vec{s}}(J), l$ and p are the extension and the restriction operators respectively. We have $L[\mathbf{v}] = B[\mathbf{u}]$, besides, the operator L is bounded from $\mathbf{h}^{-\vec{s}}(\Omega)$ into $\mathbf{h}^{-\vec{s}+\vec{\beta}}(\Omega)$. Let $l_o L[\mathbf{f}]$ be a certain continuous extention of $L[\mathbf{f}]$. Denote by $\boldsymbol{\Lambda}_{\vec{\beta}}$ the pseudo-differential operator of the form (24) with the matrix-symbol $\boldsymbol{\Lambda}(k) = ||\lambda_{ij}(k) = (1+k)^{\beta}||_{n\times n}, \; (\vec{\beta} = (\beta, \beta, \ldots, \beta))$. We have

$$L[\mathbf{v}] = p\boldsymbol{\Lambda}_{-\vec{\beta}}\Lambda_{\vec{\beta}} l_o L[\mathbf{v}]. \tag{42}$$

On the right-hand side of (42), L is bounded from $\mathbf{h}^{-\vec{s}}(\Omega)$ into $\mathbf{h}^{-\vec{s}+\vec{\beta}}(\Omega), \; l_o$ is bounded from $\mathbf{h}^{-\vec{s}+\vec{\beta}}(\Omega)$ into $\mathbf{h}^{-\vec{s}+\vec{\beta}}(J), \; \boldsymbol{\Lambda}_{\vec{\beta}}$ is bounded from $\mathbf{h}^{-\vec{s}+\vec{\beta}}(J)$ into $\mathbf{h}^{-\vec{s}}(J)$. In virtue of Theorem 4.3, the operator $\boldsymbol{\Lambda}_{-\vec{\beta}}$ is completely continuous in $\mathbf{h}^{-\vec{s}}(J)$. Finally, the operator p is continuous from $\mathbf{h}^{-\vec{s}}(J)$ into $\mathbf{h}^{-\vec{s}}(\Omega)$. Thus the operator L is completely continuous. Since $L[\mathbf{v}] = B[\mathbf{u}], B$ also is a completely continuous operator from $\mathbf{h}_o^{\vec{s}}(\Omega)$ into $\mathbf{h}^{-\vec{s}}(\Omega)$.

The proof is complete. ■

5. Solvability of systems of dual equations

5.1. Formulation of the problem. In this section we shall investigate a system of dual equations in the form

$$\begin{cases} pS^{-1}[\mathbf{A}(k)\widehat{\mathbf{u}}(k)](x) = \mathbf{f}(x), \quad x \in \Omega \subset J, \\ p'S^{-1}[\widehat{\mathbf{u}}(k)](x) = \mathbf{g}(x), \quad x \in \Omega' := J \setminus \Omega, \end{cases} \tag{43}$$

where Ω is a certain interval in $J, \; \widehat{\mathbf{u}}(k) = [\hat{u}_1(k), \hat{u}_2(k), \ldots, \hat{u}_n(k)]^T$ is a vector function to be found, $\mathbf{f}(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T \in (\mathcal{D}'(\Omega))^n, \; \mathbf{g}(x) = (g_1(x), g_2(x), \ldots, g_n(x))^T \in (\mathcal{D}'(\Omega'))^n$ are given vectors of distributions on Ω and Ω' respectively, $\mathbf{A}(k) = \|a_{ij}(k)\|_{n\times n}$ is a given square matrix and is called the symbol of the system (43), p and p' are restriction operators to Ω and Ω' respectively. The operator S^{-1} is understood in the sense of generalized functions. The dual series equations of type (43) are generalizations of some equations which are usually encountered in mixed boundary value problems of mathematical and contact problems of elasticity, for example [2, 6-9].

Note that the theory of dual series equations recently has become very developed and is the subject of numerous investigations. Formal analytical methods for finding solutions to dual equations have been studied by many authors, but much less attention has been paid to solvability question of these equations. Some results on the solvability and validation of solutions of dual series equations were considered in [1, 3-5].

