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On Polynomials over Abelian Rings

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Abstract. In this paper we study various annihilator conditions over nearrings of polynomials and formal power series over an abelian ring, which were originally used by Rickart and Kaplansky to abstract the algebraic properties of von Neumann algebra. These results are somewhat surprising since, in contrast to the polynomial ring case, the nearrings of polynomials and formal power series have substitution for its "multiplication" operation. As a consequence we obtain a generalization of [10].

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1. Introduction

Throughout this paper all rings are associative with unity and all nearrings are left nearrings. We use R and N to denote a ring and nearring, respectively. Recall that a ring or nearring is said to be *reduced* if it has no nonzero nilpotent elements. Recall from [15] that R is a *Baer* ring if the right annihilator of every nonempty subset of R is generated, as a right ideal, by an idempotent. This definition is left-right symmetric. The study of Baer rings has its roots in functional analysis [4, 15]. In [15] Kaplansky introduced Baer rings to abstract various properties of von Neumann algebras and complete $*$-regular rings.

The class of Baer rings includes the von Neumann algebras (e.g., the algebra of all bounded operators on a Hilbert space), the commutative C^*-algebra $C(T)$ of continuous complex valued functions on a Stonian space T, and the regular rings whose lattice of principal right ideals is complete (e.g., regular rings which are continuous or right self-injective). Armendariz obtained the following result: Let R be a reduced ring. Then $R[x]$ is a Baer ring if and only if R is a Baer ring [2, Theorem B].

Clark [11] defines a ring to be *quasi-Baer* if the left annihilator of every ideal is generated, as a left ideal, by an idempotent. Moreover, he shows the left-right symmetry of this condition by proving that R is quasi-Baer if and only if the right annihilator of every right ideal is generated, as a right ideal, by an idempotent. He then uses the quasi-Baer concept to characterize when a finite-dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Birkenmeier, Kim and Park proved that, a ring R is quasi-Baer if and only if $R[x]$ is a quasi-Baer ring [7, Theorem 1.8].

Birkenmeier and Huang [8] defined the *Baer-type annihilator conditions* in the class of nearrings as follows (for a nonempty $S \subseteq N$, let $r_N(S) = \{a \in N \mid Sa = 0\}$ and $\ell_N(S) = \{a \in N \mid aS = 0\}$):

(1) $N \in \mathcal{B}_{r1}$ if $r_N(S) = eN$ for some idempotent $e \in N$ for every $\phi \neq S \subseteq N$;

(2) $N \in \mathcal{B}_{r2}$ if $r_N(S) = r_N(e)$ for some idempotent $e \in N$ for every $\phi \neq S \subseteq N$;

(3) $N \in \mathcal{B}_{\ell 1}$ if $\ell_N(S) = Ne$ for some idempotent $e \in N$ for every $\phi \neq S \subseteq N$;

(4) $N \in \mathcal{B}_{\ell 2}$ if $\ell_N(S) = \ell_N(e)$ for some idempotent $e \in N$ for every $\phi \neq S \subseteq N$.

If N is a ring with unity, then $N \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$ is equivalent to N being a Baer ring. When S is an ideal of N, the *quasi-Baer annihilator conditions* on nearrings are also defined similarly except replacing $\mathcal{B}$ by $q\mathcal{B}$. If N is a ring with unity, then $N \in q\mathcal{B}_{r1} \cup q\mathcal{B}_{r2} \cup q\mathcal{B}_{\ell 1} \cup q\mathcal{B}_{\ell 2}$ is equivalent to N being a quasi-Baer ring.

Unless specifically indicated otherwise, $R[x]$ denotes the left nearring of polynomials $(R[x], +, \circ)$ with coefficients from R and

$$R_0[x] = \{f \in R[x] \mid f \text{ has zero constant term}\}$$

is the 0-symmetric left nearring of polynomials with coefficients in R. Also, the collection of all power series with positive orders using the operations of addition and substitution is denoted by $R_0[[x]]$ unless specifically indicated otherwise (i.e., $R_0[[x]]$ denotes $(R_0[[x]], +, \circ)$). Observe that the system $(R_0[[x]], +, \circ)$ is a zero-symmetric left nearring. For expositions using this approach, one may refer to [12, 16].

