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A Generalization of Hardy-Hilbert's Inequality with Multi-Parameter

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Abstract. In this paper, by introducing four parameters A, B, α and β, we give a new generalization of Hardy-Hilbert's inequality with best constant factor. As applications, the equivalent form and some particular results are derived.

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1. Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_n, b_n \geq 0$ such that $0 < \sum_{n=0}^{\infty} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^q < \infty$, then the famous *Hardy-Hilbert's inequality* (see [1]) is given by

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=0}^{\infty} a_n^p \right)^{\frac{1}{p}} \left(\sum_{n=0}^{\infty} b_n^q \right)^{\frac{1}{q}}, \tag{1}$$

and an equivalent form (see [7]) is

$$\sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_m}{m+n+1} \right)^p < \left[\frac{\pi}{\sin(\pi/p)} \right]^p \sum_{n=0}^{\infty} a_n^p, \tag{2}$$

where the constant factors $\pi/\sin(\pi/p)$ and $[\pi/\sin(\pi/p)]^p$ are the best possible. In particular, when $p = q = 2$, we have the *Hilbert's double series theorem*(see [1]), which states that: If $a_n, b_n \geq 0$ such that $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=0}^{\infty} b_n^2 < \infty$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{m+n+1} < \pi \left(\sum_{n=0}^{\infty} a_n^2 \sum_{n=0}^{\infty} b_n^2 \right)^{\frac{1}{2}}, \tag{3}$$

where the constant factor π is the best possible. These inequalities are important in analysis and its applications (see Mitrinović et al. [4]).

In the year 1999, by introducing a parameter λ and the β-function, Yang and Debnath [7] gave a generalization of (1) as: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 2 - \min\{p, q\} < \lambda \leq 2$, $a_n, b_n \geq 0$ such that $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} b_n^q < \infty$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} < k_{\lambda}(p) \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{1-\lambda} b_n^q \right\}^{\frac{1}{q}}, \tag{4}$$

where the constant factor $k_{\lambda}(p) = B\left(\frac{p+\lambda-2}{p}, \frac{q+\lambda-2}{q}\right)$ is the best possible.

In the year 2002, by introducing parameters A, B and λ, Yang [8] gave a generalization of (3) as: If $0 < A, B \leq 1$, $0 < \lambda \leq 2$, $a_n, b_n \geq 0$ such that $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{1-\lambda} a_n^2 < \infty$ and $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{1-\lambda} b_n^2 < \infty$, then

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(Am+Bn+1)^{\lambda}} \\ &< \frac{1}{(AB)^{\lambda/2}} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{1-\lambda} a_n^2 \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{1-\lambda} b_n^2 \right\}^{\frac{1}{2}}, \end{aligned} \tag{5}$$

where the constant factor $\frac{1}{(AB)^{\lambda/2}} B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible.

In the year 2003, Yang [9] gave further generalization of (1) as: If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $0 < \lambda \leq \min\{p, q\}, a_n, b_n \geq 0$ such that $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{(p-1)(1-\lambda)} a_n^p < \infty$ and $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{(q-1)(1-\lambda)} b_n^q < \infty$, then

$$\begin{aligned} &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(m+n+1)^{\lambda}} \\ &< B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{(p-1)(1-\lambda)} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right)^{(q-1)(1-\lambda)} b_n^q \right\}^{\frac{1}{q}}, \end{aligned} \tag{6}$$

where the constant factor $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ is the best possible.

In the year 2005, Yang and Rassias [10] gave a generalization of (3) which is not a generalization of (1) as: If $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ 0<\lambda\le\min\{p,q\},\ a_n,b_n\ge 0$ such that $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1-\lambda}a_n^p<\infty$ and $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q-1-\lambda}b_n^q<\infty$, then

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\lambda}}\\ &< B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q-1-\lambda}b_n^q\right\}^{\frac{1}{q}},\end{aligned}\tag{7}$$

where the constant factor $B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ is the best possible. In particular for $\lambda=1$, we have the inequality

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1}<\frac{\pi}{\sin(\pi/p)}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{p-2}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q-2}b_n^q\right\}^{\frac{1}{q}}\tag{8}$$

and the constant factor $\pi/\sin(\pi/p)$ is the best possible. This is a new generalization of (3) in (p,q)-parameter from, which is different from (3). Hence (7) is a generalization of (3), but not a generalization of (1).

In this paper, using the four parameters A, B, α and β, we give a generalization of (3), with a best constant factor, which is a more generalized inequality and from which all of the above inequalities are obtained by specialising the parameters. As application, we give the equivalent inequality and certain particular results.

2. Some lemmas

In this section we shall prove lemmas, which play crucial roles in proving our main results. We need the formula of the $\beta-$function (cf. Wang et al. [5]),

$$B(p,q)=\int_0^{\infty}\frac{1}{(1+t)^{p+q}}t^{p-1}dt=B(q,p)\tag{9}$$

and the following inequality, which is derived from Euler-Maclaurin summation formula (see [3, 6]).

