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Domain Decompositon Method for Elliptic Interface Problems

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Abstract. Interface problems arise in setting of various physical and engineering problems, where the governing differential equations have discontinuous across an interface. For solving them in recent years there are intensively developed immersed finite difference/ finite element methods which draw attention to discretization of the equations nearly the interface for ensuring accuracy. Differently from these methods in this paper we use a domain decomposition method based on updating of derivative of unknown function on interface to problem considered. It reduces the interface problem to a sequence of problems in subdomains that are easily solved by available softwares. The convergence of the iterative process is proved. Many numerical examples for rectangular and L-shape domains demonstrate the fast convergence of the method. The method can be applied especially efficiently for domains consisting of rectangles.

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1. Introduction

In the paper we consider the following boundary value problem (BVP)

$$\mathcal{L}u \equiv -\frac{\partial}{\partial x_1}\left(k_1(x)\frac{\partial u}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(k_2(x)\frac{\partial u}{\partial x_2}\right) + a(x)u = f(x), x \in \Omega, \quad (1)$$

$$[u]_{\Gamma} = \psi_1, \left[\frac{\partial u}{\partial \nu}\right]_{\Gamma} = \psi_2, \tag{2}$$

$$u = \varphi \quad \text{on} \quad \partial\Omega, \tag{3}$$

where $x = (x_1, x_2)$, Ω is a bounded domain in R^2 with the boundary $\partial\Omega$, $k_1(x)$ and $k_2(x)$ are coefficients discontinuous across the interface Γ, the notation $[u]_{\Gamma}$ stands for the jump of u across the interface, $\partial u/\partial\nu$ denotes the conormal derivative of u, defined by the formula

$$\frac{\partial u}{\partial \nu} = k_1 \frac{\partial u}{\partial x_1} \cos(n, x_1) + k_2 \frac{\partial u}{\partial x_2} \cos(n, x_2),$$

n is the unit outward normal to boundary.

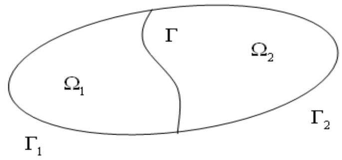

Fig. 1 Domain Ω and its subdomains

Denote $\Gamma_1 = \partial\Omega_1 \backslash \Gamma$, $\Gamma_2 = \partial\Omega_2 \backslash \Gamma$, $u_i = u|_{\Omega_i}$, $f_i = f|_{\Omega_i}$, $k_{1i} = k_1(x)$, $x \in \Omega_i$, $k_{2i} = k_2(x)$, $x \in \Omega_i$, $i = 1, 2$, and denote by n_i the outward normal to Γ relatively to Ω_i. Then the conormal derivative of u_i on Γ is

$$\frac{\partial u_i}{\partial \nu_i} = k_{1i} \frac{\partial u_i}{\partial x_1} \cos(n_i, x_1) + k_{2i} \frac{\partial u_i}{\partial x_2} \cos(n_i, x_2). \tag{4}$$

We assume that

$$0 < b_1 \le k_1(x), k_2(x) \le b_2, \quad a(x) \ge 0. \tag{5}$$

The problem (1)-(3) is known as the elliptic interface problem. It arises in setting of various physical and engineering problems.

For solving elliptic interface problems, the immersed interface method has been intensively developed in recent years (see, e.g [2, 6, 8, 9]). The method is a sharp interface method based across the interface so that the finite difference/element discretization can be accurate.

In an earlier work [3] we proposed an iterative method based on decomposition for problem (1)-(3). Recently in [4] and [5] we developed another version of domain decomposition method (DDM) for Poisson equation, which updates derivative of unknown function on interface. It is the difference from other domain decomposition methods.

In this paper we develop our DDM to problem (1)-(3), which reduces it to a sequence of problem in subdomains. The geometric progression rate of convergence of iterative process is proved. Several numerical examples demonstrate the fast convergence of the method. The method is especially efficient if applied to domains consisting of rectangles, for which we have developed a software for second order equation with different boundary conditions on sites.

2. Iterative method

Consider a method based on iterations for finding the boundary function $g = \dfrac{\partial u_1}{\partial \nu_1}$ on Γ.