According to Bell [3] a ring R is said to have the *insertion of factors property* (or simply IFP) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Reduced rings and commutative rings are IFP rings. A ring R is called *abelian*, if each idempotent is central. The following implications hold:

$$\text{reduced (resp. commutative)} \Rightarrow \text{ IFP} \Rightarrow \text{abelian.}$$

There exist some examples to show that the converse of the above implications do not hold.

In this paper first we show that a ring R is abelian if and only if the nearring $R_0[x]$ is abelian if and only if the nearring $R_0[[x]]$ is abelian. Then for an abelian ring R we show that: (1) If $R_0[x]$ or $R_0[[x]]$ satisfies any one of the Baer-type annihilator conditions, then R is a Baer ring. (2) If $R[x] \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2}$, then R is a Baer ring. Moreover for an IFP ring R it is shown that, R is quasi-Baer if and only if $R[x] \in q\mathcal{B}_{r2}$ if and only if $(R[x], +, \cdot)$ is quasi-Baer.

2. Polynomials and formal power series over abelian rings

Lemma 2.1. *Let R be an abelian ring and $(x)\mathcal{E} = \sum_{i=1}^{\infty} e_i x^i$ an idempotent of $R_0[[x]]$. Then $(x)\mathcal{E} = e_1 x$.*

Proof. Since $(x)\mathcal{E} \circ (x)\mathcal{E} = (x)\mathcal{E}$, we have

(1) $e_1^2 = e_1$,
(2) $e_1e_2 + e_2e_1^2 = e_2$,
(3) $e_1e_3 + e_2(e_1e_2 + e_2e_1) + e_3e_1^3 = e_3$,
(4) $e_1e_4 + e_2(e_1e_3 + e_3e_1 + e_2^2) + e_3(e_1^2e_2 + e_1e_2e_1 + e_2e_1^2) + e_4e_1^4 = e_4$,
(5) $\vdots$
(6) $e_1e_n + e_2 \sum_{i_1+i_2=n} e_{i_1}e_{i_2} + \cdots + e_{n-1} \sum_{i_1+\cdots+i_{n-1}=n} e_{i_1} \cdots e_{i_{n-1}} + e_n e_1^n = e_n$,
(7) $\vdots$

Since R is abelian and e_1 an idempotent, Eq.(2) implies that $e_2 = (e_2 + e_2)e_1 = e_2'e_1$. Since R is abelian and $e_1^2 = e_1$, Eq.(3) implies that $e_3 = e_1e_3 + e_2(e_1e_2 + e_2e_1) + e_3e_1^3 = (e_3 + e_2(e_2 + e_2) + e_3)e_1 = e_3'e_1$. Since $e_2 = e_2'e_1$ and $e_3 = e_3'e_1$ and R is abelian, Eq.(4) implies that $e_4 = e_4'e_1$. Continuing this process yields $e_i = e_i'e_1$ for each $i \geq 1$. Hence $(x)\mathcal{E} = e_1x + \cdots + e_nx^n + \cdots = e_1e_1x + e_2'e_1x^2 + \cdots + e_n'e_1x^n + \cdots = (e_1x) \circ (e_1x + e_2'x^2 + \cdots + e_n'x^n + \cdots)$. Thus $(x)\mathcal{E} = e_1x \circ (x)\mathcal{E} = (x)\mathcal{E} \circ e_1x$. Now by using Eqs. (2)-(7), we have $(x)\mathcal{E} = e_1x$. ■

If R is a ring, then for each $a \in R$, $a \circ a = a$. Now if $a \neq b$, then $a \circ b = b$ and $b \circ a = a$. Hence the nearring $(R[x], +, \circ)$ is not abelian.