If f has its first four derivatives on $(0,\infty)$ such that $(-1)^k f^{(k)}(x)\ge 0$ and $\lim_{x\to\infty}f^{(k)}(x)=0$, for $k=0,1,2,3,4$, then

$$\sum_{n=0}^{\infty}f(n)\le\int_0^{\infty}f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f'(0).\tag{10}$$

Lemma 2.1. *Let* $0 < A, B, \alpha, \beta \le 1$ *and for all* $m \in \mathbb{N}_0$, R_m *is defined by*

$$\begin{aligned} R_m =& \frac{1}{B(Am+\frac{1}{2})^\alpha} \int_0^{\frac{1}{2Am+1}} \frac{1}{(1+u)^{\alpha+\beta}} u^{\beta-1} du \\ & - \left[1 + \frac{(1-\beta)B}{3}\right] \frac{2^{-\beta}}{(Am+1)^{\alpha+\beta}} - \frac{(\alpha+\beta)B2^{-\beta}}{6(Am+1)^{\alpha+\beta+1}}. \end{aligned} \tag{11}$$

Then $R_m > 0$ *for all* $m \in \mathbb{N}_0$.

Proof. Integrating by parts, we obtain

$$\begin{aligned} & \int_0^{\frac{1}{2Am+1}} \frac{1}{(1+u)^{\alpha+\beta}} u^{\beta-1} du \\ &= \frac{1}{\beta} \frac{(2Am+1)^\alpha}{2^{\alpha+\beta}(Am+1)^{\alpha+\beta}} + \frac{\alpha+\beta}{\beta} \int_0^{\frac{1}{2Am+1}} \frac{1}{(1+u)^{\alpha+\beta+1}} u^\beta du \\ &> \frac{2^{-\beta}}{\beta(Am+1)^{\alpha+\beta}} \left(Am + \frac{1}{2}\right)^\alpha + \frac{\alpha+\beta}{\beta} \frac{1}{(1+\frac{1}{2Am+1})^{\alpha+\beta+1}} \int_0^{\frac{1}{2Am+1}} u^\beta du \\ &= \frac{2^{-\beta}}{\beta(Am+1)^{\alpha+\beta}} \left(Am + \frac{1}{2}\right)^\alpha + \frac{\alpha+\beta}{\beta(\beta+1)} \frac{2^{-\beta-1}}{(Am+1)^{\alpha+\beta+1}} \left(Am + \frac{1}{2}\right)^\alpha. \end{aligned}$$

Then, by (11), we have

$$R_m > \left[\frac{1}{B\beta} - 1 - \frac{(1-\beta)B}{3}\right] \frac{2^{-\beta}}{(Am+1)^{\alpha+\beta}} + \left[\frac{1}{B\beta(\beta+1)} - \frac{B}{3}\right] \frac{(\alpha+\beta)2^{-\beta-1}}{(Am+1)^{\alpha+\beta+1}}. \tag{12}$$

Since $0 < B \le 1$ and $0 < \beta \le 1$, we have

$$\frac{1}{B\beta} - 1 - \frac{(1-\beta)B}{3} \ge \frac{1}{\beta} - 1 - \frac{1-\beta}{3} = \frac{3 - 3\beta + \beta^2 - \beta}{3\beta} = \frac{(\beta-1)^2 + 2}{3\beta} > 0,$$

and

$$\frac{1}{B\beta(\beta+1)} - \frac{B}{3} = \frac{3 - \beta(\beta+1)B^2}{3B\beta(\beta+1)} \ge \frac{3-2}{3B\beta(\beta+1)} = \frac{1}{3B\beta(\beta+1)} > 0.$$

Again $A > 0$ gives $\frac{1}{Am+1} > 0$, for every $m \in \mathbb{N}_0$. Hence, by (12), we get $R_m > 0$ for all $m \in \mathbb{N}_0$. The lemma is proved. ■

Lemma 2.2. *For* $0 < A, B, \alpha, \beta \le 1$, *define the weight coefficients* $\omega(A, B, \alpha, \beta, n)$ *and* $\omega(B, A, \beta, \alpha, n)$ *as*

$$\omega(A, B, \alpha, \beta, m) = \sum_{n=0}^\infty \frac{1}{(Am+Bn+1)^{\alpha+\beta}} \left(Bn + \frac{1}{2}\right)^{\beta-1}, \quad m \in \mathbb{N}_0; \tag{13}$$

$$\omega(B,A,\beta,\alpha,n)=\sum_{m=0}^{\infty}\frac{1}{(Am+Bn+1)^{\alpha+\beta}}\left(Am+\frac{1}{2}\right)^{\alpha-1},\quad n\in\mathbb{N}_0. \tag{14}$$

Then

$$\omega(A,B,\alpha,\beta,m)<\frac{1}{B\left(Am+\frac{1}{2}\right)^{\alpha}}B(\alpha,\beta); \tag{15}$$

$$\omega(B,A,\beta,\alpha,n)<\frac{1}{A\left(Bn+\frac{1}{2}\right)^{\beta}}B(\alpha,\beta). \tag{16}$$

Proof. For $m\in\mathbb{N}_0$, define the function $f_m(x)$ by

$$f_m(x)=\frac{1}{(Am+Bx+1)^{\alpha+\beta}}\left(Bx+\frac{1}{2}\right)^{\beta-1},\ \ x\in[0,\infty).$$

Then $f_m(x)$ satisfies the conditions of (10) as follows:
Setting $g(u)=\frac{1}{(1+u)^{\alpha+\beta}}$, for $u\in\left[\frac{1}{2Am+1},\infty\right)$, we obtain

$$(-1)^k g^{(k)}(u)>0\ \text{ and }\ g^{(k)}(\infty)=0,\ \text{ for }\ k=0,1,2,3,4.$$