5.2. Uniqueness theorem. We shall consider the system of dual integral equations (43) under the following conditions

$$\begin{cases} \mathbf{A}(k) \in \boldsymbol{\Sigma}_o^{\vec{\alpha}}, \text{and } \mathbf{A}(k) \text{ is positive-definite for almost } k \in \mathbb{N}, \\ \mathbf{f}(x) \in \mathbf{h}^{-\vec{\alpha}/2}(\Omega),\ \mathbf{g}(x) \in \mathbf{h}^{\vec{\alpha}/2}(\Omega') \end{cases} \tag{44}$$

and shall find the vector $\widehat{\mathbf{u}}(k)$ in the form $\widehat{\mathbf{u}}(k) = S[\mathbf{u}](\mathbf{k})$, where $\mathbf{u} \in \mathbf{h}^{\vec{\alpha}/2}(J)$.

Theorem 5.1 (Uniqueness). *Suppose that the assumptions* (44) *are fulfilled. Then, if the system* (43) *has a solution* $\mathbf{u}(x) = S^{-1}[\widehat{\mathbf{u}}](x) \in \mathbf{h}^{\vec{\alpha}/2}(J)$ *then it is unique.*

Proof. To prove this theorem it suffices to show that the homogeneous system

$$\begin{cases} pS^{-1}[\mathbf{A}(k)\widehat{\mathbf{u}}(k)](x) = \mathbf{0}, \quad x \in \Omega, \\ p'\mathbf{u}(x) = p'S^{-1}[\widehat{\mathbf{u}}(k)](x) = \mathbf{0}, \quad x \in \Omega' := J \setminus \Omega, \end{cases}$$

has only the trivial solution. The last system may be rewritten as

$$(A\mathbf{u})(x) = \mathbf{0}, \quad x \in \Omega, \tag{45}$$

where

$$(A\mathbf{u})(x) = pS^{-1}[\mathbf{A}(k)\widehat{\mathbf{u}}(k)](x), \quad x \in \Omega. \tag{46}$$

Since $A\mathbf{u} \in \mathbf{h}^{-\vec{\alpha}/2}(\Omega) \simeq (\mathbf{h}_o^{\vec{\alpha}/2}(\Omega))^*$ (see Theorem 3.11), from (21) we have

$$[A\mathbf{u}, \mathbf{u}] = \sum_{k=0}^{\infty} \overline{\widehat{\mathbf{u}}^T(k)}.S\{lpS^{-1}[\mathbf{A}\widehat{\mathbf{u}}]\}(k).$$

Due to Theorem 3.11, the last sum does not depend upon the choice of $lpS^{-1}[\mathbf{A}\widehat{\mathbf{u}}]$, we can take the extension in the form

$$lpS^{-1}[\mathbf{A}\widehat{\mathbf{u}}] = S^{-1}[\mathbf{A}\widehat{\mathbf{u}}].$$

Hence we have

$$[A\mathbf{u}, \mathbf{u}] = \sum_{k=0}^{\infty} \overline{\widehat{\mathbf{u}}^T(k)}.\mathbf{A}(k).\widehat{\mathbf{u}}(k). \tag{47}$$

Then from (45), (46) and (47) we get

$$[A\mathbf{u},\mathbf{u}] = \sum_{k=0}^{\infty} \overline{\widehat{\mathbf{u}}^T(k)}.\mathbf{A}(k).\widehat{\mathbf{u}}(k) = 0. \tag{48}$$

Since $\mathbf{A}(k) \in \boldsymbol{\Sigma}_o^{\vec{\alpha}}$, $\overline{\widehat{\mathbf{u}}^T(k)}.\mathbf{A}(k).\widehat{\mathbf{u}}(k) \geq 0$ for almost every $k \in \mathbb{N}$. Then from (48) it follows that $\widehat{\mathbf{u}}(k) \equiv \mathbf{0}, \mathbf{u}(\mathbf{x}) \equiv \mathbf{0}$. The proof of Theorem 5.1 is complete. ■

Lemma 5.2. *The system of dual series equations* (43) *is equivalent to the following system*

$$pS^{-1}[\mathbf{A}(k)\hat{\mathbf{v}}(k)](x) = \mathbf{f}(x) - pS^{-1}[\mathbf{A}(k)\widehat{\mathbf{l}'\mathbf{g}}(k)](x),\ x \in \Omega, \tag{49}$$

where $\mathbf{v} = S^{-1}[\widehat{\mathbf{v}}] \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$ *satisfies the relation*

$$\mathbf{v} + \mathbf{l}'\mathbf{g} = \mathbf{u} \in \mathbf{h}^{\vec{\alpha}/2}(J) \tag{50}$$