Corollary 2.2. *Let R be a ring. Then the following are equivalent:*

(1) *R is an abelian ring;*
(2) *$(R_0[x], +, \circ)$ is abelian;*
(3) *$(R_0[[x]], +, \circ)$ is abelian.*

Proof. It follows from Lemma 2.1. ■

Lemma 2.3. *Let R be an abelian ring with unity and $A \subseteq R_0[x]$ or $0 \in A \subseteq R[x]$.*

(1) *If $r_{R[x]}(A) = r_{R[x]}((x)\mathcal{E})$, for an idempotent $(x)\mathcal{E} \in R[x]$, then $(x)\mathcal{E} = ex$.*

(2) *If $r_{R[x]}(A) = (x)\mathcal{E} \circ R[x]$, for an idempotent $(x)\mathcal{E} \in R[x]$, then $(x)\mathcal{E} = ex$.*

Proof. (1) Let $(x)g = b_0 + b_1x + \cdots + b_mx^m \in r_{R[x]}(A)$. Then $b_0 = 0$. Let $(x)\mathcal{E} = e_0 + e_1x + \cdots + e_nx^n$. Since $(x)\mathcal{E} \circ (x)\mathcal{E} = (x)\mathcal{E}$, we have $(x)\mathcal{E} \circ (e_0 + (e_1 - 1)x + e_2x^2 + \cdots + e_nx^n) = 0$. Hence $(e_0 + (e_1 - 1)x + e_2x^2 + \cdots + e_nx^n) \in r_{R[x]}((x)\mathcal{E}) = r_{R[x]}(A)$, and by the above argument, $e_0 = 0$. Then by Lemma 2.1, $(x)\mathcal{E} = e_1x$.

(2) Since $A \subseteq R_0[x]$ and $(x)\mathcal{E} \in r_{R[x]}(A)$, hence $(x)\mathcal{E} \in R_0[x]$. Now the result follows from Lemma 2.1. ■

Proposition 2.4. *Let R be an abelian ring with unity.*

(1) *If $R[x] \in \mathcal{B}_{r2}$, then R is a Baer ring.*

(2) *If $R[x] \in \mathcal{B}_{r1}$, then R is a Baer ring.*

Proof. (1) Let $S \subseteq R$ and $S_X = \{sx | s \in S\} \subseteq R[x]$. By Lemma 2.3 and our assumption, there exists an idempotent $(x)\mathcal{E} = ex \in R[x]$ such that $r_{R[x]}(S_X) = r_{R[x]}(ex)$.

Now we show that $\ell_R(S) = \ell_R(e)$. Let $a \in \ell_R(e)$. Then $ex \circ ax = aex = 0$. Hence $ax \in r_{R[x]}(ex) = r_{R[x]}(S_X)$. Therefore $0 = sx \circ ax = asx$ and so $as = 0$ for all $s \in S$. Hence $a \in \ell_R(S)$. Now let $b \in \ell_R(S)$. Then $sx \circ bx = bsx = 0$, for all $s \in S$. Thus $bx \in r_{R[x]}(S_X) = r_{R[x]}(ex)$, and so $0 = ex \circ bx = bex = 0$. Hence $b \in \ell_R(e)$ and $\ell_R(S) \subseteq \ell_R(e)$. Therefore $\ell_R(S) = \ell_R(e)$ and $R \in \mathcal{B}_{r2}$. Since R has unity, hence it is Baer.

Let $S \subseteq R$ and $S_X = \{sx | s \in X\} \subseteq R[x]$. By Lemma 2.3 and our assumption, there exists an idempotent $(x)\mathcal{E} = ex \in R[x]$ such that $r_{R[x]}(S_X) = ex \circ R_[x]$.

