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# Stability of Two-Step-by-Two-Step IRK Methods Based on Gauss-Legendre Collocation Points and an Application*

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**Abstract.** This paper investigates a class of IRK-type methods for solving first-order stiff initial-value problems (IVPs). The IRK-type methods are constructed by using coefficients of $s$-stage collocation Gauss-Legendre IRK methods and other $2s$-stage collocation IRK methods. The collocation points used in the $2s$-stage methods are chosen such that at $n$th integration step, their stage values can be used as the stage values of the associated collocation Gauss-Legendre IRK methods for $(n+2)$th integration step. By this way we obtain the methods in which the integration processes can be proceeded two-step-by-two-step. The resulting IRK-type methods are called two-step-by-two-step IRK methods based on Gauss-Legendre collocation points (TBTIRKG methods). Stability considerations show that these TBTIRKG methods can be $A$-stable or $A(\alpha)$-stable which can be applied to stiff IVPs with a fewer number of implicit relations that are to be solved in the integration process when compared with the traditional Gauss-Legendre IRK methods. The stability investigation results for TBTIRKG methods were applied in considerations of the asymptotic stability of a class of PC methods based on the TBTIRKG methods.

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**Key words.** Implicit Runge-Kutta methods, stability.

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## 1. Introduction

We consider numerical methods for the stiff initial-value problems (IVPs) for the system of first-order ordinary differential equations (ODEs)

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t)), \qquad \mathbf{y}(t_0) = \mathbf{y}_0, \qquad t_0 \leqslant t \leqslant T, \tag{1}$$

where $t$ denotes time values that lie on some real interval $[t_0, T] \subset [0, +\infty]$, $\mathbf{y} : [t_0, T] \to \mathbb{R}^d$, $\mathbf{f} : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, $\mathbf{y}_0 \in \mathbb{R}^d$. Among various numerical methods proposed so far, the most efficient methods for solving this stiff IVP (1) are the IRK methods. In the literature, $A$-stable IRK methods of various orders can be found in e.g., [1, 2, 4, 6]. In this paper, we investigate a particular class of IRK-type methods with the set of coefficients taken from $s$-stage collocation Gauss-Legendre IRK methods based on the abscissas $c_1, \ldots, c_s$ and $2s$-stage collocation IRK methods based on the abscissas $c_1, \ldots, c_s, 1+c_1, \ldots, 1+c_s$. The stage values of the $2s$-stage collocation IRK methods evaluated at $t_n + (1 + c_1)h, \ldots, t_n + (1 + c_s)h$ from $n$th integration step can be used as the stage values of the $s$-stage collocation Gauss-Legendre IRK methods for $(n + 2)$th integration step, so that we can apply a two-step-by-two-step (TBT) integration strategy (the integration is proceeded two-step-by-two-step). In this way we obtain IRK-type methods which will be termed *two-step-by-two-step IRK methods based on Gauss-Legendre collocation points* (TBTIRKG methods). Thus, we have achieved the new IRK-type methods with a fast integration process. Stability investigations reveal that these new TBTIRKG methods are suitable for stiff IVPs.

Section 2 investigates TBTIRKG methods with respect to order of accuracy and stability properties. Applying the stability investigation results for TBTIRKG methods, we consider the asymptotic stability of the TBTPIRKG methods (a class of PC methods based on TBTIRKG methods considered in [10]) in Section 3.

## 2. TBTIRKG methods

Let $\mathbf{c} = (c_1, \ldots, c_s)^T$ and $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_s, \tilde{c}_{s+1}, \ldots, \tilde{c}_{2s})^T := (c_1, \ldots, c_s, 1 + c_1, \ldots, 1 + c_s)^T$, $\mathbf{c}$ is the $s$-dimentional Gauss-Legendre collocation vector. Consider two collocation IRK methods defined by the following Butcher tableaux (see e.g., [7]):

$$\begin{array}{c|c} \tilde{\mathbf{c}} & A \\ \hline & \tilde{\mathbf{b}}^T \end{array}\,, \qquad\qquad \begin{array}{c|c} \mathbf{c} & \hat{A} \\ \hline & \hat{\mathbf{b}}^T \end{array}\,.$$

