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Remarks on Rings with Zero Products Commuting and Zero Insertion Properties*

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Abstract. In this paper, we continue to study the rings satisfying zero insertion property for n (abbr., ZI_n) and zero products commuting property for n (abbr., ZC_n). In particular, we prove that R satisfies ZI_n(resp., ZC_n) if I is a reduced ideal of R and R/I satisfies ZI_n(resp., ZC_n), where n is a positive integer. Furthermore, we show that $[[R^{S,\leqslant}]]$ satisfies ZI_n(resp., ZC_n) if and only if R satisfies ZI_n(resp., ZC_n) under some additional conditions.

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1. Introduction

Throughout this paper, R denotes an associative ring not necessarily with identity and n is a positive integer. A ring R is called *reduced* if it has no non-zero nilpotent elements. It is well known that if R is a reduced ring, then

$$ab = 0 \Rightarrow ba = 0 \quad \forall a, b \in R.$$

Cohn [2] called a ring R *reversible* if this condition holds, while Habeb [4] used the name *zero commutative* for this notion. Anderson and Camillo [1] extended

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the class of zero commutative rings to the class of rings satisfying zero products commuting property for $n \geq 2$. A ring R is said to *satisfy the zero products commuting property (abbr., ZC_n)* for $n \geq 2$ if for $a_1, a_2, ..., a_n \in R$, $a_1a_2...a_n = 0$ implies $a_{\sigma(1)}a_{\sigma(2)}...a_{\sigma(n)} = 0$ for each $\sigma \in S_n$, where S_n denotes the permutation group on n letters. Every reduced ring satisfies ZC_n for all $n \geq 2$ [1, Theorem I.3], but the converse is not true [1, Example II.5]. A ring R is called *symmetric* if $abc = 0$ implies $acb = 0$ for any $a, b, c \in R$. Observe that a reversible ring coincides with the ring satisfying ZC_2. Examples 1(1) of [5] showed that a symmetric ring may not be reversible, and a symmetric ring may not satisfy ZC_3, Example 1(2) of [5] showed that the ring satisfying ZC_n for all $n \geq 4$ may not be symmetric. However, it is noted that if R is a ring with identity, then R is symmetric if and only if R satisfies ZC_3 if and only if R satisfies ZC_n for all $n \geq 3$.

A ring R is called *semicommutative* if for each $a \in R$, $r(a) = \{b \in R | ab = 0\}$ is an ideal of R. Lemma 1.2 of Shin [10] showed that R is semicommutative if and only if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Habeb [4] used the name zero insertion for this notion. Hong et al. [5] extended the class of zero insertion ring to the class of ring satisfying zero insertion property for $n \geq 2$. A ring R is said to satisfy the *zero insertion property* for n (abbr., ZI_n) if for $a_1, a_2, ..., a_n \in R$, $a_1a_2...a_n = 0$ implies $a_1Ra_2R...Ra_n = 0$. Observe that a semicommutative ring coincides with the ring satisfying ZI_2. Moreover, if R is a ring with identity, then R is semicommutative if and only if R satisfies ZI_n for some $n \geq 2$ if and only if R satisfies ZI_n for all $n \geq 2$. Every ring satisfying ZC_n satisfies ZI_n for all $n \geq 2$ by [5, Corollary 7]. In this paper, we continue to study the rings satisfying ZI_n(resp., ZC_n) motivated by [5, 8].

2. On rings satisfying ZI_n(resp., ZC_n)

Lemma 2.1. [5, Corollary 7(1)] *If a ring R satisfies ZC_n, then R satisfies ZI_n for $n \geq 2$.*

Lemma 2.2. [1, Theorem I.1] *Let S be a semigroup with 0 and let $n \geq 3$. If S satisfies ZC_n, then S satisfies ZC_{n+1}.*

Lemma 2.3. [8, Proposition 3.2] *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R be an S-Armendariz ring. If $f_1, ..., f_n \in [[R^{S,\leqslant}]]$ are such that $f_1f_2...f_n = 0$, then $f_1(u_1)f_2(u_2)...f_n(u_n) = 0$ for all $u_1, u_2, ..., u_n \in S$.*

By [5, Example 13(1)], a ring R does not satisfy ZI_n (resp., ZC_n) for $n \geq 2$, even for nonzero ideal I, R/I and I satisfy ZI_n (resp., ZC_n). However, for reduced ideals I we obtain the following results.