Then for $0<\beta\le 1$, we have

$$(-1)^k\frac{d^k}{du^k}\left(g(u)u^{\beta-1}\right)>0\ \text{ and }\ \frac{d^k}{du^k}\left(g(u)u^{\beta-1}\right)|_{u=\infty}=0,\ \text{ for }\ k=0,1,2,3,4.$$

Since $f_m(x)=\frac{1}{\left(Am+\frac{1}{2}\right)^{\alpha+1}}\,g(u)\,u^{\beta-1}$, where $u=\frac{Bx+\frac{1}{2}}{Am+\frac{1}{2}}$, we have

$$f_m^{(k)}(x)=\frac{1}{\left(Am+\frac{1}{2}\right)^{\alpha+1+k}}\frac{d^k}{du^k}\left(g(u)u^{\beta-1}\right).$$

Hence $(-1)^k f^{(k)}(x)\ge 0$ and $\lim\limits_{x\to\infty}f^{(k)}(x)=0$, for $k=0,1,2,3,4$.
Now by (10), we have

$$\sum_{n=0}^{\infty}f_m(n)\le\int_0^{\infty}f_m(x)dx+\frac{1}{2}f_m(0)-\frac{1}{12}f_m'(0). \tag{17}$$

Setting $u=\frac{Bx+\frac{1}{2}}{Am+\frac{1}{2}}$, we have

$$\begin{aligned}\int_0^{\infty}f_m(x)dx&=\frac{1}{B\left(Am+\frac{1}{2}\right)^{\alpha}}\int_{\frac{1}{2Am+1}}^{\infty}\frac{1}{(1+u)^{\alpha+\beta}}u^{\beta-1}du\\&=\frac{1}{B\left(Am+\frac{1}{2}\right)^{\alpha}}\left(B(\alpha,\beta)-\int_0^{\frac{1}{2Am+1}}\frac{1}{(1+u)^{\alpha+\beta}}u^{\beta-1}du\right).\end{aligned}$$

Also

$$f_m(0) = \frac{2^{1-\beta}}{(Am+1)^{\alpha+\beta}} ;$$
$$f'_m(0) = -\frac{(\alpha+\beta)B2^{1-\beta}}{(Am+1)^{\alpha+\beta+1}} - \frac{(\beta-1)B2^{2-\beta}}{(Am+1)^{\alpha+\beta}} .$$

Hence by (13), (17) and using Lemma 2.1, we obtain

$$\begin{aligned}\omega(A,B,\alpha,\beta,m) &= \sum_{n=0}^{\infty} f_m(n) \\ &\leq \frac{1}{B\left(Am+\frac{1}{2}\right)^{\alpha}} B(\alpha,\beta) - R_m \\ &< \frac{1}{B\left(Am+\frac{1}{2}\right)^{\alpha}} B(\alpha,\beta) .\end{aligned}$$

This gives (15). Similarly, we get (16). The lemma is proved. ■

Lemma 2.3. *Let* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$. *If* $0<A,B,\alpha,\beta\leq 1$ *and* $0<\varepsilon<q\beta$, *then*

$$\begin{aligned}\sum := &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{(Am+Bn+1)^{\alpha+\beta}} \frac{1}{\left(m+\frac{1}{2A}\right)^{1-\alpha+\frac{\varepsilon}{p}}} \frac{1}{\left(n+\frac{1}{2B}\right)^{1-\beta+\frac{\varepsilon}{q}}} \\ &> \frac{1}{\varepsilon}\frac{2^{\varepsilon}}{A^{\alpha-\frac{\varepsilon}{p}}B^{\beta-\frac{\varepsilon}{q}}} B\left(\alpha+\frac{\varepsilon}{q}, \beta-\frac{\varepsilon}{q}\right) - \mathrm{O}(1).\end{aligned} \tag{18}$$

Proof. For $0<A,B,\alpha,\beta\leq 1$ and $0<\varepsilon<q\beta$, we have

$$\begin{aligned}&\sum \\ &> \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_m^{m+1}\int_n^{n+1} \frac{1}{(Ax+By+1)^{\alpha+\beta}} \frac{1}{\left(x+\frac{1}{2A}\right)^{1-\alpha+\frac{\varepsilon}{p}}} \frac{1}{\left(y+\frac{1}{2B}\right)^{1-\beta+\frac{\varepsilon}{q}}} dxdy \\ &= \int_0^{\infty}\int_0^{\infty} \frac{1}{\left(Ax+\frac{1}{2}\right)^{\alpha+\beta}\left(1+\frac{By+\frac{1}{2}}{Ax+\frac{1}{2}}\right)^{\alpha+\beta}} \frac{A^{1-\alpha+\frac{\varepsilon}{p}}}{\left(Ax+\frac{1}{2}\right)^{1-\alpha+\frac{\varepsilon}{p}}} \frac{B^{1-\beta+\frac{\varepsilon}{q}}}{\left(By+\frac{1}{2}\right)^{1-\beta+\frac{\varepsilon}{q}}} dxdy \\ &= A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}} \int_0^{\infty} \frac{1}{\left(Ax+\frac{1}{2}\right)^{1+\beta+\frac{\varepsilon}{p}}} \left[\int_0^{\infty} \frac{1}{\left(1+\frac{By+\frac{1}{2}}{Ax+\frac{1}{2}}\right)^{\alpha+\beta}} \frac{1}{\left(By+\frac{1}{2}\right)^{1-\beta+\frac{\varepsilon}{q}}} dy\right] dx.\end{aligned}$$