(i) Given a starting approximation $g^{(0)}$ on Γ, for example, $g^{(0)} = 0$ on Γ;

(ii) Knowing $g^{(k)}$ $(k = 0, 1, 2, ...)$ on Γ solve consecutively two BVPs

$$\begin{aligned} \mathcal{L}u_1^{(k)} &= f_1 \quad \text{in} \quad \Omega_1, \\ u_1^{(k)} &= \varphi \quad \text{on} \quad \Gamma_1, \\ \frac{\partial u_1^{(k)}}{\partial \nu_1} &= g^{(k)} \quad \Gamma, \end{aligned} \tag{6}$$

$$\begin{aligned} \mathcal{L}u_2^{(k)} &= f_2 \quad \text{in} \quad \Omega_2, \\ u_2^{(k)} &= \varphi \quad \text{on} \quad \Gamma_2, \\ u_2^{(k)} &= u_1^{(k)} + \psi_1 \quad \text{on} \quad \Gamma; \end{aligned} \tag{7}$$

(iii) Update the new approximation

$$g^{(k+1)} = (1-\tau)\, g^{(k)} - \tau \frac{\partial u_2^{(k)}}{\partial \nu_2} + \tau \psi_2, \tag{8}$$

where τ is an iterative parameter.

Before investigating the convergence of the iterative process (6)-(8) we make assumptions on the smoothness of data functions as follows: $f_i \in L^2(\Omega_i)(i = 1, 2)$, $\varphi \in H^{1/2}(\partial\Omega)$, $\psi_1 \in H^{1/2}(\Gamma)$, $\psi_2 \in H^{-1/2}(\Gamma)$. Here, as usual, $H^s(G)$ is a Sobolev space. Under these assumptions according to [1] the problems (6), (7) have unique solutions $u_i^{(k)} \in H^1(\Omega_i, \mathcal{L})$, where we denote

$$H^1(\Omega_i, \mathcal{L}) = \left\{ v \in H^1(\Omega_i) \mid \mathcal{L}u \in L^2(\Omega_i) \right\}$$

and due to the trace theorem [7] $g^{(k+1)} \in H^{-1/2}(\Gamma)$. We also assume that the interface problem (1)-(3) has a unique solution u and $u_i = u|_{\Omega_i} \in H^1(\Omega_i, \mathcal{L})$.

In order to study the convergence of the above iterative method we rewrite the formula (8) in the formula

$$\frac{g^{(k+1)} - g^{(k)}}{\tau} + g^{(k)} + \frac{\partial u_2^{(k)}}{\partial \nu_2} = \psi_2. \tag{9}$$

Next, we set

$$\begin{aligned} e_i^{(k)} &= u_i^{(k)} - u_i, \quad (i = 1, 2) \\ \xi^{(k)} &= g^{(k)} - g. \end{aligned} \tag{10}$$

Then it is easy to verify that the errors $e_1^{(k)}$ and $e_2^{(k)}$ satisfy the problems

$$\begin{aligned} & Le_1^{(k)} = 0 \quad \text{in} \quad \Omega_1, \\ & e_1^{(k)} = 0 \quad \text{on} \quad \Gamma_1, \\ & \frac{\partial e_1^{(k)}}{\partial \nu_1} = \xi^{(k)} \quad \text{on} \quad \Gamma, \end{aligned} \tag{11}$$

$$\begin{aligned} & Le_2^{(k)} = 0 \quad \text{in} \quad \Omega_2, \\ & e_2^{(k)} = 0 \quad \text{on} \quad \Gamma_2, \\ & e_2^{(k)} = e_1^{(k)} \quad \text{on} \quad \Gamma. \end{aligned} \tag{12}$$

From (9) and the second formula in (10) we get the relation

$$\frac{\xi^{(k+1)} - \xi^{(k)}}{\tau} + \xi^{(k)} + \frac{\partial e_2^{(k)}}{\partial \nu_2} = 0. \tag{13}$$

Now, we introduce the boundary operator S_i acting on function ξ by the formula

$$S_i \xi = \frac{\partial v_i}{\partial \nu_i}, \quad (i = 1, 2) \tag{14}$$

where v_i is the solution of the problem

$$\begin{aligned} & Lv_i = 0 \quad \text{in} \quad \Omega_i, \\ & v_i = 0 \quad \text{on} \quad \Gamma_i, \\ & v_i = \xi \quad \text{on} \quad \Gamma. \end{aligned} \tag{15}$$