Theorem 2.4. *Let R be a ring, I an ideal of R and $n \geq 2$. If R/I satisfies ZI_n and I is reduced, then R satisfies ZI_n.*

Proof. Let $a_1a_2...a_n = 0$ with $a_1, a_2, ..., a_n \in R$. Then we have $a_1Ra_2R...Ra_n \subseteq I$ and $a_nIa_1a_2...a_{n-1} = 0$ since $a_nIa_1a_2...a_{n-1} \subseteq I$, $(a_nIa_1a_2...a_{n-1})^2 = 0$ and I is reduced. Hence

$$(a_{n-1}Ra_nIa_1a_2...a_{n-2})^2 = a_{n-1}R(a_nIa_1a_2...a_{n-2}a_{n-1})Ra_nIa_1a_2...a_{n-2} = 0$$

and so $a_{n-1}Ra_nIa_1a_2...a_{n-2} = 0$. Further, we get

$$\begin{aligned} &(a_{n-2}Ra_{n-1}Ra_nIa_1...a_{n-3})^2 \\ = &\, a_{n-2}R(a_{n-1}Ra_nIa_1...a_{n-3}a_{n-2})Ra_{n-1}Ra_nIa_1...a_{n-3} = 0, \end{aligned}$$

which implies $a_{n-2}Ra_{n-1}Ra_nIa_1a_2...a_{n-3} = 0$ since $a_{n-2}Ra_{n-1}Ra_nIa_1a_2...a_{n-3} \subseteq I$ and I is reduced. Continuing as above, we get $a_1Ra_2R...Ra_nI = 0$. Hence

$$(a_1Ra_2R...Ra_n)^2 \subseteq a_1Ra_2R...Ra_nI = 0,$$

which implies $(a_1Ra_2R...Ra_n)^2 = 0$, and so $a_1Ra_2R...Ra_n = 0$. Thus R satisfies ZI_n. ■

Theorem 2.5. *Let R be a ring, I an ideal of R and $n \geq 2$. If R/I satisfies ZC_n and I is reduced, then R satisfies ZC_n.*

Proof. In fact, it suffices to show that we can interchange two elements $a_i, a_j (i < j)$ and still get a zero product. Let $a_1a_2...a_i...a_j...a_n = 0$ where $1 \leq i < j \leq n$, we need $a_1...a_{i-1}a_ja_{i+1}...a_{j-1}a_ia_{j+1}...a_n = 0$. Since R/I satisfies ZC_n, we have that $a_1...a_{i-1}a_ja_{i+1}...a_{j-1}a_ia_{j+1}...a_n \in I$ and R/I satisfies ZI_n by Lemma 2.1. Further, we have that R satisfies ZI_n by Theorem 2.4. Hence $a_1...a_i...a_j...a_n = 0$, which implies $a_1R...Ra_iR...Ra_jR...Ra_n = 0$. That is,

$$a_1u_1...a_iu_i...a_ju_j...u_{n-1}a_n = 0 \quad \forall\ u_1, u_2, ..., u_{n-1} \in R. \tag{1}$$

We have the following two cases for i, j.

Case 1. $j = i + 1$. For $i = 1$, multiply equation (1) on the left hand side by a_2, and set

$$\begin{aligned} u_1 &= a_3a_4...a_n, \\ u_2 &= a_1, \\ u_3 &= a_4a_5...a_na_2a_1a_3, \\ u_4 &= a_5a_6...a_na_2a_1a_3a_4, \\ &\vdots \\ u_{n-1} &= a_na_2a_1a_3a_4...a_{n-1}, \end{aligned}$$

we have $(a_2a_1a_3a_4...a_n)^{n-1} = 0$, which implies $a_2a_1a_3a_4...a_n = 0$. For $i = n-1$, multiply equation (1) on the right hand side by a_{n-1}, and set