For $x\in(0,\infty)$, setting $t=\frac{By+\frac{1}{2}}{Ax+\frac{1}{2}}$, we have

$$\begin{aligned}\sum &> \frac{A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}}}{B} \int_0^{\infty} \frac{1}{\left(Ax+\frac{1}{2}\right)^{1+\varepsilon}} \left[\int_{\frac{1}{2Ax+1}}^{\infty} \frac{1}{(1+t)^{\alpha+\beta}} \frac{1}{t^{1-\beta+\frac{\varepsilon}{q}}} dt\right] dx \\ &= \frac{A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}}}{B} \int_0^{\infty} \frac{1}{\left(Ax+\frac{1}{2}\right)^{1+\varepsilon}} dx \int_0^{\infty} \frac{t^{\beta-\frac{\varepsilon}{q}-1}}{(1+t)^{\alpha+\beta}} dt\end{aligned}$$

$$-\frac{A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}}}{B}\int_0^\infty \frac{1}{\left(Ax+\frac{1}{2}\right)^{1+\varepsilon}}\left[\int_0^{\frac{1}{2Ax+1}} \frac{t^{\beta-\frac{\varepsilon}{q}-1}}{(1+t)^{\alpha+\beta}}dt\right]dx$$
$$= I_1 - I_2 \ \ (\text{ say}) \ . \tag{19}$$

By (9), we get

$$\begin{aligned} I_1 &= \frac{A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}}}{B}\int_0^\infty \frac{1}{\left(Ax+\frac{1}{2}\right)^{1+\varepsilon}}dx\int_0^\infty \frac{t^{\beta-\frac{\varepsilon}{q}-1}}{(1+t)^{\alpha+\beta}}dt \\ &= \frac{1}{\varepsilon}\frac{2^\varepsilon}{A^{\alpha-\frac{\varepsilon}{p}}B^{\beta-\frac{\varepsilon}{q}}}B\left(\alpha+\frac{\varepsilon}{q},\beta-\frac{\varepsilon}{q}\right). \end{aligned}$$

Also

$$\begin{aligned} I_2 &= \frac{A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}}}{B}\int_0^\infty \frac{1}{\left(Ax+\frac{1}{2}\right)^{1+\varepsilon}}\left[\int_0^{\frac{1}{2Ax+1}} \frac{t^{\beta-\frac{\varepsilon}{q}-1}}{(1+t)^{\alpha+\beta}}dt\right]dx \\ &> \frac{A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}}}{B}\int_0^\infty \frac{1}{\left(Ax+\frac{1}{2}\right)^{1+\varepsilon}}\left[\int_0^{\frac{1}{2Ax+1}} t^{\beta-\frac{\varepsilon}{q}-1}dt\right]dx \\ &= \frac{A^{1-\alpha+\frac{\varepsilon}{p}}B^{1-\beta+\frac{\varepsilon}{q}}2^{\frac{\varepsilon}{q}-\beta}}{B\left(\beta-\frac{\varepsilon}{q}\right)}\int_0^\infty \left(Ax+\frac{1}{2}\right)^{-\beta-\frac{\varepsilon}{p}-1}dx \\ &= \frac{2^\varepsilon}{A^{\alpha-\frac{\varepsilon}{p}}B^{\beta-\frac{\varepsilon}{q}}}\frac{1}{\left(\beta-\frac{\varepsilon}{q}\right)\left(\beta+\frac{\varepsilon}{p}\right)} \\ &= \mathrm{O}(1). \end{aligned}$$

Hence by (19), we obtain (18). The lemma is proved. ∎

3. Main result

In this section, we shall establish a generalization of inequality (1) that generalizes the inequalities given in [7]-[10] and the constant factor obtained is the best possible. The equivalent form of the inequality is also obtained.

Theorem 3.1. *Let* $p > 1, \frac{1}{p}+\frac{1}{q} = 1$. *If* $0 < A, B, \alpha, \beta \leq 1$ *and* $a_n, b_n \geq 0$ *satisfy* $0 < \sum_{n=0}^\infty \left(n+\frac{1}{2A}\right)^{p(1-\alpha)-1} a_n^p < \infty$ *and* $0 < \sum_{n=0}^\infty \left(n+\frac{1}{2B}\right)^{q(1-\beta)-1} b_n^q < \infty$, *then*

$$\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{a_m b_n}{(Am+Bn+1)^{\alpha+\beta}} \tag{20}$$
$$< \frac{B(\alpha,\beta)}{A^\alpha B^\beta}\left\{\sum_{n=0}^\infty \left(n+\frac{1}{2A}\right)^{p(1-\alpha)-1} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^\infty \left(n+\frac{1}{2B}\right)^{q(1-\beta)-1} b_n^q\right\}^{\frac{1}{q}},$$

where the constant factor $\frac{B(\alpha,\beta)}{A^\alpha B^\beta}$ *is the best possible.*

