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Uniqueness of Meromorphic Functions Sharing Rational Functions with Their First Derivatives

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Abstract. In this paper, by estimating the growth of ρ_n in a theorem of Pang and Zalcman, we obtain a uniqueness theorem of meromorphic functions which share two distinct rational functions with their first derivatives. It generalizes some previous related results of Rubel and Yang, Li and Yi, Qi, Lü and Chen, and so on.

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1. Introduction and main results

In this paper, for any non-constant meromorphic function f, we adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions such as the characteristic function $T(r, f)$, the proximity function $m(r, f)$ and the counting function $N(r, f)$ (reduced form $\overline{N}(r, f)$) of poles. We refer the reader to [3, 14] and [15] for more details on those notations.

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Let f and g be two non-constant meromorphic functions, and let R be a rational function or a finite complex number. If $g(z)-R(z)=0$ whenever $f(z)-R(z)=0$, we denote the condition by $f(z)=R(z)\Rightarrow g(z)=R(z)$. If both $f(z)=R(z)\Rightarrow g(z)=R(z)$ and $g(z)=R(z)\Rightarrow f(z)=R(z)$, we denote the condition by $f(z)=R(z)\Leftrightarrow g(z)=R(z)$ and say that f and g share R IM (ignoring multiplicity). If $f-R$ and $g-R$ have the same zeros with the same multiplicities, we write $f(z)=R(z)\rightleftharpoons g(z)=R(z)$ and say that f and g share R CM (counting multiplicity).

In 1977, Rubel and Yang [12] considered the uniqueness problem of entire functions which share two distinct values CM with their first derivatives. They obtained the following well-known theorem.

Theorem 1.1. *Let a and b be complex numbers such that $b\neq a$, and let f be a non-constant entire function. If f and f' share values a and b CM, then $f=f'$.*

This result has undergone various extensions and improvements [15]. Especially, in 1979, Mues and Steinmetz [9] weakened the sharing values CM to the sharing values IM and deduced the following theorem.

Theorem 1.2. *Let a and b be complex numbers such that $b\neq a$, and let f be a non-constant entire function. If $f(z)=a\Leftrightarrow f'(z)=a$ and $f(z)=b\Leftrightarrow f'(z)=b$, then $f=f'$.*

In 2006, Li and Yi [4] further improved Theorem 1.2 and derived the following result.

Theorem 1.3. *Let a and b be complex numbers such that $b\neq a,\ 0$, and let f be a non-constant entire function. If $f(z)=a\rightleftharpoons f'(z)=a$ and $f'(z)=b\Rightarrow f(z)=b$, then $f=f'$.*

In 2009, Lü, Xu and Yi [7] generalized the previous theorems and deduced the following result.

Theorem 1.4. *Let a and b be two non-zero distinct complex numbers, and let f be a non-constant entire function. If $f(z)=a\Leftrightarrow f'(z)=a$ and $f'(z)=b\Rightarrow f(z)=b$, then $f=f'$.*

Recently, with the idea of sharing polynomials, Qi, Lü and Chen [11] proved the following result.

Theorem 1.5. *Let $Q_1(z)=a_1z^p+a_{1,p-1}z^{p-1}+\cdots+a_{1,0}$ and $Q_2(z)=a_2z^p+a_{2,p-1}z^{p-1}+\cdots+a_{2,0}$ be two polynomials such that $\deg Q_1(z)=\deg Q_2(z)=p$ (where p is a non-negative integer) and $a_1,\ a_2$ $(a_2\neq 0)$ are two distinct complex numbers. Let f be a transcendental entire function. If $f(z)=Q_1(z)\rightleftharpoons f'(z)=Q_1(z)$ and $f'(z)=Q_2(z)\Rightarrow f(z)=Q_2(z)$, then $f=f'$.*

After studying Theorem 1.5, we propose some questions as follows.

Question 1. Whether the condition that $a_1,\ a_2$ are distinct is necessary or not?

Question 2. Can the polynomials Q_1 and Q_2 be replaced by rational functions?