Notice that here, $A = (a_{ij})$ is a $2s \times 2s$ matrix and $\hat{\mathbf{b}} = (\hat{b}_i)$ is a $s$-dimentional vector. We now consider an IRK-type method which is defined as:

$$\mathbf{Y}_{n,i} = \mathbf{u}_n + h \sum_{j=1}^{2s} a_{ij} \mathbf{f}(t_n + \tilde{c}_j h, \mathbf{Y}_{n,j}), \quad i = 1, \ldots, 2s, \tag{2a}$$

$$\mathbf{u}_{n+2} = \mathbf{u}_n + h \sum_{j=1}^{s} \hat{b}_j [\mathbf{f}(t_n + \tilde{c}_j h, \mathbf{Y}_{n,j}) + \mathbf{f}(t_n + \tilde{c}_{s+j} h, \mathbf{Y}_{n,s+j})]. \tag{2b}$$

Here in (2), $\mathbf{u}_n \approx \mathbf{y}(t_n)$, $\mathbf{u}_{n+2} \approx \mathbf{y}(t_{n+2})$ and $h$ is the stepsize. Furthermore, the vector $\mathbf{Y}_n = (\mathbf{Y}_{n,1}^T, \ldots, \mathbf{Y}_{n,2s}^T)^T$ denotes the stage vector representing numerical approximations to the exact solution vector $[(\mathbf{y}(t_n+\tilde{c}_1 h))^T, \ldots, (\mathbf{y}(t_n+\tilde{c}_{2s} h))^T]^T$ at $n$th step. For a convenient presentation, we define the vector

$$\mathbf{b} = (b_1, \ldots, b_s, b_{s+1}, \ldots, b_{2s})^T := (\hat{b}_1, \ldots, \hat{b}_s, \hat{b}_1, \ldots, \hat{b}_s)^T.$$

Using the new vector $\mathbf{b}$, the method (2) can be presented in a very compact form:

$$\mathbf{Y}_{n,i} = \mathbf{u}_n + h \sum_{j=1}^{2s} a_{ij} \mathbf{f}(t_n + \tilde{c}_j h, \mathbf{Y}_{n,j}), \quad i = 1, \ldots, 2s, \tag{3a}$$

$$\mathbf{u}_{n+2} = \mathbf{u}_n + h \sum_{j=1}^{2s} b_j \mathbf{f}(t_n + \tilde{c}_j h, \mathbf{Y}_{n,j}). \tag{3b}$$

The method (3) can be conveniently presented by the Butcher tableau (see e.g., [7])

$$\begin{array}{c|c} \tilde{\mathbf{c}} & A \\ \hline \mathbf{u}_{n+2} & \mathbf{b}^T \end{array}$$

and will be called *two-step-by-two-step IRK methods based on Gauss-Legendre collocation points* (TBTIRKG methods). The matrix $A$ and the vector $\hat{\mathbf{b}}$ are defined by the simplifying conditions $C(2s)$ (based on vector $\tilde{\mathbf{c}}$) and $B(s)$ (based on vector $\mathbf{c}$), respectively (see e.g., [3, 13, 14]). They can be explicitly expressed in terms of the collocation vectors $\mathbf{c}$ and $\tilde{\mathbf{c}}$ as follows (see e.g., [7, 11])

$$A = PR^{-1}, \qquad \hat{\mathbf{b}}^T = \hat{\mathbf{g}}^T \hat{R}^{-1}, \tag{4}$$

where

$$P = (p_{ij}) = \left(\frac{\tilde{c}_i^j}{j}\right), \quad R = (r_{ij}) = (\tilde{c}_i^{j-1}), \quad i, j = 1, \ldots, 2s.$$

$$\hat{R} = (\hat{r}_{ij}) = (c_i^{j-1}), \quad \hat{\mathbf{g}} = (\hat{g}_i) = \left(\frac{1}{i}\right), \quad i, j = 1, \ldots, s.$$

### 2.1. Orders of accuracy

**Definition 2.1.** Suppose that $\mathbf{u}_n = \mathbf{y}(t_n)$, then the TBTIRKG method (3) is said to have the step point order $p$ if $\mathbf{y}(t_{n+2}) - \mathbf{u}_{n+2} = O(h^{p+1})$ and the stage order $q = \min\{p, q_1\}$ if in addition, $\mathbf{y}(t_n + \tilde{c}_i h) - \mathbf{Y}_{n,i} = O(h^{q_1+1})$, for $i = 1, \ldots, 2s$.