$$u_1 = a_2a_3...a_{n-2}a_na_{n-1}a_1,$$

$$\begin{aligned}
u_2 &= a_3a_4...a_{n-2}a_na_{n-1}a_1a_2,\\
&\vdots\\
u_{n-3} &= a_{n-2}a_na_{n-1}a_1...a_{n-3},\\
u_{n-2} &= a_n,\\
u_{n-1} &= a_1a_2...a_{n-2},
\end{aligned}$$

we have $(a_1a_2...a_{n-2}a_na_{n-1})^{n-1} = 0$, which implies $a_1a_2...a_{n-2}a_na_{n-1} = 0$. For $i \neq 1$ and $i \neq n-1$, set

$$\begin{aligned}
u_1 &= a_2a_3...a_{i-1}a_{i+1}a_ia_{i+2}...a_na_1,\\
u_2 &= a_3a_4...a_{i-1}a_{i+1}a_ia_{i+2}...a_na_1a_2,\\
&\vdots\\
u_{i-2} &= a_{i-1}a_{i+1}a_ia_{i+2}...a_na_1...a_{i-2},\\
u_{i-1} &= a_{i+1},\\
u_i &= a_{i+2}...a_na_1...a_{i-1},\\
u_{i+1} &= a_i,\\
u_{i+2} &= a_{i+3}...a_na_1a_2...a_{i-1}a_{i+1}a_ia_{i+2},\\
u_{i+3} &= a_{i+4}...a_na_1a_2...a_{i-1}a_{i+1}a_ia_{i+2}a_{i+3},\\
&\vdots\\
u_{n-1} &= a_na_1a_2...a_{i-1}a_{i+1}a_ia_{i+2}...a_{n-1},
\end{aligned}$$

we have $(a_1a_2...a_{i-1}a_{i+1}a_ia_{i+2}...a_n)^{n-2} = 0$, which implies $a_1a_2...a_{i-1}a_{i+1}a_i a_{i+2}...a_n = 0$.

Case 2. $j \neq i+1$. By using the case 1 again and again, the case is true.

Therefore we have $a_1a_2...a_j...a_i...a_n = 0$ for any $1 \leqslant i < j \leqslant n$. ■

Remark 2.6. [6, Theorem 6] and [7, Proposition 1.12] are obtained from Theorem 2.4 and Theorem 2.5, respectively, when setting $n = 2$.

An example was given in [5] to show that the conditions ZI_n and ZC_n for $n \geq 2$ are not preserved by homomorphic images. But it was shown in [7, Proposition1.14(1)] that for a symmetric ring R if I is an ideal of R that is an annihilator in R then R/I is a reversible ring. More generally we have the following Proposition 2.7 and Proposition 2.8.

Proposition 2.7. *Let R be a ring and I be an ideal of R that is an annihilator in R. If R satisfies ZI_n for $n \geq 2$, then R/I satisfies ZI_n.*

Proof. Set $I = r_R(J)$ with $J \subseteq R$ and $\bar{r} = r + I$ for $r \in R$. Let $\bar{a_1}\bar{a_2}\ldots\bar{a_n} = 0$ then $a_1a_2\ldots a_n \in I$ and so $Ja_1a_2\ldots a_n = 0$. Since R satisfies ZI_n, we have that $Ja_1Ra_2R\ldots Ra_n = 0$ by [5, Proposition 4(2)], which implies $\bar{a_1}(R/I)\bar{a_2}(R/I)\ldots (R/I)\bar{a_n} = 0$. Thus R/I satisfies ZI_n. ■

Proposition 2.8. *Let R be a ring and I be an ideal of R that is an annihilator in R. If R satisfies ZC_n for $n \geq 3$, then R/I satisfies ZC_n.*

Proof. Set $I = r_R(J)$ with $J \subseteq R$ and $\bar{r} = r + I$ for $r \in R$. Let $\bar{a_1}\bar{a_2}\dots\bar{a_n} = 0$ then $a_1 a_2 \dots a_n \in I$ and so $Ja_1a_2\dots a_n = 0$. Since R satisfies ZC_n, we have that R satisfies ZC_{n+1} by Lemma 2.2. Hence $Ja_1a_2\dots a_n = 0$ implies $Ja_{\sigma(1)}a_{\sigma(2)}\dots a_{\sigma(n)} = 0$ by [5, Proposition (2)] and so $\bar{a}_{\sigma(1)}\bar{a}_{\sigma(2)}\dots\bar{a}_{\sigma(n)} = 0$. Therefore R/I satisfies ZC_n. ■

Let $(S, \leqslant)$ be an ordered set. Recall that $(S, \leqslant)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S, \leqslant)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to [3].