Now we show that $\ell_R(S) = Re$. Since $r_{R[x]}(S_X) = (ex) \circ R[x]$, we have $0 = (sx) \circ (ex) = esx$, for each $s \in S$. Hence $e \in \ell_R(S)$ and $Re_1 \subseteq \ell_R(S)$. Let $a \in \ell_R(S)$. Then $as = 0$ and so $(sx) \circ (ax) = asx = 0$, for each $s \in S$. Thus $(ax) \in r_{R[x]}(S_X) = (ex) \circ R[x]$ and $ax = (ex) \circ (ax)$. Hence $a = ae \in Re$ and $\ell_R(S) \subseteq Re$. Therefore $\ell_R(S) = Re$ and R is a Baer ring. ■

We now turn to the problem of extending Baer-type annihilator conditions from R to $R_0[x]$ and $R_0[[x]]$.

Proposition 2.5. *Let R be an abelian ring with unity. Then*

(1) *$R \in \mathcal{B}_{r1}$ if and only if $R_0[x] \in \mathcal{B}_{\ell 1}$;*

(2) *$R \in \mathcal{B}_{r2}$ if and only if $R_0[x] \in \mathcal{B}_{\ell 2}$;*

(3) *$R \in \mathcal{B}_{r1}$ if and only if $R_0[[x]] \in \mathcal{B}_{\ell 1}$;*

(4) $R \in \mathcal{B}_{r2}$ *if and only if* $R_0[[x]] \in \mathcal{B}_{\ell 2}$.

Proof. (1) Assume $R_0[x] \in \mathcal{B}_{\ell 1}$. Let S be a subset of R and define $S_X = \{sx | s \in S\}$ a subset of $R_0[x]$. By Lemma 2.1 and our assumption, there exists an idempotent $(x)\mathcal{E} = ex \in R[x]$ such that $\ell_{R_0[x]}(S_X) = R_0[x] \circ ex$.

Now we show that $r_R(S) = eR$. Since $ex \in \ell_{R_0[x]}(S_X)$, hence $0 = ex \circ sx = sex$ for each $s \in S$. Then $eR \subseteq r_R(S)$. Let $b \in r_R(S)$. Then $sb = 0$ and so $bx \circ sx = 0$, for each $s \in S$. Hence $bx \in \ell_{R_0[x]}(S_X) = R_0[x] \circ ex$. Thus $bx = bx \circ ex$ and that $b = eb \in eR$. Thus $r_R(S) \subseteq eR$ and hence $r_R(S) = eR$. Therefore $R \in \mathcal{B}_{r1}$.

Now, assume $R \in \mathcal{B}_{r1}$. Since R is abelian, hence R is reduced. Thus the result follows from [8, Proposition 3.7].

(2) Assume $R_0[x] \in \mathcal{B}_{\ell 2}$. Let S be a subset of R and define $S_X = \{sx | s \in S\}$ a subset of $R_0[x]$. By Lemma 2.1 and our assumption, there exists an idempotent $(x)\mathcal{E} = ex \in R[x]$ such that $\ell_{R_0[x]}(S_X) = \ell_{R_0[x]}(ex)$.

Now by a similar argument as used in (1), we can show that $r_R(S) = r_R(e)$. Therefore $R \in \mathcal{B}_{r2}$.

Conversely, assume that $R \in \mathcal{B}_{r2}$. Since R is abelian and has unity, hence R is reduced. Thus the result follows from [8, Proposition 3.7].

(3) Since R is abelian, hence if $R \in \mathcal{B}_{r1}$, then R is reduced. Thus by [9, Proposition 3.4], $R_0[[x]] \in \mathcal{B}_{\ell 1}$.