The Definition 2.1 above is more general than the one given in [10]. For the step point order, and stage order of the TBTIRKG method (3), we have the following theorem:

**Theorem 2.2.** *If the function* $\mathbf{f}$ *is Lipschitz continuous, then the TBTIRKG method* (3) *has the step point order* $p = 2s$ *and the stage order* $q = 2s$.

*Proof.* The collocation principle applied to the $2s$-stage collocation IRK methods defined by $(\tilde{\mathbf{c}}, A, \tilde{\mathbf{b}})$ gives the following local order relation

$$\mathbf{y}(t_n + \tilde{c}_i h) - \mathbf{Y}_{n,i} = O(h^{2s+1}), \quad i = 1, \ldots, 2s. \tag{5}$$

The step point order $p = 2s$ can be proved by using Definition 2.1, the order relation (5) and the step point order $2s$ of Gauss-Legender IRK method defined by $(\mathbf{c}, \hat{A}, \hat{\mathbf{b}})$. Thus, we suppose that $\mathbf{u}_n = \mathbf{y}(t_n)$ and consider

$$\begin{aligned}
\mathbf{y}(t_{n+2}) - \mathbf{u}_{n+2} &= \mathbf{y}(t_{n+2}) - \mathbf{y}(t_n) - h\sum_{j=1}^{2s} b_j \mathbf{f}(t_n + \tilde{c}_j h, \mathbf{Y}_{n,j}) \\
&= \mathbf{y}(t_{n+1}) - \mathbf{y}(t_n) - h\sum_{j=1}^{s} \hat{b}_j \mathbf{f}(t_n + c_j h, \mathbf{y}(t_n + c_j h)) \\
&\quad + h\sum_{j=1}^{s} \hat{b}_j [\mathbf{f}(t_n + c_j h, \mathbf{y}(t_n + c_j h)) - \mathbf{f}(t_n + c_j h, \mathbf{Y}_{n,j})] \\
&\quad + \mathbf{y}(t_{n+2}) - \mathbf{y}(t_{n+1}) - h\sum_{j=1}^{s} \hat{b}_j \mathbf{f}(t_{n+1} + c_j h, \mathbf{y}(t_{n+1} + c_j h)) \\
&\quad + h\sum_{j=1}^{s} \hat{b}_j [\mathbf{f}(t_{n+1} + c_j h, \mathbf{y}(t_{n+1} + c_j h)) - \mathbf{f}(t_{n+1} + c_j h, \mathbf{Y}_{n+1,j})] \\
&= O(h^{2s+1}) + O(h^{2s+2}) + O(h^{2s+1}) + O(h^{2s+2}).
\end{aligned}$$

From here, we obtain

$$\mathbf{y}(t_{n+2}) - \mathbf{u}_{n+2} = O(h^{2s+1}). \tag{6}$$

By Definition 2.1, the order relations (5) and (6) prove Theorem 2.2. ■

### 2.2. Linear stability

The linear stability of the TBTIRKG method (3) is investigated by using the model test equation $y'(t) = \lambda y(t)$, where $\lambda$ is assumed to be lying in the left half-plane. For the model test equation, the method (3) has the form

$$\mathbf{Y}_n = \mathbf{e} u_n + zA\mathbf{Y}_n, \qquad u_{n+2} = u_n + z\mathbf{b}^T \mathbf{Y}_n, \tag{7}$$

where $z := h\lambda$, $\mathbf{Y}_n = (Y_{n,1}, \ldots, Y_{n,2s})^T$ and $\mathbf{e} = (1, \ldots, 1)^T$ are $2s$-dimensional vectors. The relation (7) leads us to the recursion

$$u_{n+2} = [1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{e}]u_n = \mathbb{R}(z)u_n. \tag{8}$$

The function $\mathbb{R}(z) = 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{e}$ defined by (8) which determines the stability of the TBTIRKG method, will be called the *stability function*. Similarly to the case of IRK methods, we can find an alternative expression for the stability function $\mathbb{R}(z)$ (cf. e.g., [12]).