Let $(S, \leqslant)$ be a strictly ordered monoid (that is, $(S, \leqslant)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R be a ring. Let $[[R^{S,\leqslant}]]$ be the set of all maps $f : S \to R$ such that $supp(f) = \{s \in S | f(s) \neq 0\}$ is artinian and narrow with pointwise addition, and the operation of convolution

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v),$$

where $X_s(f, g) = \{(u, v) \in S \times S | s = u + v, f(u) \neq 0, g(v) \neq 0\}$ is a finite set by Ribenboin [9] for every $s \in S$ and $f, g \in [[R^{S,\leqslant}]]$, $[[R^{S,\leqslant}]]$ becomes a ring, which is called the *generalized power series ring* with coefficients in R and exponents in S.

A monoid S is called *torsion-free* if the following property holds: if $s, t \in S$ and $k \geq 1$ are such that $ks = kt$, then $s = t$. An element $t \in S$ is *cancellative* whenever $t + s = t + s'$ (with $s, s' \in S$) implies $s = s'$. A monoid S is called *cancellative* if every element is cancellative. A ring R is called *S-Armendariz* if whenever $f, g \in [[R^{S,\leqslant}]]$ satisfy $fg = 0$, then $f(u)g(v) = 0$ for each $u, v \in S$.

Similar to [8, Proposition 2.7], we get the following result.

Proposition 2.9. *Let $(S, \leq)$ be a strictly ordered monoid and R be an S-Armendariz ring. Then $[[R^{S,\leqslant}]]$ is reversible if and only if R is reversible.*

Proof. Suppose that R is reversible and $f, g \in [[R^{S,\leqslant}]]$ are such that $fg = 0$. Since R is S-Armendariz and reversible, we have that $g(u)f(v) = 0$ for any $u \in supp(g)$ and $v \in supp(f)$. Now for any $s \in S$,

$$(gf)(s) = \sum_{(u,v) \in X_s(g,f)} g(u)f(v) = 0.$$

Thus $gf = 0$. This shows that $[[R^{S,\leqslant}]]$ is reversible.

If $[[R^{S,\leqslant}]]$ is reversible, then R is reversible since subrings of reversible rings are also reversible. ■

Based on Proposition 2.9 and Lemma 2.3, we have a more general result as follows.

Theorem 2.10. *Let S be a torsion-free and cancellative monoid, $\leq$ a strict order on S and R be an S-Armendariz ring. Then*

(1) $[[R^{S,\leqslant}]]$ *satisfies* ZI_n *if and only if* R *satisfies* ZI_n *for* $n \geq 3$;

(2) $[[R^{S,\leqslant}]]$ *satisfies* ZC_n *if and only if* R *satisfies* ZC_n *for* $n \geq 3$.

In particular, $[[R^{S,\leqslant}]]$ *is symmetric if and only if* R *is symmetric.*

Proof. (1) Suppose that R satisfies ZI_n and $f_1, f_2, ..., f_n \in [[R^{S,\leqslant}]]$ are such that $f_1 f_2 ... f_n = 0$. By Lemma 2.3, we have that $f_1(u_1)f_2(u_2)...f_n(u_n) = 0$ for any $u_i \in supp(f_i)(i = 1, 2, ..., n)$, which implies $f_1(u_1)Rf_2(u_2)R...Rf_n(u_n) = 0$ since R satisfies ZI_n. Now for any $g_1, g_2, ..., g_{n-1} \in [[R^{S,\leqslant}]]$ and any $s \in S$,