These operators are known as the Steklov-Poincare operators (see [13]).
Clearly, v_i is $L-$*extension* of ξ from Γ to Ω_i. For brevity we write $v_i = L_i \xi$.
It is clear that the inverse operator S_i^{-1} of S_i is determined as follows

$$S_i^{-1} \eta = w_i, \tag{16}$$

where w_i is a solution of the problem

$$\begin{aligned} & Lw_i = 0 \quad \text{in} \quad \Omega_i, \\ & w_i = 0 \quad \text{on} \quad \Gamma_i, \\ & \frac{\partial w_i}{\partial \nu_i} = \eta \quad \text{on} \quad \Gamma. \end{aligned} \tag{17}$$

In view of the definition of the above operators, from (11) and (12) we obtain

$$e_1^{(k)} = S_1^{-1}\xi^{(k)}, \tag{18}$$

$$S_2 e_1^{(k)} = \frac{\partial e_2^{(k)}}{\partial \nu_2}. \tag{19}$$

Therefore we can rewrite (13) in the formula

$$\frac{\xi^{(k+1)} - \xi^{(k)}}{\tau} + \xi^{(k)} + S_2 e_1^{(k)} = 0.$$

Acting S_1^{-1} on two sides of the above equality we have

$$\frac{e_1^{(k+1)} - e_1^{(k)}}{\tau} + e_1^{(k)} + S_1^{-1} S_2 e_1^{(k)} = 0.$$

Hence from here it follows that

$$e_1^{(k+1)} = (I - \tau B)\, e_1^{(k)}, \tag{20}$$

where we denoted

$$B = I + S_1^{-1} S_2. \tag{21}$$

In order to establish the convergence of the iterative process (6) – (8), or equivalently, the iterative scheme (20) we shall consider the operator B in an appropriate functional space. First we remark that the operator S_i $(i = 1, 2)$ acts between the space $\mathcal{H} = H_{00}^{1/2}(\Gamma) = \left\{ v|_\Gamma : v \in H_0^1(\Omega) \right\}$ and its dual $\mathcal{H}' = H_{00}^{-1/2}(\Gamma)$ (see [13]).

It can be noticed that starting from the weak formulation of (15) one is led to the following equivalent definition of the operators S_i

$$\langle S_i \xi, \eta \rangle_{\mathcal{H},\mathcal{H}'} = \int_{\Omega_i} \left(k_{1i} \frac{\partial (L_i \xi)}{\partial x_1} \frac{\partial (L_i \eta)}{\partial x_1} + k_{2i} \frac{\partial (L_i \xi)}{\partial x_2} \frac{\partial (L_i \eta)}{\partial x_2} \right) dx \;\; \forall \xi, \eta \in \mathcal{H}. \tag{22}$$

In the case if $S_i \xi \in L^2(\Gamma)$ we have the representation

$$\langle S_i \xi, \eta \rangle_{\mathcal{H}',\mathcal{H}} = (S_i \xi, \eta)_{L^2(\Gamma)}.$$

Therefore S_i is symmetric and positive definite because for all $\xi \in \mathcal{H}$

$$\begin{aligned} \langle S_i \xi, \xi \rangle_{\mathcal{H},\mathcal{H}'} &= \int_{\Omega_i} \left(k_{1i} \left(\frac{\partial v_i}{\partial x_1} \right)^2 + k_{2i} \left(\frac{\partial v_i}{\partial x_2} \right)^2 \right) dx \\ &\geq k_0 \int_{\Omega_i} \left[\left(\frac{\partial v_i}{\partial x_1} \right)^2 + \left(\frac{\partial v_i}{\partial x_2} \right)^2 \right] dx \\ &\geq C_{1i}^2 b_1 \|v_i\|_{H^1(\Omega_i)}^2 \geq C_{2i}^2 \|\xi\|_{H^{1/2}(\Gamma)}^2 \end{aligned} \tag{23}$$

due to the Poincare inequality and the trace theorem (see [13]). In the above estimates $v_i = L_i \xi$ and v_i is the solution of (15), C_{1i} and C_{2i} are positive

constants independent of ξ and η. In the sequel we also use C_{3i}, C_0, C to denote constants.
On the other hand, in view of the estimate for the solution of (15) we have

$$\langle S_i\xi, \xi\rangle_{\mathcal{H},\mathcal{H}'} \le b_2 \|v_i\|^2_{H^1(\Omega_i)} \le C^2_{3i} \|\xi\|^2_{H^{1/2}(\Gamma)}. \tag{24}$$