**Lemma 2.3.** *The stability function* $\mathbb{R}(z)$ *of TBTIRKG method* (3) *is given by*

$$\mathbb{R}(z) = \frac{\det[I - z(A - \mathbf{e}\mathbf{b}^T)]}{\det[I - zA]}. \tag{9}$$

*Proof.* This lemma can be proved by applying the approach used in e.g., [19, p. 200] for the case of (step-by-step) IRK methods. First, we write the equations (7) in the following form

$$\begin{pmatrix} 1 - za_{1,1} & -za_{1,2} & \cdots & za_{1,2s} & 0 \\ -za_{2,1} & 1 - za_{2,2} & \ldots & -za_{2,2s} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ -za_{2s,1} & -za_{2s,2} & \ldots & 1 - za_{2s,2s} & 0 \\ -zb_1 & -zb_2 & \ldots & -zb_{2s} & 1 \end{pmatrix} \begin{pmatrix} Y_{n,1} \\ Y_{n,2} \\ \cdot \\ \cdot \\ \cdot \\ Y_{n,2s} \\ u_{n+2} \end{pmatrix} = \begin{pmatrix} u_n \\ u_n \\ \cdot \\ \cdot \\ \cdot \\ u_n \\ u_n \end{pmatrix}.$$

The solution for $u_{n+2}$ can be defined by Cramer's rule as $u_{n+2} = N/D$, where

$$N = \det \begin{pmatrix} 1 - za_{1,1} & -za_{1,2} & \cdots & za_{1,2s} & u_n \\ -za_{2,1} & 1 - za_{2,2} & \ldots & -za_{2,2s} & u_n \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ -za_{2s,1} & -za_{2s,2} & \ldots & 1 - za_{2s,2s} & u_n \\ -zb_1 & -zb_2 & \ldots & -zb_{2s} & u_n \end{pmatrix},$$

$$D = \det \begin{pmatrix} 1 - za_{1,1} & -za_{1,2} & \cdots & za_{1,2s} & 0 \\ -za_{2,1} & 1 - za_{2,2} & \ldots & -za_{2,2s} & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ -za_{2s,1} & -za_{2s,2} & \ldots & 1 - za_{2s,2s} & 0 \\ -zb_1 & -zb_2 & \ldots & -zb_{2s} & 1 \end{pmatrix}.$$

Subtracting the last row of $N$ from each of the upper rows leaves $N$ unaltered, whence

$$N = \det \begin{pmatrix} 1 - za_{1,1} + zb_1 & -za_{1,2} + zb_2 & \cdots & za_{1,2s} + zb_{2s} & 0 \\ -za_{2,1} + zb_1 & 1 - za_{2,2} + zb_2 & \dots & -za_{2,2s} + zb_{2s} & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ -za_{2s,1} + zb_1 & -za_{2s,2} + zb_2 & \dots & 1 - za_{2s,2s} + zb_{2s} & 0 \\ -zb_1 & -zb_2 & \dots & -zb_{2s} & u_n \end{pmatrix}$$

$$= u_n \det[I - z(A - \mathbf{e}\mathbf{b}^T)].$$

Clearly that $D = \det[I - zA]$, and we obtain $u_{n+2} = N/D = \mathbb{R}(z)u_n$, where $\mathbb{R}(z)$ is defined by (9) in Lemma 2.3. ■

The stability region denoted by $\mathbb{S}$ of the TBTIRKG method (3) is defined as

$$\mathbb{S} := \big\{z \in \mathbb{C} : |\mathbb{R}(z)| \leqslant 1, \mathrm{Re}(z) \leqslant 0\big\}.$$

**Definition 2.4.** Let $\mathbb{C}^- := \big\{z \in \mathbb{C} : \mathrm{Re}(z) \leqslant 0\big\}$. If $\mathbb{S} = \mathbb{C}^-$, then the TBTIRKG method (3) is said to be A-stable.

$A$-stability is a highly desirable property for numerical methods to possess when solving stiff IVPs. However, this property is not always possible to obtain because other qualities such as efficiency, robustness and reliability are also necessary and these can be in conflict with $A$-stability (cf. [3]).

There exist numerical methods, which are not $A$-stable, but also are very useful for stiff IVPs, that is $A(\alpha)$-stable methods.