Question 3. Can f be a meromorphic function in the above theorem?

In the present paper, we investigate the above questions. Actually, with the theory of normal families, we completely solve the Question 1 and partially solve the other two questions.

Theorem 1.6. *Let f be a transcendental meromorphic function with finitely many poles, and let R_1 and R_2 be two distinct rational functions such that*

$$\frac{|R_1(z)|}{|R_2(z)|} \to M \text{ as } |z| \to \infty,$$

where M is a positive constant. If

(1) *f and R_1 have no common poles,*
(2) *$f(z) = R_1(z) \rightleftharpoons f'(z) = R_1(z)$ and*
(3) *$f'(z) = R_2(z) \Rightarrow f(z) = R_2(z)$,*

then $f = f'$.

Remark 1.7. Obviously, Theorem 1.6 is an improvement of Theorem 1.5.

Remark 1.8. The following example shows that the hypothesis that f is transcendental cannot be omitted in Theorem 1.6.

Example 1.9. Consider the function $f(z) = z^2$. Let $R_1(z) = 4z - z^2$ and $R_2(z) = 2z + z^2$. It is not difficult to see that

$$\frac{f' - R_1}{f - R_1} = \frac{1}{2} \text{ and } f'(z) = R_2(z) \Rightarrow f(z) = R_2(z).$$

But $f \neq f'$.

Remark 1.10. In our proof of Theorem 1.6, we need the condition that $\frac{|R_1(z)|}{|R_2(z)|} \to M$ as $|z| \to \infty$. But, we don't know whether the condition is necessary or not. Unfortunately, we neither give a negative example nor prove that Theorem 1.6 is still valid without the condition.

Remark 1.11. The proof of Theorem 1.6 is similar to that of Theorem 1.5. It is crucial to deduce that f is of finite order. With the theory of normal families and a tiny improvement of a Pang and Zalcman's lemma, we obtain the conclusion. In fact, we deduce the following result which is of independent interest.

Theorem 1.12. *Let f be a transcendental meromorphic function with finitely many poles, and let R_1 and R_2 be two distinct rational functions such that*

$$\frac{|R_1(z)|}{|R_2(z)|} \to M \text{ as } |z| \to \infty,$$

where M is a positive constant. If

(1) $f(z) = R_1(z) \Rightarrow f'(z) = R_1(z)$ *and*

(2) $f'(z) = R_2(z) \Rightarrow f(z) = R_2(z)$,

then f is of finite order.

Remark 1.13. Obviously, Theorem 1.12 is an improvement of several results in some sense, such as [4, Theorem 3] and [11, Theorem 1.6].

For the proof of our results, we need to introduce the theory of normal families. Let $\mathcal{F}$ be a family of meromorphic functions on a domain D. We say that $\mathcal{F}$ is normal in D if every sequence of functions $\{f_n\} \subset \mathcal{F}$ contains either a subsequence which converges to a meromorphic function f uniformly on each compact subset of D or a subsequence which converges to ∞ uniformly on each compact subset of D. There is a famous criterion, called Marty's criterion. It says that $\mathcal{F}$ is normal if and only if for each $f \in \mathcal{F}$, $f^\sharp(\xi)$ is locally bounded in D. Here

$$f^\sharp(\xi) = \frac{|f'(\xi)|}{1 + |f(\xi)|^2}$$

is the spherical derivative of f (see [1, 13]).

2. Some lemmas

In order to prove our theorems, we need several lemmas. For the convenience of the reader, we recall these lemmas here.

About the famous lemma of Pang and Zalcman [10, Lemma 2], we make a slight modification. By estimating the growth of ρ_n, we deduce the following lemma, which plays an important role in the proof of Theorem 1.12 below. And the idea are due to Liu, Nevo and Pang [5].