From (23) and (24) we obtain the two-sides estimate

$$C_{2i}\|\xi\|_{H^{1/2}(\Gamma)} \le \langle S_i\xi, \xi\rangle^{1/2}_{\mathcal{H}',\mathcal{H}} \le C_{3i}\|\xi\|_{H^{1/2}(\Gamma)}. \tag{25}$$

Therefore $\langle S_1\xi, \eta\rangle_{\mathcal{H},\mathcal{H}'}$ defines a scalar product of $\xi, \eta \in \mathcal{H}$ and the norm generated by this scalar product is equivalent to the usual norm of $H^{1/2}(\Gamma)$. Denote this scalar product and the corresponding norm by $(\cdot,\cdot)_{S_1}$ and $\|\cdot\|_{S_1}$, respectively. So, by definition $(\xi,\eta)_{S_1} = \langle S_1\xi, \eta\rangle_{\mathcal{H}',\mathcal{H}}$. In this product we have

$$(B\xi, \eta)_{S_1} = \left\langle S_1\left(I + S_1^{-1}S_2\right)\xi, \eta\right\rangle_{\mathcal{H}',\mathcal{H}} = \langle S_1\xi, \eta\rangle_{\mathcal{H}',\mathcal{H}} + \langle S_2\xi, \eta\rangle_{\mathcal{H}',\mathcal{H}}.$$

Since S_1 and S_2 are symmetric, as shown above, the operator B is symmetric. Further, we suppose that for the partition of Ω into the subdomains Ω_1 and Ω_2 there exist the constants $0 < m \le M$ such that

$$m \le \frac{\langle S_2\xi, \xi\rangle_{\mathcal{H}',\mathcal{H}}}{\langle S_1\xi, \xi\rangle_{\mathcal{H}',\mathcal{H}}} \le M \quad \forall \xi \in \mathcal{H}. \tag{26}$$

Then we have

$$(1+m)\|\xi\|^2_{S_1} \le (B\xi, \xi)_{S_1} \le (1+M)\|\xi\|^2_{S_1}.$$

It means that

$$(1+m)I \le B \le (1+M)I$$

in the energetic space of S_1. From the general theory of two-layer iterative scheme [10] it follows that if

$$0 < \tau < \frac{2}{1+M} \tag{27}$$

then $\|I - \tau B\|_{S_1} < 1$. Therefore the iterative process (20) converges, namely $\|e_1^{(k)}|_\Gamma\|_{S_1} \to 0$ as $k \to \infty$. Due to the mentioned above equivalence of the norms $\|\cdot\|_{S_1}$ and $\|\cdot\|_{H^{1/2}(\Gamma)}$ we also have $\|e_1^{(k)}|_\Gamma\|_{H^{1/2}(\Gamma)} \to 0$. Now, using the estimate for the solution of the elliptic problems (11) and (12) [7] we have

$$\left\|e_i^{(k)}\right\|_{H^1(\Omega_i)} \le C_0 \left\| e_1^{(k)}\Big|_\Gamma \right\|_{H^{1/2}(\Gamma)}. \tag{28}$$

Hence we come to the result

$$\|e_i^{(k)}\|_{H^1(\Omega_i)} \to 0 \tag{29}$$

or

$$u_i^{(k)} \to u_i \text{ in } H^1(\Omega_i) \text{ as } k \to \infty. \tag{30}$$

Moreover, we can also prove that

$$\frac{\partial u_i^{(k)}}{\partial \nu_i} \to \frac{\partial u_i}{\partial \nu_i} \text{ as } k \to \infty \quad (i = 1, 2) \text{ in } H^{-1/2}(\Gamma) \tag{31}$$

or equivalently

$$\frac{\partial e_i^{(k)}}{\partial \nu_i} \to 0 \text{ as } k \to \infty \quad (i = 1, 2) \text{ in } H^{-1/2}(\Gamma).$$

In view of (19), (18), (10), (6) and the notation of g the latter limits are the same as

$$S_i e_1^{(k)} \to 0 \text{ as } k \to \infty \text{ in } H^{-1/2}(\Gamma).$$

This follows from the inequality

$$|(S_i \xi, \eta)| \le b_2 \|L_i \xi\|_{H^1(\Omega_i)} \|L_i \eta\|_{H^1(\Omega_i)} \quad \forall \xi, \eta \in \mathcal{H},$$

which is a consequence of (22) and (5) when setting $\xi = e_1^{(k)}|_\Gamma$ and taking into account (29).