**Definition 2.5.** Let $\alpha$ denote an angle satisfying $0^o < \alpha \leqslant 90^o$. The TBTIRKG method (3) is said to be $A(\alpha)$-stable if its stability region $\mathbb{S}$ contains the infinite wedge $\mathbb{W}(\alpha) := \big\{z \in \mathbb{C} : -\alpha < \pi - \arg(z) < \alpha\big\}$.

The notion of $A(\alpha)$-stability was firstly introduced in [21]. It also was considered in e.g., [3, 20]). Definition 2.4 and Definition 2.5 show that $A(90^o)$-stability implies $A$-stability.

**Theorem 2.6.** *The TBTIRKG method* (3) *is* $A(\alpha)$*-stable if and only if* (i) *all poles of* $\mathbb{R}(z)$ *lie outside the wedge* $\mathbb{W}(\alpha)$ *and* (ii) $\mathbb{R}(z)$ *is bounded by 1 on the sides* (*boundaries*) *of the wedge* $\mathbb{W}(\alpha)$.

*Proof. Necessity.* (i) follows from the fact that if $z^* \in \mathbb{W}(\alpha)$ is a pole of $\mathbb{R}(z)$ then $\lim_{z \to z^*} |\mathbb{R}(z)| = \infty$, hence $|\mathbb{R}(z)| > 1$ for some $z$ close enough to $z^*$. (ii) follows from the fact that if there exists $z = x + iy$ on the sides of the wedge $\mathbb{W}(\alpha)$ such that $|\mathbb{R}(z)| > 1$, then $|\mathbb{R}(z_\epsilon)| > 1$ for some $z_\epsilon = -\epsilon + x + iy \in \mathbb{W}(\alpha)$. *Sufficiency.* (i) and (ii) imply that $\mathbb{R}(z)$ is analytic in the wedge $\mathbb{W}(\alpha)$ and bounded by 1 on the sides of the wedge $\mathbb{W}(\alpha)$ so that by the maximum modulus principle, $|\mathbb{R}(z)| \leqslant 1$ on the wedge $\mathbb{W}(\alpha)$ and the TBTIRKG method (3) is $A(\alpha)$-stable. ■

Lemma 2.3 indicates that the stability function $\mathbb{R}(z)$ of the TBTIRKG method (3) is a rational function (the ratio of two polynomials) with the denominator polynomial $\det[I - zA]$ (see (9)). It can be seen that the eigenvalues of the matrix $A$ are the poles of the stability function $\mathbb{R}(z)$. Thus we have the following corollary

**Corollary 2.7.** *The TBTIRKG method* (3) *is $A(\alpha)$-stable if and only if* (i) *all the eigenvalues of the matrix $A$ lie outside the wedge $\mathbb{W}(\alpha)$ and* (ii) *$\mathbb{R}(z)$ is bounded by 1 on the sides of the wedge $\mathbb{W}(\alpha)$.*

Numerical computations of the spectrum of the matrices $A$ defining the TBTIRKG methods and numerical searches for their stability functions $\mathbb{R}(z)$ along the sides of the wedges $\mathbb{W}(\alpha)$ give the following results:

- For $s = 2, 3$, all the eigenvalues of $A$ are lying outside the wedge $\mathbb{W}(90^o)$ and the stability functions $\mathbb{R}(z)$ are bounded by 1 on its sides;
- For $s = 4$, all the eigenvalues of $A$ are lying outside the wedge $\mathbb{W}(89.99^o)$ and the stability function $\mathbb{R}(z)$ is bounded by 1 on its sides;
- For $s = 5$, all the eigenvalues of $A$ are lying outside the wedge $\mathbb{W}(87.79^o)$ and the stability function $\mathbb{R}(z)$ is bounded by 1 on its sides.

Thus, we have achieved 4th-order and 6th-order TBTIRKG methods which are $A(90^o)$-stable ($A$-stable); 8th-order TBTIRKG method which is $A(89.99^o)$-stable (almost $A$-stable) and 10th-order TBTIRKG method which is $A(87.79^o)$-stable.