Lemma 2.1. *Let $\{f_n\}$ be a family of functions meromorphic (or holomorphic) on the unit disc $\triangle$, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f \in \mathcal{F}$ and $f(z) = 0$. If $a_n \to a$, $|a| < 1$, and $f_n^\sharp(a_n) \to \infty$, then there exist, for each $0 \leq \alpha \leq k$,*

(a) *a subsequence of $\{f_n\}$ (which we still write as $\{f_n\}$);*

(b) *points z_n $|z_n| < 1$;*

(c) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a non-constant meromorphic (or entire) function on $\mathbb{C}$ with order at most 2 (or 1), and all the zeros of g have multiplicity at least k. Meanwhile, $g^\sharp(\xi) \leq g^\sharp(0) = kA + 1$ and

$$\rho_n \leq \frac{M}{\sqrt[1+\alpha]{f_n^\sharp(a_n)}},$$

where M is a positive constant which is independent of n.

Proof. The proof of this lemma is similar to that of Lemma 2 in [10]. Thus, we just give the main points. Comparing it to Lemma 2 of [10], we just need to estimate the size of the ρ_n's. Noting that the assumption $f_n^\sharp(a_n) \to \infty$, we can replace z_n^* by a_n in Lemma 2 of [10]. Then, in Lemma 2 of [10], it follows from (9) and the first inequality on p.329 that

$$kA + 1 \geq t_n^{1+\alpha} \frac{(1 - \frac{|a_n|^2}{r^2})^{1+\alpha} |f_n'(a_n)|}{(1 - \frac{|a_n|^2}{r^2})^{2\alpha} + |f_n(a_n)|^2} \geq t_n^{1+\alpha} (1 - \frac{|a_n|^2}{r^2})^{1+\alpha} f_n^\sharp(a_n). \tag{1}$$

On the other hand, in Lemma 2 of [10], we have

$$\rho_n = (1 - \frac{|z_n|^2}{r^2}) t_n = \frac{1 - \frac{|z_n|^2}{r^2}}{1 - \frac{|a_n|^2}{r^2}} (1 - \frac{|a_n|^2}{r^2}) t_n. \tag{2}$$

In view of (1), we deduce that

$$\rho_n \leq \frac{1 - \frac{|z_n|^2}{r^2}}{1 - \frac{|a_n|^2}{r^2}} \frac{\sqrt[1+\alpha]{kA + 1}}{\sqrt[1+\alpha]{f_n^\sharp(a_n)}} \leq \frac{M}{\sqrt[1+\alpha]{f_n^\sharp(a_n)}},$$

where M is a positive number which is independent of n. Thus, we finish the proof of this lemma. ■

Next, we need to introduce a result of Liu, Nevo and Pang [5].

Lemma 2.2. *Let f be a meromorphic function of infinite order on $\mathbb{C}$. Then there exist points $z_n \to \infty$, such that for every $N > 0$, $f^\sharp(z_n) > |z_n|^N$ if n is sufficiently large.*

Finally, we recall a result of Lü and Yi [8, Lemma 2.6], which is crucial to the proof of Theorem 1.6.

Lemma 2.3. *Let R and H be two non-zero rational functions, let Q be a polynomial, and let F be a transcendental meromorphic function with finite order. If F is a solution of the following differential equation*

$$F' - Re^Q F = H, \tag{3}$$

then Q is a constant. In particular, if $R = \frac{1}{P}$, where P is a polynomial, then R is also a constant.

3. Proof of Theorem 1.12

Obviously, $R_1 \neq 0$. Let

$$F = \frac{f}{R_1} - 1.$$

In the following, we proceed our proof into two cases.

Case 1. F is of finite order.

Then, $f = (F+1)R_1$ is of finite order as well.

Case 2. F is of infinite order.

It follows from Lemma 2.2 that there exists a sequence $\{w_n\}$ such that $w_n \to \infty$ and for every $N > 0$ (for n sufficiently large)

$$F^{\sharp}(w_n) > |w_n|^N. \tag{4}$$

We first construct a family of holomorphic functions. Since R_1 is a rational function, then there exists a $r_1 > 0$ such that for $|z| \geq r_1$,

$$R_1(z) \neq 0 \ \text{ and } |\frac{R_1'(z)}{R_1(z)}| \leq \frac{B}{|z|} \leq 1, \tag{5}$$

where B is a positive constant. Noting that f has finitely many poles, so F has finitely many poles. Thus, there is a $r_2 > 0$ such that F is analytic in $D_0 = \{z : |z| \geq r_2\}$.