Now, taking into account (30), (31) we see that the limit of the approximate solution computed by the iterative process (6)-(8) is the solution of the original problem (1)-(3).

According to [10] the optimal value of τ for the iterative process (20) is

$$\tau_{opt} = \frac{2}{2 + m + M}. \tag{32}$$

For this value of τ there holds the estimate

$$\left\| e_1^{(k)} \Big|_\Gamma \right\|_{S_1} \le \rho^{(k)} \left\| e_1^{(0)} \Big|_\Gamma \right\|_{S_1}, \tag{33}$$

where

$$\rho = \frac{M - m}{2 + m + M}. \tag{34}$$

In view of the equivalence of the norms $\|\cdot\|_{S_1}$ and $\|\cdot\|_{H^{1/2}(\Gamma)}$ due to (28) we obtain

$$\left\| e_i^{(k)} \right\|_{H^1(\Omega_i)} \le C \rho^k \left\| e_1^{(0)} \Big|_\Gamma \right\|_{H^{1/2}(\Gamma)}, \tag{35}$$

for some constant C.

Now, we sum up the obtained result of the convergence of the iterative method for problem (1)-(3) in the following

Theorem 2.1. *Under the assumption* (26) *about the subdomains* Ω_1 *and* Ω_2 *the iterative method* (6)-(8) *for problem* (1)-(3) *converges if the iterative parameter* τ *satisfies the condition* (27). *Moreover, for the optimal value* τ_{opt} *given by* (32) *we* *have the estimate* (35) *for the errors, where* $e_i^{(k)} = u_i^{(k)} - u_i$ *and* ρ *is calculated by* (34).

3. A particular case

Now consider a simple case when:

- The domain Ω is the rectangle $[0,1] \times [0,b]$ divided into two subdomains by a line segment $\Gamma = \{x_1 = r, \quad 0 \le x_2 \le b\}, \quad 0 < r < 1$. We denote the left subdomain by Ω_1 and the right one by Ω_2.
- The coefficient $a(x) = 0$ and $k_1(x), k_2(x)$ are given as follows

$$\begin{aligned} &k_1(x) = \begin{cases} k_{11}, & x \in \Omega_1 \\ k_{12}, & x \in \Omega_2, \end{cases} \\ &k_2(x) = 1, \quad x \in \Omega, \end{aligned} \tag{36}$$

  where k_{11}, k_{12} are positive constants.

In this case by the method of separation of variables we find the solution of problem (15) in the form

$$v_1(x) = \sum_{n=1}^{\infty} \xi_n \frac{\sinh\left(\frac{\lambda_n}{\sqrt{k_{11}}} x_1\right)}{\sinh\left(\frac{\lambda_n}{\sqrt{k_{11}}} r\right)} e_n(x_2),$$

$$v_2(x) = -\sum_{n=1}^{\infty} \xi_n \frac{\sinh\left(\frac{\lambda_n}{\sqrt{k_{12}}}(1-x_1)\right)}{\sinh\left(\frac{\lambda_n}{\sqrt{k_{12}}}(1-r)\right)} e_n(x_2),$$

where

$$\lambda_n = \frac{n\pi}{b}, \quad e_n(x_2) = \sqrt{\frac{2}{b}} \sin(\lambda_n x_2),$$
$$\xi_n = (\xi, e_n)_{L^2(\Gamma)} = \int_0^b \xi(x_2) e_n(x_2) dx_2, \quad n = 1, 2, ...$$