We summarize the main characteristics of the achieved TBTIRKG methods in the Table 1, in which $p$ denotes the step point order, $q$ denotes the stage order and $s^*$ denotes the number of implicit stages. From the Table 1, we see that for

**Table 1** Summary of the main characteristics for various TBTIRKG methods

| Methods | $p$ | $q$ | $s^*$ | Stability |
|---|---|---|---|---|
| TBTIRKG4 | 4 | 4 | 4 | $A$-stable |
| Gauss-Legendre IRK4 | 4 | 2 | 2 | $A$-stable |
| TBTIRKG6 | 6 | 6 | 6 | $A$-stable |
| Gauss-Legendre IRK6 | 6 | 3 | 3 | $A$-stable |
| TBTIRKG8 | 8 | 8 | 8 | $A(89.99^o)$-stable |
| Gauss-Legendre IRK8 | 8 | 4 | 4 | $A$-stable |
| TBTIRKG10 | 10 | 10 | 10 | $A(87.79^o)$-stable |
| Gauss-Legendre IRK10 | 10 | 5 | 5 | $A$-stable |

a given order of accuracy $p$ with competitive stability property, the TBTIRKG methods have the stage order $q$ two times larger than the stage order of the Gauss-Legender IRK methods. We observe that the TBTIRKG methods also have the stage number $s^*$ larger than that of Gauss-Legendre IRK methods. The larger $s^*$ is a disadvantage of the TBTIRKG methods. This disadvantage will be not important on parallel machines when a suitable parallel iteration approach (functional iteration for nonstiff problems, diagonally implicit iteration for stiff problems) is applied (cf. e.g., [8, 9, 10, 11, 15, 16, 17]).

Applying to TBTIRKG methods defined by (3) a parallel PC iteration scheme, we obtain TBTPIRKG methods, which have been investigated in [10]. The investigations in [10] show that the TBTPIRKG methods are very efficient numerical integration methods for nonstiff IVPs. The stability consideration results for the TBTIRKG methods allow us to investigate the asymptotic stability regions of the TBTPIRKG methods as an application.

### 2.3. Numerical illustrations

In this section, we illustrate the efficiency of the new investigated methods. For this purpose, we apply TBTIRKG methods and Gauss-Legendre IRK methods to the numerical solution of two test stiff IVPs taken from ODE literature. These two IVPs possess exact solutions in closed form. Initial conditions are taken from the exact solutions. The first test stiff IVP is taken from [20] and has the following form

$$\frac{dy_1}{dt} = -\alpha y_2 + (1+\alpha)\cos(t),\ \frac{dy_2}{dt} = \alpha y_1 - (1+\alpha)\sin(t),\ 0 \leqslant t \leqslant 100, \quad (10)$$

with $\alpha = 10$ and exact solution $y_1 = \sin(t)$ and $y_2 = \cos(t)$ for all values of the parameter $\alpha$. The second test stiff IVP is taken from [18] and has the following form

$$\frac{dy_1}{dt} = -(2+\epsilon^{-1})y_1 + \epsilon^{-1}(y_2)^2, \quad \frac{dy_2}{dt} = y_1 - y_2(1+y_2), \quad 0 \leqslant t \leqslant 1, \quad (11)$$

with $\epsilon = 10^{-8}$. The exact solution of this IVP is given by $y_1 = \exp(-2t)$ and $y_2 = \exp(-t)$ for all values of the parameter $\epsilon$.

We confine on the consideration of TBTIRKG and Gauss-Legendre IRK methods of order 4 and order 6 respectively denoted by TBTIRKG4, IRK4, TBTIRKG6 and IRK6.

The absolute error obtained at the end point of the integration interval is presented in the form $10^{-NCD}$ so that $NCD$ indicates the accuracy and may be interpreted as the average number of correct decimal digits. The values of $NCD$ are plotted as a function of the stepsize $h$. The numerical results presented in Figure 1 and Figure 2 show that the TBTIRKG methods clearly are suitable for stiff IVPs. These numerical results also show that the TBTIRKG methods are superior to the original Gauss-Legendre IRK methods.