($\mathbf{l}'\mathbf{g}$ *being an arbitrary extension of the generalized vector-function* $\mathbf{g}$ *from* Ω' *into* J*).*

Proof. Assume that $\mathbf{u} \in \mathbf{h}^{\vec{\alpha}/2}(J)$ satisfies the system of dual equations (43) and $\mathbf{l}'\mathbf{g} \in \mathbf{h}^{\vec{\alpha}/2}(J)$ is an arbitrary extension of $\mathbf{g} \in \mathbf{h}^{\vec{\alpha}/2}(\Omega')$. Taking $\mathbf{v} = \mathbf{u} - \mathbf{l}'\mathbf{g}$, we get $\mathbf{v} \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$. Putting (50) into (43) we have (49). The right-hand side of (49) belongs to $\mathbf{h}^{-\vec{\alpha}/2}(\Omega)$ in view of Theorem 4.3.

Conversely, assume that $\mathbf{v} \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$ satisfies the equation (49). Then obviously, the vector-function $\mathbf{u}$ defined by (50) belongs to $\mathbf{h}^{\vec{\alpha}}(J)$. We shall prove that this function satisfies the system of dual equations (43) in the sense of distributions. Indeed, by transfering the second term in the right-hand side of (49) to the left-hand side and using (50) we obtain the first equality in (43). Further, from (50) the second equality of this system of dual equations follows. ■

Denote

$$\mathbf{h}(x) = \mathbf{f}(x) - pS^{-1}[\mathbf{A}(k)\widehat{\mathbf{l}'\mathbf{g}}(k)](x). \tag{51}$$

Using (46), (51) we rewrite (49) in the form

$$(A\mathbf{v})(x) = \mathbf{h}(x), \quad x \in \Omega. \tag{52}$$

Our purpose now is to establish the existence of the solution of the system (52) in the space $\mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$. We shall consider the following cases.

5.3. The case $\mathbf{A}(k) \in \boldsymbol{\Sigma}_+^{\vec{\alpha}}$. In this case due to Theorem 4.2 the scalar product and the norm in $\mathbf{h}^{\vec{\alpha}/2}(J)$ can be defined by the formulas (27) and (28)

$$(\mathbf{v},\mathbf{w})_{\mathbf{A}_+,\tilde{\alpha}/\mathbf{2}} = \sum_{k=0}^{\infty} \overline{S[\mathbf{w}^T](k)}\mathbf{A}_+(k)F[\mathbf{v}](k),$$

$$\|\mathbf{v}\|_{\mathbf{A}_+,\tilde{\alpha}/\mathbf{2}} = \Big(\sum_{k=0}^{\infty} \overline{S[\mathbf{v}^T](k)}\mathbf{A}_+(k)S[\mathbf{v}](k)\Big)^{1/2},$$

respectively. We shall also write $A_+\mathbf{v}$ instead of $A\mathbf{v}$.

Theorem 5.3 (Existence). *If* $\mathbf{h} \in \mathbf{h}^{-\vec{\alpha}/2}(\Omega)$, $\mathbf{A}(k) = \mathbf{A}_+(k) \in \Sigma_+^{\vec{\alpha}}$, *then the system* (52) *has a unique solution* $\mathbf{v} \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$.

Proof. By an argument similar to that used in the proof of Theorem 3.11 we have

$$[A_+\mathbf{v}, \mathbf{w}] = \sum_{k=0}^{\infty} \overline{S[\mathbf{w}^T](k)}\mathbf{A}_+(k)S[\mathbf{v}](k) = (\mathbf{v}, \mathbf{w})_{\mathbf{A}_+, \tilde{\alpha}/\mathbf{2}}$$

for arbitrary vector-functions $\mathbf{v}$ and $\mathbf{w}$ belonging to $\mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$, where $[A_+\mathbf{v}, \mathbf{w}]$ is defined by the formula (21). Therefore if $\mathbf{v} \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$ satisfies (52), then the following equality holds

$$(\mathbf{v}, \mathbf{w})_{\mathbf{A}_+, \tilde{\alpha}/\mathbf{2}} = [\mathbf{h}, \mathbf{w}], \quad \forall \mathbf{w} \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega). \tag{53}$$