Let $r = \max\{r_1,\ r_2\}$ and $D = \{z : |z| \geq r\}$. Then F is analytic in D. In view of $w_n \to \infty$, without loss of generality, we may assume $|w_n| \geq r+1$ for all n. Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{f(w_n+z)}{R_1(w_n+z)} - 1.$$

Then all $F_n(z)$ are analytic in D_1. Thus, we obtain a family of holomorphic functions $(F_n)_n$. Moreover,

$$F_n^{\sharp}(0) = F^{\sharp}(w_n) \to \infty \text{ as } n \to \infty.$$

It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Now, let $z \in D_1$ be fixed. If $F_n(z) = 0$, then $f(w_n+z) = R_1(w_n+z)$. It is clear from the assumption (1) that $f'(w_n+z) = R_1(w_n+z)$. Thus, with (5), we deduce that (for n large enough)

$$\begin{aligned} |F_n'(z)| &= \left| \frac{f'(w_n+z)}{R_1(w_n+z)} - \frac{f(w_n+z)}{R_1(w_n+z)} \frac{R_1'(w_n+z)}{R_1(w_n+z)} \right| \\ &\leq \left| \frac{f'(w_n+z)}{R_1(w_n+z)} \right| + \left| \frac{f(w_n+z)}{R_1(w_n+z)} \frac{R_1'(w_n+z)}{R_1(w_n+z)} \right| \leq 2. \end{aligned} \tag{6}$$

Therefore, applying Lemma 2.1 with $\alpha = k = 1$ and choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n \in D_1$ and $(\rho_n)_n$ such that $\rho_n \to 0$ and

$$g_n(\zeta) = \rho_n^{-1} F_n(z_n + \rho_n\zeta) = \rho_n^{-1}[\frac{f(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} - 1] \to g(\zeta) \tag{7}$$

locally uniformly in $\mathbb{C}$, where g is a non-constant entire function of order at most 1, $g^\sharp(\zeta) \le g^\sharp(0) = 3$ for all $\zeta \in \mathbb{C}$ and

$$\rho_n \le \frac{M_1}{\sqrt{F_n^\sharp(0)}} = \frac{M_1}{\sqrt{F^\sharp(w_n)}} \tag{8}$$

for a positive number M_1. With (4), for every $N > 0$, if n is sufficiently large, we have

$$\rho_n \le \frac{M_1}{\sqrt{F_n^\sharp(0)}} = \frac{M_1}{\sqrt{F^\sharp(w_n)}} \le |w_n|^{-\frac{N}{2}}. \tag{9}$$

Differentiating (7) yields

$$\begin{aligned} g_n'(\zeta) &= \frac{f'(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} - \frac{f(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} \frac{R_1'(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} \\ &= \frac{f'(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} - [\rho_n g_n(\zeta) + 1] \frac{R_1'(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} \to g'(\zeta) \end{aligned} \tag{10}$$

locally uniformly in $\mathbb{C}$. In view of (10) and $\frac{R_1'(w_n+z_n+\rho_n\zeta)}{R_1(w_n+z_n+\rho_n\zeta)} \to 0$ as $n \to \infty$, we deduce that

$$\frac{f'(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} \to g'(\zeta) \tag{11}$$

locally uniformly in $\mathbb{C}$.

Noting that $\frac{|R_1(z)|}{|R_2(z)|} \to M$ as $|z| \to \infty$, where M is a positive number, we can assume that

$$\frac{R_1(z)}{R_2(z)} \to \frac{1}{a} \text{ as } z \to \infty, \tag{12}$$

where $\frac{1}{a}$ is a constant with $|\frac{1}{a}| = M$. We claim

(I) $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$,

(II) $g'(\zeta) \ne a$.