## 3. Asymptotic stability of TBTPIRKG methods

In this section, we investigate the asymptotic stability region of the TBTPIRKG methods based on the TBTIRKG methods of the form (3) considered in [10],

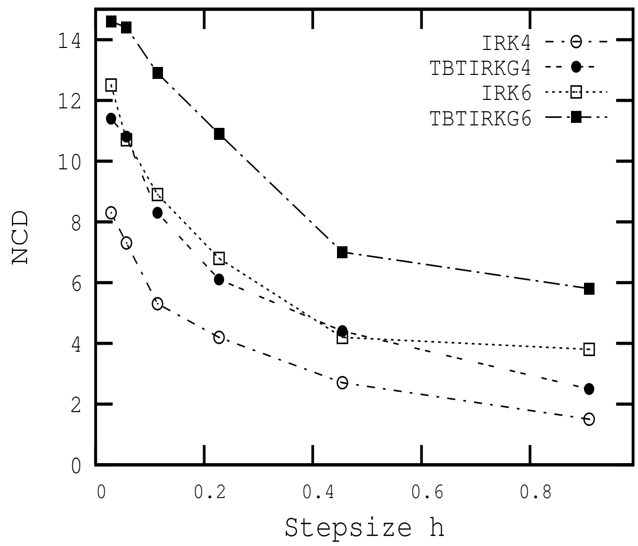


**Fig. 1** Numerical results for the problem (10)

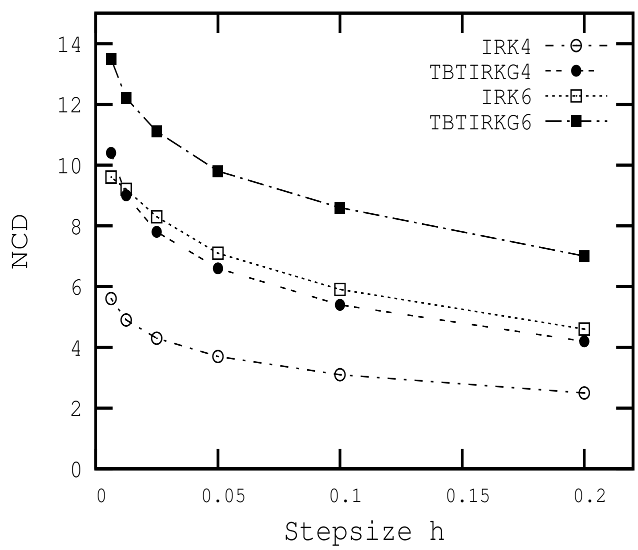


**Fig. 2** Numerical results for the problem (11)

which are of the following form

$$\mathbf{Y}_{n,i}^{(0)} = \mathbf{y}_n + h \sum_{j=1}^{2s} v_{ij} \mathbf{f}(t_{n-2} + \tilde{c}_j h, \mathbf{Y}_{n-2,j}^{(m)}), \quad i = 1, \ldots, 2s, \tag{12a}$$

$$\mathbf{Y}_{n,i}^{(k)} = \mathbf{y}_n + h\sum_{j=1}^{2s} a_{ij}\mathbf{f}(t_n + \tilde{c}_j h, \mathbf{Y}_{n,j}^{(k-1)}), \quad i = 1, \ldots, 2s,\ k = 1, \ldots, m, \tag{12b}$$

$$\mathbf{y}_{n+2} = \mathbf{y}_n + h\sum_{j=1}^{2s} b_j\mathbf{f}(t_n + \tilde{c}_j h, \mathbf{Y}_{n,j}^{(m)}). \tag{12c}$$

It is evident that if the iteration process defined by (12b) converges, then $\mathbf{Y}_{n,i}^{(m)}$ converges to $\mathbf{Y}_{n,i}$ as $m$ tends to $\infty$. The convergence conditions for (12b) as defined in [10, Section 3.2] are given by

$$|z| < \frac{1}{\rho(A)} \quad \text{or} \quad h < \frac{1}{\rho(\partial\mathbf{f}/\partial\mathbf{y})\rho(A)}. \tag{13}$$

The values $1/\rho(A)$ are called convergence boundaries of the TBTPIRKG methods. The convergence region $\mathbb{S}_{conv}$ is defined as

$$\mathbb{S}_{conv} := \{z : z \in \mathbb{C}, |z| < 1/\rho(A)\}. \tag{14}$$

The convergence boundaries $1/\rho(A)$ of our four TBTPIRKG methods are listed in Table 2.