Proof. By Hölder's inequality and (13), (14), we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(Am+Bn+1)^{\alpha+\beta}}\\
&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(Am+Bn+1)^{\alpha+\beta}}\left\{\frac{\left(Bn+\frac{1}{2}\right)^{\frac{\beta-1}{p}}}{\left(Am+\frac{1}{2}\right)^{\frac{\alpha-1}{q}}}a_m\right\}\left\{\frac{\left(Am+\frac{1}{2}\right)^{\frac{\alpha-1}{q}}}{\left(Bn+\frac{1}{2}\right)^{\frac{\beta-1}{p}}}b_n\right\}\\
&\le\left\{\sum_{m=0}^{\infty}\left[\sum_{n=0}^{\infty}\frac{1}{(Am+Bn+1)^{\alpha+\beta}}\left(Bn+\frac{1}{2}\right)^{\beta-1}\right]\left(Am+\frac{1}{2}\right)^{(p-1)(1-\alpha)}a_m^p\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\sum_{n=0}^{\infty}\left[\sum_{m=0}^{\infty}\frac{1}{(Am+Bn+1)^{\alpha+\beta}}\left(Am+\frac{1}{2}\right)^{\alpha-1}\right]\left(Bn+\frac{1}{2}\right)^{(q-1)(1-\beta)}b_n^q\right\}^{\frac{1}{q}}\\
&=\left\{\sum_{m=0}^{\infty}\omega(A,B,\alpha,\beta,m)\left(Am+\frac{1}{2}\right)^{(p-1)(1-\alpha)}a_m^p\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\sum_{n=0}^{\infty}\omega(B,A,\beta,\alpha,n)\left(Bn+\frac{1}{2}\right)^{(q-1)(1-\beta)}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence by (15) and (16), we get (20).

For $0<\varepsilon<q\beta$, setting $\tilde{a}_n=\left(n+\frac{1}{2A}\right)^{\alpha-1-\frac{\varepsilon}{p}}$ and $\tilde{b}_n=\left(n+\frac{1}{2B}\right)^{\beta-1-\frac{\varepsilon}{q}}$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{p(1-\alpha)-1}\tilde{a}_n^p&=\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{-1-\varepsilon}\\
&<(2A)^{1+\varepsilon}+\sum_{n=1}^{\infty}\int_{n-1}^{n}\left(x+\frac{1}{2A}\right)^{-1-\varepsilon}dx\\
&=(2A)^{1+\varepsilon}+\int_{0}^{\infty}\left(x+\frac{1}{2A}\right)^{-1-\varepsilon}dx\\
&=\frac{1}{\varepsilon}(2A)^{\varepsilon}(2A\varepsilon+1).
\end{aligned}$$

Similarly, we have

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{q(1-\beta)-1}\tilde{b}_n^q<\frac{1}{\varepsilon}(2B)^{\varepsilon}(2B\varepsilon+1).$$

Hence we obtain

$$\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{p(1-\alpha)-1}\tilde{a}_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{q(1-\beta)-1}\tilde{b}_n^q\right\}^{\frac{1}{q}}$$

$$< \frac{1}{\varepsilon} (2A)^{\frac{\varepsilon}{p}} (2B)^{\frac{\varepsilon}{q}} (2A\varepsilon + 1)^{\frac{1}{p}} (2B\varepsilon + 1)^{\frac{1}{q}} . \tag{21}$$

If the constant factor $\frac{B(\alpha,\beta)}{A^\alpha B^\beta}$ in (20) is not the best possible, then there exists a positive constant K such that $K < \frac{B(\alpha,\beta)}{A^\alpha B^\beta}$ and (20) still remains valid if $\frac{B(\alpha,\beta)}{A^\alpha B^\beta}$ is replaced by K. In particular by (18) and (21), we have

$$\begin{aligned}
&\frac{2^\varepsilon}{A^{\alpha-\frac{\varepsilon}{p}} B^{\beta-\frac{\varepsilon}{q}}} B\left(\alpha + \frac{\varepsilon}{q}, \beta - \frac{\varepsilon}{q}\right) - \varepsilon \mathrm{O}(1) \\
&< \varepsilon \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(Am + Bn + 1)^{\alpha+\beta}} \tilde{a}_m \tilde{b}_n \\
&< \varepsilon K \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{p(1-\alpha)-1} \tilde{a}_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{q(1-\beta)-1} \tilde{b}_n^q \right\}^{\frac{1}{q}} \\
&< K (2A)^{\frac{\varepsilon}{p}} (2B)^{\frac{\varepsilon}{q}} (2A\varepsilon + 1)^{\frac{1}{p}} (2B\varepsilon + 1)^{\frac{1}{q}} .
\end{aligned}$$

Then $\frac{B(\alpha,\beta)}{A^\alpha B^\beta} \le K$ as $\varepsilon \to 0^+$. This contradiction shows that the constant factor $\frac{B(\alpha,\beta)}{A^\alpha B^\beta}$ in (20) is the best possible. This proves the theorem. ■