First we prove (I). Suppose that $g(\zeta_0) = 0$. By Hurwitz's theorem and (7), there exists a sequence $\{\zeta_n\}$ such that $\zeta_n \to \zeta_0$ and (for n sufficiently large)

$$\frac{f(w_n + z_n + \rho_n\zeta_n)}{R_1(w_n + z_n + \rho_n\zeta_n)} = 1.$$

From the assumption (1), we have

$$f'(w_n + z_n + \rho_n\zeta_n) = R_1(w_n + z_n + \rho_n\zeta_n). \tag{13}$$

Combining (11) and (13) yields

$$g'(\zeta_0) = \lim_{n\to\infty} \frac{f'(w_n + z_n + \rho_n\zeta_n)}{R_1(w_n + z_n + \rho_n\zeta_n)} = 1,$$

which implies (I). Next, we prove $g'(\zeta) \neq a$.

Obviously, $g' \not\equiv a$. Otherwise, we assume that $g' = a$. Then it is clear from (I) that $a = 1$, while

$$3 = g^{\sharp}(0) \leq g'(0) = 1 < 3,$$

a contradiction. Thus, we get $g' \not\equiv a$.

On the other hand, it follows from (11) and (12) that

$$\frac{f'(w_n + z_n + \rho_n\zeta) - R_2(w_n + z_n + \rho_n\zeta)}{R_1(w_n + z_n + \rho_n\zeta)} \to g'(\zeta) - a. \tag{14}$$

Now, we return to the proof of (II). Suppose $g'(\xi_0) = a$. Noting that $g' \not\equiv a$, by Hurwitz's theorem and (14), there exists a sequence $\{\xi_n\}$ such that $\xi_n \to \xi_0$ and (for n sufficiently large)

$$f'(w_n + z_n + \rho_n\xi_n) = R_2(w_n + z_n + \rho_n\xi_n).$$

By the condition (2) of Theorem 1.12, we have

$$f(w_n + z_n + \rho_n\xi_n) = R_2(w_n + z_n + \rho_n\xi_n).$$

Suppose that

$$\frac{R_2' - R_1}{R_1} = \frac{Q_1}{Q_2},$$

where Q_1 and Q_2 are two polynomials with no common factors, and suppose $\deg(Q_2) = p$. Noting that $\frac{|R_1(z)|}{|R_2(z)|} \to M$ as $|z| \to \infty$, we know $R_2' \not\equiv R_1$. That is $Q_1 \not\equiv 0$. Then, combining (9) and (14) yields

$$\begin{aligned}
|g(\xi_0)| &= \lim_{n\to\infty} \rho_n^{-1} \left| \frac{f'(w_n + z_n + \rho_n\xi_n)}{R_1(w_n + z_n + \rho_n\xi_n)} - 1 \right| \\
&= \lim_{n\to\infty} \rho_n^{-1} \left| \frac{R_2'(w_n + z_n + \rho_n\xi_n) - R_1(w_n + z_n + \rho_n\xi_n)}{R_1(w_n + z_n + \rho_n\xi_n)} \right| \\
&\geq \lim_{n\to\infty} |w_n|^{p+1} \left| \frac{R_2'(w_n + z_n + \rho_n\xi_n) - R_1(w_n + z_n + \rho_n\xi_n)}{R_1(w_n + z_n + \rho_n\xi_n)} \right| \\
&= \lim_{n\to\infty} |w_n|^{p+1} \left| \frac{Q_1(w_n + z_n + \rho_n\xi_n)}{Q_2(w_n + z_n + \rho_n\xi_n)} \right| = \infty,
\end{aligned} \tag{15}$$

which is a contradiction. Thus, we finish the proof of (II).