We shall demonstrate that if (53) holds for any $\mathbf{w} \in \mathbb{H}_o^{\vec{\alpha}}(\Omega)$, then the vector-function $\mathbf{v}$ will satisfy the system of equations (52) in the sense of generalized functions on Ω. In fact, noting that (53) holds for $\mathbf{w} = \boldsymbol{\varphi} \in (C_o^{\infty}(\Omega))^n$ and using the formula

$$\sum_{k=0}^{\infty} \widehat{f}(k)\overline{\widehat{\varphi}(k)}d = \langle f, \overline{\varphi}\rangle = (f, \varphi),$$

we get

$$[\mathbf{h}, \boldsymbol{\varphi}] = \sum_{k=0}^{\infty} \widehat{\mathbf{lh}}(k)\overline{\widehat{\boldsymbol{\varphi}}(\xi)} = \langle \mathbf{lh}, \overline{\boldsymbol{\varphi}}\rangle = (\mathbf{lh}, \boldsymbol{\varphi}),$$

$$(\mathbf{v}, \boldsymbol{\varphi})_{\mathbf{A}_+, \tilde{\alpha}/\mathbf{2}} = \sum_{k=0}^{\infty} \overline{S[\boldsymbol{\varphi}^T](k)}\mathbf{A}_+(k)S[\mathbf{v}](k)$$

$$= \sum_{k=0}^{\infty} \overline{S[\boldsymbol{\varphi}^T](k)}SS^{-1}[\mathbf{A}_+\widehat{\mathbf{v}}](k) = (S^{-1}[\mathbf{A}_+\hat{\mathbf{v}}], \boldsymbol{\varphi}).$$

Hence, from (53) we have

$$(S^{-1}[\mathbf{A}_+\hat{\mathbf{v}}], \boldsymbol{\varphi}) = (\mathbf{lh}, \boldsymbol{\varphi}), \quad \forall \boldsymbol{\varphi} \in (C_o^{\infty}(\Omega))^n,$$

i.e.

$$pS^{-1}[\mathbf{A}_+\hat{\mathbf{v}}](x) = p\mathbf{lh}(x) = \mathbf{h}(x), \quad x \in \Omega.$$

We now return to the relation (53). Since $[\mathbf{h}, \mathbf{w}]$ is a linear continuous functional on the Hilbert space $\mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$, then by virtue of Riesz theorem there exists a unique element $\mathbf{v}_o \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$ such that

$$[\mathbf{h}, \mathbf{w}] = (\mathbf{v_o}, \mathbf{w})_{\mathbf{A}_+, \vec{\alpha}/2}, \quad \mathbf{w} \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega),$$

and moreover, there holds the estimation

$$\|\mathbf{v_o}\|_{\mathbf{A}_+,\vec{\alpha}/2} \leqslant C\|\mathbf{h}\|_{\mathbf{h}^{-\vec{\alpha}/2}(\Omega)}, \tag{54}$$

where C is a positive constant.

Since (53) is equivalent to (52), the system (52) has a unique solution $\mathbf{v} = \mathbf{v_o} \in \mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$, and this completes the proof of Theorem 5.3. ■

Remark 5.4. It is easy to see that the inverse operator A_+^{-1} from $\mathbf{h}^{-\vec{\alpha}}(\Omega)$ into $\mathbf{h}_o^{\vec{\alpha}}(\Omega)$ is bounded.

This affirmation follows from Theorem 5.3 and the inequality (54).

Remark 5.5. The solution $\mathbf{u}$ of the system of the dual integral equations (43) expressed in terms of the solution $\mathbf{v}$ of the system (52) by the formula (50) does not depend on the choice of the extension $\mathbf{lg}$.

This fact follows from the uniqueness of the solution of the system of dual equations (43). Hence we can choose an extension $\mathbf{lg}$ such that

$$\|\mathbf{lg}\|_{\vec{\alpha}/2} \leqslant 2\|\mathbf{g}\|_{\mathbf{h}^{\vec{\alpha}/2}(\Omega')}.$$

In this case, from (50), (51) and (54) it is easy to obtain the following estimate

$$\|\mathbf{u}\|_{\vec{\alpha}/2} \leqslant C(\|\mathbf{f}\|_{\mathbf{h}^{-\vec{\alpha}/2}(\Omega)} + \|\mathbf{g}\|_{\mathbf{h}^{\vec{\alpha}/2}(\Omega')}), \tag{55}$$

where $C = \text{const} > 0$. Therefore the solution of the system of dual equations (43) depends continuously upon the vector-functions given on the right-hand side of this system.