If $R_0[[x]] \in \mathcal{B}_{\ell 1}$, then by using Lemma 2.1 and an argument as used in the proof of (1), we can show that $R \in \mathcal{B}_{r1}$. ■

Theorem 2.6. *Let R be an abelian ring with unity.*

(1) *If R is Baer, then* $R_0[x] \in \mathcal{B}_{r1} \cap \mathcal{B}_{r2} \cap \mathcal{B}_{\ell 1} \cap \mathcal{B}_{\ell 2}$*;*

(2) *If* $R_0[x] \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$*, then R is Baer;*

(3) *If R is Baer, then* $R_0[[x]] \in \mathcal{B}_{r1} \cap \mathcal{B}_{r2} \cap \mathcal{B}_{\ell 1} \cap \mathcal{B}_{\ell 2}$*;*

(4) *If* $R_0[[x]] \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$*, then R is Baer.*

Proof. (1) Since R is an abelian Baer ring, hence it is reduced. Thus the result follows from [8, Theorem 3.8].

(2) Assume that $R_0[x] \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$. From [8, Lemma 2.2] and Proposition 2.5, we need only to prove for $R_0[x] \in \mathcal{B}_{r2}$. Assume $R_0[x] \in \mathcal{B}_{r2}$. Let S be a subset of R and define $S_X = \{sx | s \in S\}$ a subset of $R_0[x]$. By Lemma 2.1 and our assumption there exists an idempotent $(x)\mathcal{E} = ex \in R_0[x]$ such that $r_{R_0[x]}(S_X) = r_{R_0[x]}(ex)$. Now by a similar argument as used in the proof of Proposition 2.4, we can show that $\ell_R(S) = \ell_R(e)$. Therefore R is a Baer ring.

(3) Since R is an abelian Baer ring, hence it is reduced. Thus the result follows from [9, Theorem 3.5].

(4) By using Lemma 2.1 and a similar argument as used in the proof of (2), we can prove it. ■

The following example shows that the hypothesis, assuming R is abelian in Theorem 2.6, is not superfluous.

Example 2.7. There exists a Baer ring R such that $R_0[x] \notin \mathcal{B}_{r2}$ and $R_0[[x]] \notin \mathcal{B}_{r2}$. Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ be the upper 2-by-2 matrix ring over the field of integers $\mathbb{Z}_2$. The ring R is Baer by [15, p.16] and R has 8 elements namely, $\{0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_3 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, q = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\}$. Note that $q^2 = 0, \alpha^2 = 1$ and $0, 1, e_1, e_2, e_3, e_4$ are idempotents in R. Let $(x)\varepsilon = \sum_{i=1}^{\infty} \varepsilon_i x^i$ be a nonconstant idempotent in $R_0[[x]]$. If $\varepsilon_i \in \{0, q\}$ for all $i \geq 1$, then $(x)\varepsilon \circ (x)\varepsilon = 0$, since $q^2 = 0$. Therefore there exists at least one ε_j such that $\varepsilon_j \notin \{0, q\}$ for some $j \geq 1$. Let k be the smallest number such that $\varepsilon_k \notin \{0, q\}$. Observe that $qa + aq \in \{0, q\}$ for all $a \in R$, and $a^2 \in \{1, e_1, e_2, e_3, e_4\}$, if $a \notin \{0, q\}$. Hence the coefficient of x^{2k} in the power series $(x)\varepsilon.(x)\varepsilon$ is $\varepsilon_k\varepsilon_k + (\varepsilon_{k+1}\varepsilon_{k-1} + \varepsilon_{k-1}\varepsilon_{k+1}) + \cdots + (\varepsilon_{2k-1}\varepsilon_1 + \varepsilon_1\varepsilon_{2k-1}) = \varepsilon_k^2 + (0 \text{ or } q) + \cdots + (0 \text{ or } q) \neq 0$. Therefore $x^2 \notin r_{R_0[[x]]}((x)\varepsilon)$ for any nonconstant idempotent $(x)\varepsilon \in R_0[[x]]$. Let $I = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$. Observe that $x^2 \in r_{R_0[[x]]}(I_0[[x]])$, since $(x)f.(x)f = 0$, for each $(x)f \in I_0[[x]]$. Since $r_{R_0[[x]]}(I_0[[x]]) \neq R_0[[x]] = r_{R_0[[x]]}(0)$, hence $r_{R_0[[x]]}(I_0[[x]]) \neq r_{R_0[[x]]}((x)\varepsilon)$ for each idempotent $(x)\varepsilon \in R_0[[x]]$. Therefore $R_0[[x]] \notin \mathcal{B}_{r2}$.