Observing that g is of order at most 1 and (II), we can set

$$g'(z) = a + A\lambda e^{\lambda z}, \tag{16}$$

where $A,\ \lambda$ are two non-zero constants. Integrating the above function yields

$$g(z) = az + c + Ae^{\lambda z}, \tag{17}$$

where c is a constant. By (I), (16) and (17), it is not difficult to deduce a contradiction, which implies that the case 2 cannot occur.

Hence, we complete the proof of Theorem 1.12. ■

4. Proof of Theorem 1.6

It follows from Theorem 1.12 that f is of finite order. Define

$$\mu = \frac{f' - R_1}{f - R_1}. \tag{18}$$

The fact that f is transcendental implies $\mu \neq 0$.

From the conditions (1) and (2) in Theorem 1.6, we obtain μ has no zeros. The assumption that f has finitely many poles implies that μ has finitely many poles. Moreover, from (18), we have $\rho(\mu) \leq \rho(f)$, which implies that μ is of finite order. Therefore, we set

$$\mu = \frac{1}{P_1} e^Q, \tag{19}$$

where $P_1 \neq 0$ and Q are two polynomials.

Let $F = f - R_1$ and $\phi = R_1 - R_1'$. Then ϕ is a non-zero rational function. Rewrite (18) as

$$F' - \phi = \frac{1}{P_1} e^Q F.$$

It follows from Lemma 2.3 that Q and P_1 are two constants. Thus μ is a non-zero constant, say $\mu = \lambda$. We rewrite (18) as

$$f' = \lambda f + (1 - \lambda) R_1. \tag{20}$$

Combining condition (1) and the above equation yields that f is an entire function and R_1 reduces to a polynomial. Then, we can proceed our proof as that of Theorem E in [11]. For the convenience of the reader, we present our proof in detail.

If $\lambda = 1$, then $f = f'$. We now assume $\lambda \neq 1$. By solving the differential equation (20), we have

$$f(z) = Ae^{\lambda z} + P(z), \tag{21}$$

where A is a non-zero constant and P is a polynomial. Differentiating (21) yields

$$f'(z) = A\lambda e^{\lambda z} + P'(z). \tag{22}$$

By substituting the forms of f and f' into (20), we deduce that

$$(1 - \lambda) R_1 + \lambda P - P' = 0. \tag{23}$$

In the following, we prove $P' = R_2$. On the contrary, suppose $P' \neq R_2$.
Since $P' - R_2$ is a rational function, we obtain that

$$\overline{N}(r, \frac{1}{f' - R_2}) = \overline{N}(r, \frac{1}{A\lambda e^{\lambda z} + P' - R_2}) = T(r, e^{\lambda z}) + S(r, f) \neq S(r, f). \quad (24)$$

Let z_0 be a zero of $f' - R_2$. The assumption (3) implies that $f(z_0) = R_2(z_0)$. Plugging z_0 into (21) and (22), we have

$$(1 - \lambda)R_2(z_0) + \lambda P(z_0) - P'(z_0) = 0.$$

Suppose that $(1 - \lambda)R_2 + \lambda P - P' \neq 0$. Noting that $(1 - \lambda)R_2 + \lambda P - P'$ is a rational function, we deduce

$$\begin{aligned}\overline{N}(r, \frac{1}{f' - R_2}) &\leq \overline{N}(r, \frac{1}{(1 - \lambda)R_2 + \lambda P - P'}) \\ &\leq T(r, (1 - \lambda)R_2 + \lambda P - P') = S(r, f),\end{aligned}$$

which contradicts (24). Hence,

$$(1 - \lambda)R_2 + \lambda P - P' = 0.$$

Comparing it to (23), we have $R_1 = R_2$, a contradiction.
Thus we prove that $P' = R_2$ and R_2 is also a polynomial. It is easy to deduce that $\deg R_2 = \deg P'$ and $\deg R_1 = \deg P$. Thus

$$\frac{|R_1(z)|}{|R_2(z)|} \to \infty, \text{ as } |z| \to \infty,$$

which contradicts the assumption. So the case $\lambda \neq 1$ cannot occur.

Hence, we complete the proof of Theorem 1.6. ■

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