**Table 2** Convergence boundaries for various TBTPIRKG methods

| TBTPIRKG method of order $p$ | $p = 4$ | $p = 6$ | $p = 8$ | $p = 10$ |
|---|---|---|---|---|
| Convergence boundary $1/\rho(A)$ | 2.506 | 3.443 | 4.392 | 5.345 |

Asymptotic stability regions of TBTPIRKG methods is the stability regions when the number $m$ in (12b) tends to $\infty$. For defining the asymptotic stability regions, first, we consider the stability regions for a fixed number $m$ by applying the methods (12) to the model test equation $y'(t) = \lambda y(t)$ and obtain (cf. [10, Section 3.3])

$$\begin{pmatrix} \mathbf{Y}_n^{(m)} \\ y_{n+2} \end{pmatrix} = M_m(z) \begin{pmatrix} \mathbf{Y}_{n-2}^{(m)} \\ y_n \end{pmatrix}, \tag{15a}$$

where $M_m(z)$ is the $(2s+1) \times (2s+1)$ matrix defined by

$$M_m(z) = \begin{pmatrix} z^{m+1}A^mV & [I + zA + \cdots + (zA)^m]\mathbf{e} \\ z^{m+2}\mathbf{b}^TA^mV & 1 + z\mathbf{b}^T[I + zA + \cdots + (zA)^m]\mathbf{e} \end{pmatrix}. \tag{15b}$$

The matrix $M_m(z)$ in (15) which determines the stability of the TBTPIRKG methods, will be called the *amplification matrix*, its spectral radius $\rho\big(M_m(z)\big)$ the *stability function*. For a given number of iterations $m$, the stability region denoted by $\mathbb{S}_{stab}(m)$ of the TBTPIRKG methods is defined as

$$\mathbb{S}_{stab}(m) := \big\{z : \rho\big(M_m(z)\big) < 1, \mathrm{Re}(z) \leqslant 0\big\}.$$

The asymptotic stability regions denoted by $\mathbb{S}_{stab}(\infty)$ is estimated by the following theorem:

**Theorem 3.1.** *The asymptotic stability region* $\mathbb{S}_{stab}(\infty)$ *of the TBTPIRKG methods* (12) *for* $m \to \infty$ *contains the intersection in the left-half plane of the stability region of the TBTIRKG methods defined by* (3) *and the convergence region defined by* (14) *of the TBTPIRKG methods* (12).

*Proof.* From (15b), we see that if $z$ satisfies the convergence condition defined by (13), then as $m$ tends to $\infty$, $M_m(z)$ converges to a matrix denoted by $M_\infty(z)$, which is given by

$$M_\infty(z) = \begin{pmatrix} O_{2s\times 2s} & (I - zA)^{-1}\mathbf{e} \\ O_{2s\times 2s} & 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{e} \end{pmatrix},$$

where $O_{2s\times 2s}$ is the $2s \times 2s$ matrix with zero entries. Thus, $\rho(M_m(z))$ converges to $\rho(M_\infty(z))$ as $m$ tends to $\infty$. Since

$$\rho(M_\infty(z)) = |\mathbb{R}(z)| = |1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{e}|$$

(see Section 2.2.), the stability function of the TBTPIRKG methods (12) converges to absolute value of the stability function of the TBTIRKG methods defined by (3) as $m$ tends to $\infty$. Hence, the asymptotic stability region of the TBTPIRKG methods (12) as $m$ tends to $\infty$, $\mathbb{S}_{stab}(\infty)$, contains the intersection on the left-half plane of the stability region of the TBTIRKG methods defined by (3) and convergence region defined by (14) of the TBTPIRKG methods (12). Theorem 3.1 is then completely proved. ■

## 4. Concluding remarks

In this paper, we considered a class of new methods, that is the two-step-by-two-step IRK methods based on Gauss-Legendre collocation points (TBTIRKG methods). We have showed that the investigated methods have a good stability property and are suitable for solving stiff IVPs. We have also estimated the asymptotic stability regions of a class of PC methods based on these new TBTIRKG methods (i.e. TBTPIRKG methods considered in [10]).

In forthcoming papers, we will pursue investigations of these TBTIRKG methods with respect to parallel diagonally implicit iterations of TBTIRKG methods for stiff IVPs.

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