Corollary 3.2. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < A, B \le C, 0 < \alpha, \beta \le 1$ *and* $a_n, b_n \ge 0$ *satisfy* $0 < \sum_{n=0}^{\infty} \left(n + \frac{C}{2A}\right)^{p(1-\alpha)-1} a_n^p < \infty$ *and* $0 < \sum_{n=0}^{\infty} \left(n + \frac{C}{2B}\right)^{q(1-\beta)-1} b_n^q < \infty$*, then*

$$\begin{aligned}
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m b_n}{(Am + Bn + C)^{\alpha+\beta}} &< \frac{B(\alpha, \beta)}{A^\alpha B^\beta} \left\{ \sum_{n=0}^{\infty} \left(n + \frac{C}{2A}\right)^{p(1-\alpha)-1} a_n^p \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \sum_{n=0}^{\infty} \left(n + \frac{C}{2B}\right)^{q(1-\beta)-1} b_n^q \right\}^{\frac{1}{q}} ,
\end{aligned} \tag{22}$$

where the constant factor $\frac{B(\alpha,\beta)}{A^\alpha B^\beta}$ *is the best possible.*

Theorem 3.3. *Let* $p > 1, \frac{1}{p} + \frac{1}{q} = 1$. *If* $0 < A,\ B,\ \alpha,\ \beta \le 1$ *and* $a_n \ge 0$ *satisfy* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{p(1-\alpha)-1} a_n^p < \infty$*, then we obtain the equivalent inequality of* (20) *as follows:*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{p\beta-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(Am + Bn + 1)^{\alpha+\beta}} \right]^p \\
&\qquad < \left[\frac{B(\alpha, \beta)}{A^\alpha B^\beta} \right]^p \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{p(1-\alpha)-1} a_n^p ,
\end{aligned} \tag{23}$$

where the constant factor $\left[\frac{B(\alpha,\beta)}{A^\alpha B^\beta}\right]^p$ *is the best possible.*

Proof. Setting $b_n = \left(n+\frac{1}{2B}\right)^{p\beta-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\alpha+\beta}}\right]^{p-1}$, we obtain from (20) that

$$\begin{aligned}
0 &< \sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{q(1-\beta)-1} b_n^q \\
&= \sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{p\beta-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\alpha+\beta}}\right]^p \\
&= \sum_{n=0}^{\infty} b_n \sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\alpha+\beta}} \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(Am+Bn+1)^{\alpha+\beta}} \\
&\le \frac{B(\alpha,\beta)}{A^{\alpha}B^{\beta}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{p(1-\alpha)-1} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=n_0}^{\infty}\left(n+\frac{1}{2B}\right)^{q(1-\beta)-1} b_n^q\right\}^{\frac{1}{q}}.
\end{aligned} \tag{24}$$

Thus

$$\begin{aligned}
0 &< \left[\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{q(1-\beta)-1} b_n^q\right]^{\frac{1}{p}} \\
&= \left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{p\beta-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\alpha+\beta}}\right]^p\right\}^{\frac{1}{p}} \\
&\le \frac{B(\alpha,\beta)}{A^{\alpha}B^{\beta}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{p(1-\alpha)-1} a_n^p\right\}^{\frac{1}{p}} < \infty.
\end{aligned} \tag{25}$$

It follows that (24) takes the form of strict inequality by using (20), so does (25). Hence we get (23).

On the other hand, if (23) holds, then by Hölder's inequality, we have

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(Am+Bn+1)^{\alpha+\beta}} \\
&= \sum_{n=0}^{\infty}\left[\left(n+\frac{1}{2B}\right)^{\beta-\frac{1}{p}}\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\alpha+\beta}}\right]\left[\left(n+\frac{1}{2B}\right)^{\frac{1}{p}-\beta} b_n\right] \\
&\le \left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{p\beta-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\alpha+\beta}}\right]^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{q(1-\beta)-1} b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}$$

Hence by (23), (20) yields. Thus it follows that (20) and (23) are equivalent. Since the constant factor in (20) is the best possible, hence the constant factor in (23) is the best possible. The theorem is proved. ■

Corollary 3.4. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ 0 < A,\ B \le C,\ 0 < \alpha,\ \beta \le 1$ *and* $a_n \ge 0$ *satisfy* $0 < \sum_{n=0}^{\infty} \left(n + \frac{C}{2A}\right)^{p(1-\alpha)-1} a_n^p < \infty$ *then*

$$\sum_{n=0}^{\infty} \left(n + \frac{C}{2B}\right)^{p\beta-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(Am + Bn + C)^{\alpha+\beta}}\right]^p < \left[\frac{B(\alpha,\beta)}{A^{\alpha}B^{\beta}}\right]^p \sum_{n=0}^{\infty} \left(n + \frac{C}{2A}\right)^{p(1-\alpha)-1} a_n^p, \tag{26}$$

where the constant factor $\left[\frac{B(\alpha,\beta)}{A^{\alpha}B^{\beta}}\right]^p$ *is the best possible.*

4. Some particular results

In this section, we shall derive some particular inequalities, by considering different values of the parameters α and β.

Setting $\alpha = \beta = 1$ in Theorem 3.1 and Theorem 3.3, we get the following inequalities.

Theorem 4.1. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < A, B \le 1$ *and* $a_n, b_n \ge 0$ *satisfy* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{-1} a_n^p < \infty$ *and* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{-1} b_n^q < \infty$*, then the following two equivalent inequalities hold*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(Am+Bn+1)^2} < \frac{1}{AB} \left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2A}\right)^{-1} a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=0}^{\infty} \left(n+\frac{1}{2B}\right)^{-1} b_n^q\right\}^{\frac{1}{q}}; \tag{27}$$

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{p-1} \left[\sum_{m=0}^{\infty} \frac{a_m}{(Am + Bn + 1)^2}\right]^p < \frac{1}{(AB)^p} \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{-1} a_n^p, \tag{28}$$

where the constant factors $(AB)^{-1}$ *and* $(AB)^{-p}$ *are the best possible.*

Setting $\alpha = \frac{\lambda}{p}, \beta = \frac{\lambda}{q}$ in Theorem 3.1 and Theorem 3.3, we get the following inequalities.