Hence the following theorem has been proved.

Theorem 5.6 (Existence). *Let* $\mathbf{A}(k) \in \Sigma_+^{\vec{\alpha}}$, $\mathbf{f} \in \mathbf{h}^{-\vec{\alpha}/2}(\Omega)$, $\mathbf{g} \in \mathbf{h}^{\vec{\alpha}/2}(\Omega')$. *Then the system of dual integral equations* (43) *has a unique solution* $\mathbf{u} = S^{-1}[\hat{\mathbf{u}}] \in \mathbf{h}^{\vec{\alpha}/2}(J)$ *satisfying estimation* (55).

5.4. The case $\mathbf{A}(k) \in \Sigma_o^{\vec{\alpha}}$. We assume in addition that the set Ω is bounded and there exists a square matrix $\mathbf{A}_+(k) \in \Sigma_+^{\vec{\alpha}}$ such that

$$\mathbf{B}(k) := \mathbf{A}(k) - \mathbf{A}_+(k) \in \Sigma^{\vec{\alpha}-\vec{\beta}}, \tag{56}$$

where $\vec{\beta} = (\beta, \beta, \ldots, \beta) \in \mathbb{R}^n$, $\beta > 0$. Now we represent the operator A defined by (46) in the form $A = A_+ + B$, where

$$A_+\mathbf{v} = pS^{-1}[\mathbf{A}_+\widehat{\mathbf{v}}], \quad B\mathbf{v} = pS^{-1}[\mathbf{B}\widehat{\mathbf{v}}], \quad \widehat{\mathbf{v}} = S[\mathbf{v}]. \tag{57}$$

Theorem 5.7 (Existence). *Under conditions* (44) *and* (56), *for every* $\mathbf{f} \in \mathbf{h}^{-\vec{\alpha}/2}(\Omega)$, $\mathbf{g} \in \mathbf{h}^{\vec{\alpha}/2}(\Omega')$ *the system of dual equations* (43) *has a unique solution* $\mathbf{u} = S^{-1}[\hat{\mathbf{u}}] \in \mathbf{h}^{\vec{\alpha}/2}(J)$.

Proof. According to Lemma 5.2 the system of dual equations (43) is equivalent to system (52). In virtue of Remark 5.4, the operator A_+^{-1} is bounded from $\mathbf{h}^{-\vec{\alpha}/2}(\Omega)$ into $\mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$ and due to Theorem 4.3, the operator B defined by (57) is completely continuous from $\mathbf{h}_o^{\vec{\alpha}/2}(\Omega)$ into $\mathbf{h}^{-\vec{\alpha}/2}(\Omega)$. In this case we represent system (52) in the form

$$A_+\mathbf{v} + B\mathbf{v} = \mathbf{h}.$$

Hence we have

$$\mathbf{v} + A_+^{-1}B\mathbf{v} = A_+^{-1}\mathbf{h}. \tag{58}$$

Thus the operator $A_+^{-1}B$ is completely continuous. It follows that system (58) is Fredholm. Due to the uniqueness of its solution (Theorem 5.1) it follows that this system has a unique solution $\mathbf{v} \in \mathbf{h}_o^{\vec{\alpha}/2}$. Therefore, in this case, system (43) has a unique solution $\mathbf{u} \in \mathbf{h}^{\vec{\alpha}/2}(J)$. The proof is complete. ■

6. Applications

6.1. A mixed boundary value problem for the harmonic equation. Consider the following problem: find a solution of the Laplace equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0, \quad (0 < x < l,\ 0 < y < h) \tag{59}$$

subject to the boundary conditions

$$\Phi(0, y) = \Phi(l, y) = 0,\ \ 0 < y < h, \tag{60}$$

$$\begin{cases} -\Phi(x,0) = f_1(x), & x \in (0,a), \\ \dfrac{\partial \Phi}{\partial y}(x,0) = g_1(x), & x \in (a,l), \end{cases} \tag{61}$$

$$\begin{cases} \dfrac{\partial \Phi}{\partial y}(x,h) = f_2(x), & x \in (0,a), \\ \Phi(x,h) = g_2(x), & x \in (a,l), \end{cases} \tag{62}$$

where f_1, f_2, g_1, g_2 are given functions.