By a similar argument we can show that $R_0[x] \notin \mathcal{B}_{r2}$.

Corollary 2.8. *Let R be an abelian ring with unity. Then the following are equivalent*

(1) *R is Baer;*
(2) *$(R[x], +, \cdot)$ is Baer;*
(3) *$(R_0[x], +, \circ) \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$;*
(4) *$(R[[x]], +, \cdot)$ is Baer;*
(5) *$(R_0[[x]], +, \circ) \in \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$.*

Proof. This follows from [7, Corollary 1.10] and Theorem 2.6. ∎

The following example shows that our Corollary 2.8 is not implied from [8, Theorem 3.8] and [9, Theorem 3.5].

Example 2.9. Let R be an abelian ring and $n \geq 2$. Let

$$S = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} | a, a_{ij} \in R \right\}.$$

Clearly S is not reduced, but S is abelian by [14, Lemma 2]. Since S is not

Baer, hence $(S[x],+,\circ) \notin \mathcal{B}_{r1} \cup \mathcal{B}_{r2}$, by Proposition 2.4. Also $(S_0[x],+,\circ) \notin \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$ and $(S_0[[x]],+,\circ) \notin \mathcal{B}_{r1} \cup \mathcal{B}_{r2} \cup \mathcal{B}_{\ell 1} \cup \mathcal{B}_{\ell 2}$ by Theorem 2.6.

Proposition 2.10. *Assume R is an abelian ring with unity. Let S be the subnearring of $R_0[x]$ generated by the set $\{ex | e = e^2 \in R\}$ and T a subnearring of $R_0[x]$. If $R_0[x] \in \mathcal{B}_{vi}$, where $v \in \{r, \ell\}$ and $i \in \{1, 2\}$, and $S \subseteq T$, then $T \in \mathcal{B}_{vi}$.*

Proof. (1) Assume that $R_0[x] \in \mathcal{B}_{r1}$. Let A be a nonempty subset of T. There exists an idempotent $(x)\mathcal{E} = ex \in R_0[x]$ such that $r_{R_0[x]}(A) = ex \circ R_0[x]$. Note that $(ex) \circ T \subseteq T$ and $(ex) \circ T \subseteq (ex) \circ R_0[x] = r_{R[x]}(A)$, and so $(ex) \circ T \subseteq r_T(A)$. On the other hand, let $(x)f \in r_T(A)$. Then $(x)f \in r_{R_0[x]}(A) = (ex) \circ R_0[x]$ and so $(x)f = (ex) \circ (x)f \in (ex) \circ T$, and thus $r_T(A) \subseteq (ex) \circ T$. Hence $T \in \mathcal{B}_{r1}$.

(2) Assume that $R_0[x] \in \mathcal{B}_{r2}$. Let A be a nonempty subset of T. There exists an idempotent $(x)\mathcal{E} = ex \in R_0[x]$ such that $r_{R_0[x]}(A) = r_{R_0[x]}(ex)$. We want to show that $r_T(A) = r_T(ex)$. If $(x)g \in r_T(A)$, then $(x)g \in r_{R_0[x]}(A) = r_{R_0[x]}(ex)$, and so $(e_1x) \circ (x)g = 0$. Therefore $(x)g \in r_T(ex)$. On the other hand, if $(x)h \in r_T(ex)$, then $(x)h \in r_{R_0[x]}(ex) = r_{R_0[x]}(A)$. That is $\alpha \circ (x)h = 0$ for all $\alpha \in A$, and so $(x)h \in r_T(A)$. Hence $r_T(A) = r_T(ex)$ and $T \in \mathcal{B}_{r2}$.