Theorem 4.2. *If* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, 0 < A, B \le 1, 0 < \lambda \le \min\{p, q\}$ *and* $a_n, b_n \ge 0$ *satisfy* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{p-\lambda-1} a_n^p < \infty$ *and* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{q-\lambda-1} b_n^q < \infty$*, then the following two equivalent inequalities hold*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(Am + Bn + 1)^{\lambda}} < \frac{1}{A^{\frac{\lambda}{p}} B^{\frac{\lambda}{q}}} B\left(\frac{\lambda}{p}, \frac{\lambda}{q}\right) \left\{\sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{p-\lambda-1} a_n^p\right\}^{\frac{1}{p}}$$

$$\times\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{q-\lambda-1}b_n^q\right\}^{\frac{1}{q}}; \tag{29}$$

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{\lambda(p-1)-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\lambda}}\right]^p < \left[\frac{1}{A^{\frac{\lambda}{p}}B^{\frac{\lambda}{q}}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{p-\lambda-1}a_n^p, \tag{30}$$

where the constant factors $\frac{1}{A^{\frac{\lambda}{p}}B^{\frac{\lambda}{q}}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ *and* $\left[\frac{1}{A^{\frac{\lambda}{p}}B^{\frac{\lambda}{q}}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p$ *are the best possible.*

Remark 4.3. For $p=q=2$, (29) reduces to (5) and for $A=B=1$, it reduces to (7). Hence it is a generalization of (3) and (8), but not a generalization of (1).

Setting $\alpha=\frac{\lambda}{q},\beta=\frac{\lambda}{p}$ in Theorem 3.1 and Theorem 3.3, we get the following inequalities.

Theorem 4.4. *If* $p>1, \frac{1}{p}+\frac{1}{q}=1$, $0<A,B\leq 1$, $0<\lambda\leq\min\{p,q\}$ *and* a_n, $b_n\geq 0$ *satisfy* $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{(p-1)(1-\lambda)}a_n^p<\infty$ *and* $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{(q-1)(1-\lambda)}b_n^q$ $<\infty$, *then the following two equivalent inequalities hold*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{(Am+Bn+1)^{\lambda}}<\frac{1}{A^{\frac{\lambda}{q}}B^{\frac{\lambda}{p}}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{(p-1)(1-\lambda)}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{(q-1)(1-\lambda)}b_n^q\right\}^{\frac{1}{q}}; \tag{31}$$

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{\lambda-1}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\lambda}}\right]^p < \left[\frac{1}{A^{\frac{\lambda}{q}}B^{\frac{\lambda}{p}}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{(p-1)(1-\lambda)}a_n^p, \tag{32}$$

where the constant factors $\frac{1}{A^{\frac{\lambda}{q}}B^{\frac{\lambda}{p}}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)$ *and* $\left[\frac{1}{A^{\frac{\lambda}{q}}B^{\frac{\lambda}{p}}}B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\right]^p$ *are the best possible.*

Remark 4.5. For $A=B=1$, (31) reduces to (6) and for $p=q=2$, (31) reduces to (5). Hence it is a generalization of (1).

Setting $\alpha=\frac{p+\lambda-2}{p},\beta=\frac{q+\lambda-2}{q}$ in Theorem 3.1 and Theorem 3.3, we get the following inequalities.

Theorem 4.6. *If* $p>1, \frac{1}{p}+\frac{1}{q}=1$, $0<A,B\le 1$, $2-\min\{p,q\}<\lambda\le 2$ *and* $a_n, b_n \ge 0$ *satisfy* $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{1-\lambda} a_n^p<\infty$ *and* $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{1-\lambda} b_n^q<\infty$, *then the following two equivalent inequalities hold*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(Am+Bn+1)^{\lambda}}<C_1\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{1-\lambda}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{1-\lambda}b_n^q\right\}^{\frac{1}{q}}; \tag{33}$$

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{(p-1)(\lambda-1)}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\lambda}}\right]^p<[C_1]^p\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{1-\lambda}a_n^p, \tag{34}$$

where $C_1=A^{\frac{2-p-\lambda}{p}}B^{\frac{2-q-\lambda}{q}}B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *and the constant factors* C_1 *and* $[C_1]^p$ *are the best possible.*

Remark 4.7. For $A=B=1$, (33) reduces to (4) and for $p=q=2$, (33) reduces to (5). Hence it is a generalization of (1).

Setting $\alpha=\frac{q+\lambda-2}{q}, \beta=\frac{p+\lambda-2}{p}$ in Theorem 3.1 and Theorem 3.3, we get the following inequalities.