We put $\psi_k(x) = \sqrt{\dfrac{2}{l}} \sin \dfrac{k\pi x}{l}$. It is clear that the functions $\psi_k(x)$ generate an orthonormal sequence in $L_2(0,l)$. Let $\mathcal{A}' = \mathcal{A}'(\psi_k(x))$ be the space of generalized functions generated by the functions $\psi_k(x)$. For a generalized function $f \in \mathcal{A}'$ we put

$$\widehat{f}(k) := S[f](k) := (f, \psi_k), \tag{63}$$

$$f(x) = S^{-1}[\widehat{f}](x) = \sum_{k=1}^{\infty} \widehat{f}(k)\psi_k(x). \tag{64}$$

Problem (59)-(62) is reduced to the following system of dual integral equations

$$\begin{cases} pS^{-1}[\mathbf{A}(\xi)\widehat{\mathbf{u}}(\xi)](x) = \mathbf{f}(x), & x \in \Omega := (0,a), \\ p'S{-}1[\widehat{\mathbf{u}}(\xi)](x) = \mathbf{g}(x), & x \in \Omega' := (a,l), \end{cases} \tag{65}$$

where

$$\mathbf{f}(x) = (f_1(x), f_2(x))^T, \quad \mathbf{g}(x) = (g_1(x), g_2(x))^T,$$

$$\mathbf{u} = (u_1, u_2)^T = (\frac{\partial \Phi}{\partial y}(x,0), \Phi(x,h))^T,$$

$$\widehat{\mathbf{u}}(\xi) = S[\mathbf{u}] = (\widehat{\mathbf{u}}_\mathbf{1}(\xi), \widehat{\mathbf{u}}_\mathbf{2}(\xi))^\mathbf{T},$$

$$\mathbf{A}(\xi) = \begin{pmatrix} \dfrac{\tanh(kh)}{\xi} & -\dfrac{1}{\cosh(kh)} \\ \dfrac{1}{\cosh(kh)} & \xi\tanh(kh) \end{pmatrix},$$

p and p' denote resriction operators to Ω and Ω' respectively. Denote

$$\vec{\alpha} = (\alpha_1, \alpha_2)^T = (-1, 1)^T$$

and introduce the matrices

$$\mathbf{A}_+(k) = \begin{pmatrix} \dfrac{\tanh(kh)}{k} & 0 \\ 0 & k\coth(kh) \end{pmatrix},$$

$$\mathbf{B}(k) = \mathbf{A}(k) - \mathbf{A}_+(k) = \begin{pmatrix} 0 & \dfrac{-1}{\cosh(kh)} \\ \dfrac{1}{\cosh(kh)} & \dfrac{-k}{\sinh(kh)\cosh(kh)} \end{pmatrix}.$$

We have

$$\frac{\tanh(kh)}{k} \in \sigma_+^{-1}, \ \ k\coth(kh) \in \sigma_+^1 \ \ \mathbf{A}_+(k) \in \Sigma_+^{\vec{\alpha}}, \ \vec{\alpha} = (-1,1)^T.$$

It is not difficult to show that

$$\mathbf{A}(k) \in \Sigma_o^{\vec{\alpha}}, \ \ \mathbf{B}(k) \in \Sigma^{-\vec{\beta}}, \quad \vec{\beta} = (\beta,\beta)^T, \quad \beta > 1.$$

Due to Theorem 5.7 we have the following result:

Theorem 6.1. *For every* $\mathbf{f}(\mathbf{x}) \in \mathbf{h}^{-\vec{\alpha}/2}(0,a), \quad \mathbf{g}(\mathbf{x}) \in \mathbf{h}^{\vec{\alpha}/2}(a,l)$*, the system of dual equations* (65) *has a unique solution* $\mathbf{u} = S^{-1}[\hat{\mathbf{u}}] \in \mathbf{h}^{\vec{\alpha}/2}$*, i.e.*

$$u_1(x) = \frac{\partial \Phi}{\partial y}(x,0) \in h^{-1/2}(0,l), \quad u_2(x) = \Phi(x,h) \in H^{1/2}(0,l).$$