Hence we have

$$S_1 \xi = \left.\frac{\partial v_1}{\partial \nu_1}\right|_{\Gamma} = \left.\frac{\partial v_1}{\partial x_1}\right|_{x_1 = r} = \sum_{n=1}^{\infty} \xi_n \frac{\lambda_n}{\sqrt{k_{11}}} \coth\left(\frac{\lambda_n}{\sqrt{k_{11}}} r\right) e_n(x_2),$$

$$S_2 \xi = \left.\frac{\partial v_2}{\partial \nu_2}\right|_{\Gamma} = -\left.\frac{\partial v_2}{\partial x_1}\right|_{x_1 = r} = \sum_{n=1}^{\infty} \xi_n \frac{\lambda_n}{\sqrt{k_{12}}} \coth\left(\frac{\lambda_n}{\sqrt{k_{12}}}(1-r)\right) e_n(x_2)$$

and

$$(S_1\xi,\xi)=\sum_{n=1}^{\infty}\frac{\lambda_n}{\sqrt{k_{11}}}\xi_n^2\coth\left(\frac{\lambda_n}{\sqrt{k_{11}}}r\right), \tag{37}$$

$$(S_2\xi,\xi)=\sum_{n=1}^{\infty}\frac{\lambda_n}{\sqrt{k_{12}}}\xi_n^2\coth\left(\frac{\lambda_n}{\sqrt{k_{12}}}(1-r)\right). \tag{38}$$

Now, in order to get the constants m and M in (26) we represent $(S_2\xi,\xi)$ in the form

$$(S_2\xi,\xi)=\sum_{n=1}^{\infty}\frac{\lambda_n}{\sqrt{k_{11}}}\xi_n^2\coth\left(\frac{\lambda_n}{\sqrt{k_{11}}}r\right)\sqrt{\frac{k_{11}}{k_{12}}}q_n, \tag{39}$$

where

$$q_n=\frac{\tanh\left(\frac{\lambda_n}{\sqrt{k_{11}}}r\right)}{\tanh\left(\frac{\lambda_n}{\sqrt{k_{12}}}(1-r)\right)}. \tag{40}$$

Further we need the following

Lemma 3.1. *Let b,k_{11},k_{12} and r be positive numbers, and $r<1,\lambda_n=n\pi/b$. Let q_n be a sequence given by the formula* (40)*. Denote*

$$\gamma_1=\inf_{n\geq 1}q_n,\quad \gamma_2=\sup_{n\geq 1}q_n. \tag{41}$$

Then we have

$$\gamma_1+\gamma_2=1+\frac{\tanh\left(\frac{\pi r}{b\sqrt{k_{11}}}\right)}{\tanh\left(\frac{\pi(1-r)}{b\sqrt{k_{12}}}\right)}. \tag{42}$$

Proof. Consider the function

$$Q(t)=\frac{\tanh(C_1t)}{\tanh(C_2t)}\quad (C_1,C_2,t>0).$$

We have

$$Q^{'}(t)=\frac{\cosh(C_2t-C_1t)}{\sinh^2(C_2t)\cosh^2(C_1t)}.\sinh(C_2t-C_1t).$$

So, if $C_2>C_1$ then $Q^{'}(t)>0$ and the function $Q(t)$ is increasing. Otherwise, if $C_2<C_1$ the function $Q(t)$ is decreasing.
Now, setting

$$C_1=\frac{r}{\sqrt{k_{11}}},\quad C_2=\frac{1-r}{\sqrt{k_{12}}},$$

in view of the function $Q(t)$, we can write $q_n=Q(\lambda_n)$. In the case $C_2>C_1$, or equivalently, $r<\frac{\sqrt{k_{11}}}{\sqrt{k_{11}}+\sqrt{k_{12}}}$, due to the increase of the function $Q(t)$ we have

$$q_1\leq q_n<q_\infty=1.$$

Therefore

$$\gamma_1 = \inf_{n\geq 1} q_n = q_1, \quad \gamma_2 = \sup_{n\geq 1} q_n = 1.$$

In the case $C_2 < C_1$, which is equivalent to $\dfrac{\sqrt{k_{11}}}{\sqrt{k_{11}}+\sqrt{k_{12}}} < r < 1$, due to the descrease of $Q(t)$, we have

$$1 = q_\infty < q_n \leq q_1 \quad \text{and} \quad \gamma_1 = 1, \gamma_2 = q_1.$$

Thus in the both cases we have

$$\gamma_1 + \gamma_2 = 1 + q_1 = 1 + \frac{\tanh\left(\dfrac{\pi r}{b\sqrt{k_{11}}}\right)}{\tanh\left(\dfrac{\pi(1-r)}{b\sqrt{k_{12}}}\right)}.$$