Theorem 4.8. *If* $p>1, \frac{1}{p}+\frac{1}{q}=1$, $0<A,B\le 1$, $2-\min\{p,q\}<\lambda\le 2$ *and* $a_n, b_n\ge 0$ *satisfy* $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{(p-1)(2-\lambda)-1}a_n^p<\infty$ *and* $0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{(q-1)(2-\lambda)-1}b_n^q<\infty$, *then the following two equivalent inequalities hold*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(Am+Bn+1)^{\lambda}}<C_2\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{(p-1)(2-\lambda)-1}a_n^p\right\}^{\frac{1}{p}} \times\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{(q-1)(2-\lambda)-1}b_n^q\right\}^{\frac{1}{q}}; \tag{35}$$

$$\sum_{n=0}^{\infty}\left(n+\frac{1}{2B}\right)^{p+\lambda-3}\left[\sum_{m=0}^{\infty}\frac{a_m}{(Am+Bn+1)^{\lambda}}\right]^p <[C_2]^p\sum_{n=0}^{\infty}\left(n+\frac{1}{2A}\right)^{(p-1)(2-\lambda)-1}a_n^p, \tag{36}$$

where $C_2=A^{\frac{2-q-\lambda}{q}}B^{\frac{2-p-\lambda}{p}}B\left(\frac{p+\lambda-2}{p},\frac{q+\lambda-2}{q}\right)$ *and the constant factors* C_2 *and* $[C_2]^p$ *are the best possible.*

Remark 4.9. For $p=q=2$, (35) reduces to (5) and for $A=B=\lambda=1$, it reduces to (8). Hence it is a generalization of (3) and (8), but not a generalization of (1).

Setting $\alpha = \frac{\lambda-1}{2} + \frac{1}{p}, \beta = \frac{\lambda-1}{2} + \frac{1}{q}$ in Theorem 3.1 and Theorem 3.3, we get the following inequalities.

Theorem 4.10. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1,\ 0 < A, B \leq 1,\ 1 - 2\min\{\frac{1}{p}, \frac{1}{q}\} < \lambda \leq 1 + 2\min\{\frac{1}{p}, \frac{1}{q}\}$ *and* $a_n, b_n \geq 0$ *satisfy* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{\frac{1}{2}p(3-\lambda)-2} a_n^p < \infty$ *and* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{\frac{1}{2}q(3-\lambda)-2} b_n^q < \infty$, *then the following two equivalent inequalities hold*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(Am + Bn + 1)^{\lambda}} < C_3 \left\{\sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{\frac{1}{2}p(3-\lambda)-2} a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{\frac{1}{2}q(3-\lambda)-2} b_n^q\right\}^{\frac{1}{q}}; \tag{37}$$

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{\frac{1}{2}p(\lambda+1)-2} \left[\sum_{m=0}^{\infty} \frac{a_m}{(Am + Bn + 1)^{\lambda}}\right]^p < [C_3]^p \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{\frac{1}{2}p(3-\lambda)-2} a_n^p, \tag{38}$$

where $C_3 = A^{\frac{1-\lambda}{2} - \frac{1}{p}} B^{\frac{1-\lambda}{2} - \frac{1}{q}} B\left(\frac{\lambda-1}{2} + \frac{1}{p}, \frac{\lambda-1}{2} + \frac{1}{q}\right)$ *and the constant factors* C_3 *and* $[C_3]^p$ *are the best possible.*

Remark 4.11. For $p = q = 2$, (37) reduces to (5) and for $A = B = \lambda = 1$, (37) reduces to (8) . Hence it is a generalization of (3) and (8), but not a generalization of (1).

Setting $\alpha = \frac{\lambda-1}{2} + \frac{1}{q}, \beta = \frac{\lambda-1}{2} + \frac{1}{p}$ in Theorem 3.1 and Theorem 3.3, we get the following inequalities.

Theorem 4.12. *If* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, 0 < A, B \leq 1, 1 - 2\min\{\frac{1}{p}, \frac{1}{q}\} < \lambda \leq 1 + 2\min\{\frac{1}{p}, \frac{1}{q}\}$ *and* $a_n, b_n \geq 0$ *satisfy* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{\frac{1}{2}p(1-\lambda)} a_n^p < \infty$ *and* $0 < \sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{\frac{1}{2}q(1-\lambda)} b_n^q < \infty$, *then the following two equivalent inequalities hold*

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{a_m b_n}{(Am + Bn + 1)^{\lambda}} < C_4 \left\{\sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{\frac{1}{2}p(1-\lambda)} a_n^p\right\}^{\frac{1}{p}} \times \left\{\sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{\frac{1}{2}q(1-\lambda)} b_n^q\right\}^{\frac{1}{q}}; \tag{39}$$

$$\sum_{n=0}^{\infty} \left(n + \frac{1}{2B}\right)^{\frac{1}{2}p(\lambda-1)} \left[\sum_{m=0}^{\infty} \frac{a_m}{(Am + Bn + 1)^{\lambda}}\right]^p$$

$$< [C_4]^p \sum_{n=0}^{\infty} \left(n + \frac{1}{2A}\right)^{\frac{1}{2}p(1-\lambda)} a_n^p, \qquad (40)$$

where $C_4 = A^{\frac{1-\lambda}{2}-\frac{1}{q}} B^{\frac{1-\lambda}{2}-\frac{1}{p}} B\left(\frac{\lambda-1}{2}+\frac{1}{p}, \frac{\lambda-1}{2}+\frac{1}{q}\right)$ *and the constant factors* C_4 *and* $[C_4]^p$ *are the best possible.*

Remark 4.13. For $p = q = 2$, (39) reduces to (5) and for $A = B = \lambda = 1$, (39) reduces to (1). Hence it is a generalization of both (1) and (3).

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