6.2. A mixed boundary value problem for the biharmonic equation. We study the solution $\Phi(x,y)$ of a boundary value problem for the biharmonic equation

$$\Delta^2\Phi(x,y) = \frac{\partial^4\Phi}{\partial x^4} + 2\frac{\partial^4\Phi}{\partial x^2\partial} + \frac{\partial^4\Phi}{\partial y^4} = 0 \tag{66}$$

in the rectangle

$$\Pi = \{(x,y) : 0 < x < l, 0 < y < h\}$$

with boundary conditions

$$\Phi(0,y) = \Phi_{xx}(0,y) = 0, \;\; 0 < y < h, \tag{67}$$

$$\Phi(l,y) = \Phi_{xx}(l,y) = 0, \;\; 0 < y < h. \tag{68}$$

$$\Phi(x,0) = 0, \;\; \Phi(x,h) = 0, \;\; -\infty < x < \infty. \tag{69}$$

$$\begin{cases} -\Phi_y(x,0) = f_1(x), \;\; -a < x < a, \\ \Phi_{yy}(x,0) + \nu\Phi_{xx}(x,0) = g_1(x), \;\; |x| > a, \end{cases} \tag{70}$$

$$\begin{cases} \Phi_y(x,h) = f_2(x), \;\; -a < x < a, \\ \Phi_{yy}(x,h) + \nu\Phi_{xx}(x,h) = g_2(x), \;\; |x| > a, \end{cases} \tag{71}$$

where $0 < \nu < 1$.

The solution of (66) is given in the form

$$\Phi(x,y) = \sum_{k=1}^{\infty} \widehat{\Phi}(k,y)\psi_k(x), \tag{72}$$

where $\psi_k(x) = \sqrt{\frac{2}{l}}\sin\frac{k\pi x}{l}$ and

$$\widehat{\Phi}(k,y) = A_k\cosh(ky) + B_k y\cosh(ky) + C_k\sinh(ky) + D_k y\sinh(ky). \tag{73}$$

Here A_k, B_k, C_k, D_k are arbitrary coefficients. We see that the function $\Phi(x,y)$ determined by the formula (73) satisfies the boundary conditions (67) and (68) for arbitrary coefficients A_k, B_k, C_k, D_k. We put

$$\hat{u}_1(k) = \widehat{\Phi}_{yy}(k,0) + \nu\widehat{\Phi}_{xx}(k,0), \tag{74}$$

$$\hat{u}_2(k) = \widehat{\Phi}_{yy}(k,h) + \nu\widehat{\Phi}_{xx}(k,h). \tag{75}$$

Thus we have

$$A_k = 0, \quad D_k = \frac{\widehat{u}_1(k)}{2k},$$

$$C_k = -\frac{\widehat{u}_1(k)}{2k}.\frac{h}{\sinh^2(kh)} - \frac{\widehat{u}_2(k)}{2k}.\frac{h\cosh(kh)}{\sinh^2(kh)},$$

$$B_k = \frac{\widehat{u}_1(k)}{2k}\Big[\frac{1-\sinh^2(kh)}{\sinh(kh)\cosh(kh)}\Big] + \frac{\widehat{u}_2(k)}{2k\sinh(kh)}.$$

Satisfying the boundary conditions (70) and (71) we have the following system of dual series equations

$$\begin{cases} pS^{-1}[\mathbf{A}(k)\widehat{\mathbf{u}}(k)](x) & = \mathbf{f}(x), \quad x \in \Omega := (0,a), \\ p'S^{-1}[\widehat{\mathbf{u}}(k)](x) & = \mathbf{g}(x), \quad x \in \Omega' := (a,l), \end{cases} \tag{76}$$

where

$$\mathbf{f}(x) = (f_1(x), f_2(x))^T, \quad \mathbf{g}(x) = (g_1(x), g_2(x))^T,$$

$$\mathbf{u} = (u_1, u_2)^T = (\Phi_{yy}(x,0) + \nu\Phi_{xx}(x,0), \Phi_{yy}(x,h) + \nu\Phi_{xx}(x,h))^T,$$

$$\widehat{\mathbf{u}}(k) = S[\mathbf{u}](k) = (\widehat{u}_1(k), \widehat{u}_2(k))^T,$$

$$\mathbf{A}(k) = \begin{pmatrix} a_{11}(k) & a_{12(k)} \\ a_{21}(k) & a_{22}(k) \end{pmatrix}$$