The other cases follow a similar argument to that used in (1) and (2). ■

Example 2.11. Assume R is an abelian ring with unity. Let T be any following nearring. If $R_0[x] \in \mathcal{B}_{vi}$, where $v \in \{r, \ell\}$ and $i \in \{1, 2\}$, then by using Propositions 2.10 and 2.5, $T \in \mathcal{B}_{vi}$.

(1) $\{ax | a \in R\}$,
(2) $\{(x)f = \sum_{i=1}^{n} a_{2i-1}x^{2i-1} \in R_0[x] | n \in \mathbb{N}\}$,
(3) $\{(x)f = \sum_{i=1}^{\infty} a_{2i-1}x^{2i-1} \in R_0[[x]]\}$,
(4) $E_0[x]$, where E is a subring containing all idempotents of R,
(5) $E_0[[x]]$, where E is a subring containing all idempotents of R.

Theorem 2.12. *Let R be an IFP ring with unity. Then the following are equivalent*

(1) *R is quasi-Baer;*
(2) *$R[x] \in q\mathcal{B}_{r2}$;*
(3) *$(R[x],+,\cdot)$ is quasi-Baer.*

Proof. The equivalence of (1) and (3) follows from [7, Theorem 1.8].

(1)⇒(2). Assume R is a quasi-Baer ring. Since R is IFP, hence it is Baer. Now the result follows from [10, Theorem 2.4].

(2)⇒(1). Assume $R[x] \in q\mathcal{B}_{r2}$. Let A be an ideal of R and $A[x]$ the nearring of polynomials over A. Let $(x)a = \sum\limits_{i=0}^{m} a_i x^i \in A[x]$, and $(x)f, (x)g = \sum\limits_{j=0}^{n} g_j x^j \in R[x]$. Observe that $(x)f \circ (x)a = \sum\limits_{i=0}^{m} a_i((x)f)^i \in A[x]$ and $((x)a + (x)f) \circ (x)g - (x)f \circ$

$(x)g = \sum_{j=0}^{n} g_j((x)a + (x)f)^j - \sum_{j=0}^{n} g_j((x)f)^j = \sum_{j=1}^{n} g_j[((x)a + (x)f)^j - ((x)f)^j] \in A[x]$. Therefore $A[x]$ is an ideal of $R[x]$. Since $R[x] \in q\mathcal{B}_{r2}$, there exists an idempotent $(x)\varepsilon \in R[x]$ such that $r_{R[x]}(A[x]) = r_{R[x]}((x)\varepsilon)$. Note that $(x)\varepsilon = ex$, by Lemma 2.3. We will show that $\ell_R(A) = \ell_R(e) = R(1-e)$. Let $a \in A$ and $\alpha \in R$ be arbitrary. Since $\alpha(1-e)x \in r_{R[x]}((x)\varepsilon) = r_{R[x]}(A[x])$, we have $0 = ax \circ \alpha(1-e)x = \alpha(1-e)ax$. Therefore $\alpha(1-e)a = 0$ and so $R(1-e) \subseteq \ell_R(A)$. On the other hand, if $\alpha \in \ell_R(A)$, then $\alpha a = 0$ for all $a \in A$. Hence $\alpha x \in r_{R[x]}(A[x]) = r_{R[x]}((x)\varepsilon)$. Then $0 = ex \circ \alpha x = \alpha ex$, and so $\alpha \in \ell_R(e) = R(1-e)$. Hence $\ell_R(A) = R(1-e)$. ■

Example 2.7 also shows that the hypothesis, assuming R is abelian in Theorem 2.12, is not superfluous.

Since each reduced ring is IFP, hence we have the following:

Corollary 2.13. [10, Theorem 2.4] *Let R be a reduced ring. Then the following are equivalent*

(1) *R is Baer;*

(2) *$R[x] \in q\mathcal{B}_{r2}$;*

(3) *$(R[x], +, \cdot)$ is Baer.*

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