The lemma is proved. ■

Now, let us return to the formula (39). We can get the estimates

$$\frac{\sqrt{k_{11}}}{\sqrt{k_{12}}}\left(\inf_{n\geq 1} q_n\right)(S_1\xi,\xi) \leq (S_2\xi,\xi) \leq \left(\sup_{n\geq 1} q_n\right)(S_1\xi,\xi)\frac{\sqrt{k_{11}}}{\sqrt{k_{12}}}, \tag{43}$$

or in notations (41)

$$\gamma_1\frac{\sqrt{k_{11}}}{\sqrt{k_{12}}}(S_1\xi,\xi) \leq (S_2\xi,\xi) \leq \gamma_2\frac{\sqrt{k_{11}}}{\sqrt{k_{12}}}(S_1\xi,\xi).$$

Hence

$$m \leq \frac{(S_2\xi,\xi)}{(S_1\xi,\xi)} \leq M, \tag{44}$$

where $m = \gamma_1\sqrt{\dfrac{k_{11}}{k_{12}}}, \quad M = \gamma_2\sqrt{\dfrac{k_{11}}{k_{12}}}$.

According to (28) we choose the iterative parameter τ by the formula

$$\tau_{opt} = \frac{2}{2+\sqrt{\dfrac{k_{11}}{k_{12}}}(\gamma_1+\gamma_2)}.$$

Using the formula (42) in the above lemma we have

$$\tau_{opt} = \frac{2}{2+\sqrt{\dfrac{k_{11}}{k_{12}}}\left(1+\dfrac{\tanh\left(\dfrac{\pi r}{b\sqrt{k_{11}}}\right)}{\tanh\left(\dfrac{\pi(1-r)}{b\sqrt{k_{12}}}\right)}\right)}. \tag{45}$$

In the particular case when $k_{11} = k_{12} = 1$ we obtain the known formula in [4].

4. Numerical examples

We performed numerical experiments for some examples for illustrating the convergence of the proposed iterative method in dependence of the choice of iterative parameter τ. In all examples for solving BVPs (6)-(7) we use difference schemes of second order truncation error on grids of 65×65 nodes. For the difference problems we use the method of complete reduction [11]. We also use a formula with second order approximation for calculating the derivative in (8). Below we report some results of experiments.

Example 4.1. Consider a problem in the domain $\Omega = [0,1] \times [0,1]$ with the exact solution

$$u(x_1, x_2) = \begin{cases} \left(x_1^2 + 1\right) e^{x_2} & \text{in} \quad \Omega_1 = [0, r] \times [0,1] \\ \left(x_1^2 + x_1 + 0.5\right) e^{x_2} & \text{in} \quad \Omega_2 = [r, 1] \times [0,1] \end{cases}$$

and suitable Dirichlet boundary condition and right-hand side and the piecewise constant coefficients

$$k_1(x) = \begin{cases} 2, & x \in \Omega_1 \\ 1, & x \in \Omega_2 \end{cases}, \quad k_2(x) = 1 \quad \text{in} \quad \Omega.$$

The exact solution and its conormal derivative are continuous or have jumps across the interface $\Gamma = \{x_1 = r, \quad 0 \le x_2 \le 1\}$ in dependence on the value of r. Namely, for $r = 0.5$ they are continuous, and for other r they have some jumps across Γ.
For stopping iterative process we use the criterion

$$\max\left\{ \left\| u_1^{(k)} - u_1^{(k-1)} \right\|_\infty, \left\| u_2^{(k)} - u_2^{(k-1)} \right\|_\infty \right\} < TOL.$$

Below, we report the result of experiments for $TOL = 10^{-4}$. In all tables k is the number of iterations, $error = \max\left\{ \left\| u_1 - u_1^{(k)} \right\|_\infty, \left\| u_2 - u_2^{(k)} \right\|_\infty \right\}$, the value τ marked by the star is the value computed by the formula (45).