$$a_{11}(k) = \frac{1}{2k}\Big[\frac{-1+\sinh^2(kh)}{\sinh(kh)\cosh(kh)} + \frac{kh}{\sinh^2(kh)}\Big],$$

$$a_{12}(k) = -\frac{1}{2k}.\frac{\sinh(kh) - kh}{\sinh^2(kh)},$$

$$a_{21}(k) = \frac{1}{2k}\Big[-\frac{kh\cosh(kh)}{\sinh^2(kh)} + \frac{\cosh(kh) + 2kh\sinh(kh)}{\sinh(kh)\cosh(kh)}\Big],$$

$$a_{22}(k) = \frac{1}{2k}.\frac{\sinh(kh)\cosh(kh) - kh}{\sinh^2(kh)}.$$

Proposition 6.2. *The matrix* $\mathbf{A}(k)$ *is a positive-definite matrix for all* $k = 1, 2, ...$

Proof. Indeed, putting $kh = t$ we have

$$2ka_{11} = \frac{-1+\sinh^2 t}{\sinh t\cosh t} + \frac{t}{\sinh^2 t} = \frac{\sinh^3 t - \sinh t + t\cosh t}{\sinh^2 t\cosh t}.$$

$$f(t) := \sinh^3 t - \sinh t + t\cosh t, \quad f(0) = 0,$$

$$f'(t) = 3\sinh^2 t\cosh t - \cosh t + \cosh t + t\sinh t = 3\sinh^2 t\cosh t + t\sinh t > 0, \forall t > 0.$$

Hence $a_{11}(k) > 0$. It is clear that $a_{22}(k) > 0, -a_{12}(k) < 0 \ \forall k = 1, 2, ...$ So $a_{11}(k)a_{22}(k) > 0$. We have

$$2ka_{21}(k) = \frac{\sinh t\cosh t + 2t\sinh^2 t - t\cosh^2 t}{\sinh^2 t\cosh t} =$$

$$= \frac{\sinh t\cosh t + t\sinh^2 t - t(\cosh^2 t - \sinh^2 t)}{\sinh^2 t\cosh t} =$$

$$= \frac{(\sinh t\cosh t - t) + t\sinh^2 t}{\sinh^2 t\cosh t} > 0.$$

So we have

$\varDelta_1 = a_{11}(k) > 0,$

$\varDelta_2 = a_{11}(k)a_{22}(k) - a_{12}(k)a_{21}(k) > 0.$

The proposition is proved. ■

According to Proposition 6.2, $\mathbf{A}(k) \in \boldsymbol{\Sigma}_o^{\vec{\alpha}/2}, \vec{\alpha} = (-1,-1)^T$. Then, due to Theorem 5.1, the dual series equations (76) has at most one solution in the space $\mathbf{h}^{\vec{\alpha}/2}(0.l), \vec{\alpha} = (-1,-1)^T$.

Next, we introduce the matrices

$$\mathbf{A}_+(k) = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \ \ \mathbf{B}(k) = \mathbf{A}(k) - \mathbf{A}_+(k).$$

It is not difficult to show that

$$\mathbf{A}(k) \in \boldsymbol{\Sigma}_+^{\vec{\alpha}}, \ \ \vec{\alpha} = (-1,-1)^T; \ \ \mathbf{B}(k) \in \boldsymbol{\Sigma}^{-\vec{\beta}}, \ \ \vec{\beta} = (\beta,\beta)^T, \beta > 1.$$

Theorem 6.3. *For every* $\mathbf{f}(\mathbf{x}) \in \mathbf{h}^{-\vec{\alpha}/2}(0,a), \ \ \mathbf{g}(\mathbf{x}) \in \mathbf{h}^{\vec{\alpha}/2}(a,l)(\vec{\alpha} = (-1,-1)^T)$, *the system of dual equations* (76) *has a unique solution* $\mathbf{u} = S^{-1}[\hat{\mathbf{u}}] \in \mathbf{h}^{\vec{\alpha}/2}$, *i.e.*

$$u_1(x) = \varPhi_{yy}(x,0) + \nu\varPhi_{xx}(x,0) \in h^{-1/2}(0,l),$$

$$u_2(x) = \varPhi_{yy}(x,h) + \nu\varPhi_{xx}(x,h) \in h^{-1/2}(0,l).$$

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