$$
\begin{aligned}
&(f_1 g_1 f_2 g_2 ... g_{n-1} f_n)(s) \\
&= \sum_{(u_1, v_1, u_2, v_2, ..., v_{n-1}, u_n)} f_1(u_1)g_1(v_1)f_2(u_2)g_2(v_2)...g_{n-1}(v_{n-1})f_n(u_n) \\
&= 0,
\end{aligned}
$$

where $(u_1, v_1, u_2, v_2, ..., v_{n-1}, u_n) \in X_s(f_1, g_1, f_2, g_2, ..., g_{n-1}, f_n)$. Thus $f_1 g_1 f_2 g_2 ... g_{n-1} f_n = 0$. This shows that $f_1[[R^{S,\leqslant}]]f_2[[R^{S,\leqslant}]]...[[R^{S,\leqslant}]]f_n = 0$. Thus $[[R^{S,\leqslant}]]$ satisfies ZI_n.

If $[[R^{S,\leqslant}]]$ satisfies ZI_n, then R satisfies ZI_n since subrings of rings satisfying ZI_n also satisfy ZI_n.

(2) Suppose that R satisfies ZC_n and $f_1, f_2, ..., f_n \in [[R^{S,\leqslant}]]$ are such that $f_1 f_2 ... f_n = 0$, we next show that $f_{\sigma(1)} f_{\sigma(2)} ... f_{\sigma(n)} = 0$ for any $\sigma \in S_n$. By Lemma 2.3, we have that $f_1(u_1)f_2(u_2)...f_n(u_n) = 0$ for any $u_i \in supp(f_i)(i = 1, 2, ..., n)$, which implies

$$f_{\sigma(1)}(v_1)f_{\sigma(2)}(v_2)...f_{\sigma(n)}(v_n) = 0,$$

for any $v_i \in supp(f_{\sigma(i)})$ and any $\sigma \in S_n$ since R satisfies ZC_n. Now for any $s \in S$,

$$(f_{\sigma(1)} f_{\sigma(2)} ... f_{\sigma(n)})(s) = \sum_{(v_1, v_2, ..., v_n)} f_{\sigma(1)}(v_1)f_{\sigma(2)}(v_2)...f_{\sigma(n)}(v_n) = 0,$$

where $(v_1, v_2, ..., v_n) \in X_s(f_{\sigma(1)}, f_{\sigma(2)}, ... f_{\sigma(n)})$. Thus $f_{\sigma(1)} f_{\sigma(2)} ... f_{\sigma(n)} = 0$. This shows that $[[R^{S,\leqslant}]]$ satisfies ZC_n.

If $[[R^{S,\leqslant}]]$ satisfies ZC_n, then R satisfies ZC_n since subrings of rings satisfying ZC_n also satisfy ZC_n. ■

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References

1. D. D. Anderson and V.Camillo, Semigroups and rings whose zero products commute, *Commun. Algebra* **27** (1999), 2847-2852.
2. P. M. Cohn, Reversible rings, *Bull. Lond. Math. Soc.* **31** (1999), 641-648.
3. G. A. Elliott and P. Ribenboim, Fields of generalized power series, *Arch. Math.* **54** (1990), 365-371.
4. J. M. Habeb, A note on zero commutative and duo rings, *Math. J. Okayama Univ.* **32** (1990), 73-76.
5. C. Y. Hong, N. K. Kim and T. K. Kwak, Extensions of generalized reduced rings, *Algebra Colloq.* **12** (2005), 229-240.
6. C. Huh, Y. Lee and A. Smoktunowicz, Armendariz rings and semicommutative rings, *Commun. Algebra* **30** (2002), 751-761.
7. N. K. Kim and Y. Lee, Extensions of reversible rings, *J. Pure Appl. Algebra* **185** (2003), 207-223.
8. Z. K. Liu, Special properties of rings of generalized power series, *Commun. Algebra* **32** (2004), 3215-3226.
9. P. Ribenboim, Semisimple rings and Von Neumann regular rings of generalized power series, *J. Algebra* **198** (1997), 327-338.
10. G. Y. Shin, Prime ideals and sheaf representation of a pseudo symmetric ring, *Trans. Am. Math. Soc.* **184** (1973), 43-60.