Table 1 Convergence of iterative process for $r = 0.5$

τ	k	error
0.3	15	6.3896e-05
0.4	11	1.9146e-05
0.4297*	10	1.3894e-05
0.5	8	4.1084e-05
0.6	5	2.8750e-06
0.7	7	1.2959e-05
0.8	11	9.9911e-06

The graphs of the found solution are given in corresponding figures

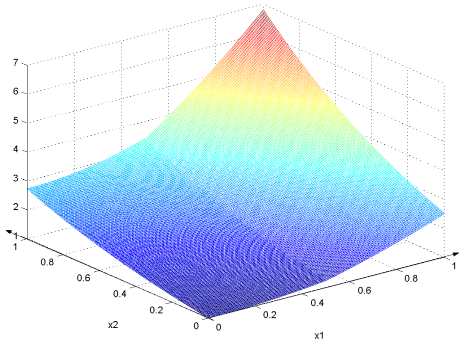

Fig. 2 Graph of the solution in Example 1 for $r = 0.5$.

Table 2 Convergence for $r = 0.3$

τ	k	error
0.3	16	7.7293e-05
0.4	11	7.2464e-05
0.4696*	9	4.7987e-05
0.5	9	1.4580e-05
0.6	7	2.4241e-06
0.7	7	5.0697e-06
0.8	10	1.2143e-05

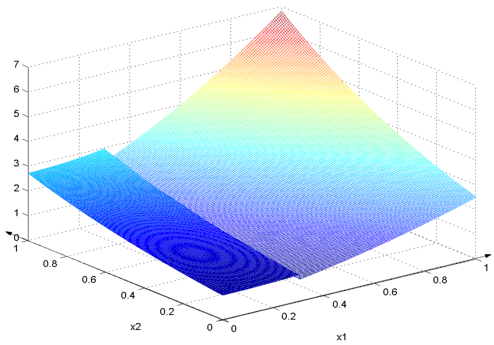

Fig. 3 Graph of the solution in Example 1 for $r = 0.3$.

From the above tables and the results of experiments for other value of r we see that the star value of τ is good but not optimal, meanwhile $\tau = 0.6$

appears optimal. The experiments for other piecewise constant function $k_1(x)$ also support the above observation. This may be explained by the fact that the numbers m and M given by (42) are not best bounds for $(S_2\xi,\xi)\,/\,(S_1\xi,\xi)$ due to the rather raw estimates (43).

Example 4.2. Consider a problem in the L-shape domain shown in Figure 4 with the same exact solution as in Example 1 in the corresponding subdomains Ω_1 and Ω_2.

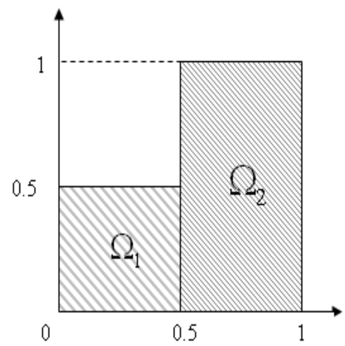

Fig. 4 L-shape domain

Table 3 Example 1 with L-shape domain

τ	k	error
0.3	15	5.4122e-06
0.4	11	2.1876e-06
0.5	7	5.0309e-06
0.6	5	1.5524e-06
0.7	6	1.5843e-06
0.8	9	1.7808e-06
0.9	13	2.9831e-06

The results of experiments show that the iterative method (6)-(7) with the iterative parameter $\tau = 0.6$ give the best convergence and this convergence is the same as in Example 1.

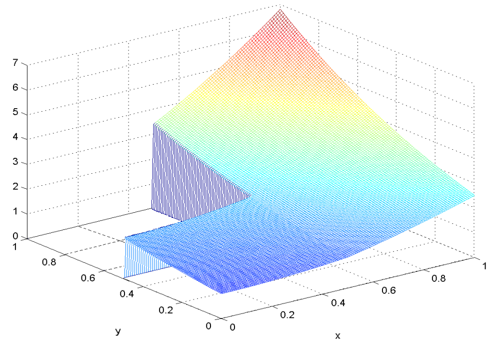

Fig. 5 Graph of the solution in Example 1 for L-shape domain.

5. Concluding remarks

In the paper we proposed an approach to the solution of an interface problem, which is based on the domain decomposition method. It reduces the problem to a sequence of problems in subdomains, separated by the interface. This allows to use many available high accuracy methods to the latter problems. The fast convergence of the sequence of these problems is proved and confirmed on some examples. The approach is especially effective when the computational domain consists of rectangles since for problems in domains of this type there are many efficient software packages for second order problems.

The proposed approach can be applied to interface problems for parabolic equations and higher order equations with discontinuous coefficients.

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