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New Criteria for Stability and Stabilization of Neural Networks with Mixed Interval Time-Varying Delays

Mai Viet Thuan[1] **and Vu Ngoc Phat**[2]

[1] *Department of Mathematics, Thai Nguyen University, Thai Nguyen, Vietnam*

[2] *Institute of Mathematics, VAST, 18 Hoang Quoc Viet, Hanoi, Vietnam*

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Abstract. This paper considers the global exponential stability and stabilization for a class of neural networks with mixed interval time-varying delays. The time delay is assumed to be a continuous function belonging to a given interval, but not necessary to be differentiable. By constructing a set of new Lyapunov-Krasovskii functionals combined with Newton-Leibniz formula, new delay-dependent criteria for exponential stability and stabilization of the system are established in terms of linear matrix inequalities (LMIs), which allows to compute simultaneously the two bounds that characterize the exponential stability of the solution. Numerical examples are included to illustrate the effectiveness of the results.

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Key words. Neural networks, stability, stabilization, non-differentiable delays, Lyapunov function, linear matrix inequalities.

1. Introduction

Neural networks have found a lot of successful applications in many fields, such as signal processing, pattern recognition and association. Some of these applications require that the equilibrium points of the designed network are stable. In both biological and artificial neural systems, time delays due to integration and communication are ubiquitous, and often become a source of instability. The time delays in electronic neural networks are usually time-varying, and sometimes vary violently with respect to time due to the finite switching speed of amplifiers and faults in the electrical circuitry. Therefore, stability analysis of delayed neural networks is a very important issue, and many stability criteria have been developed in the literature [2, 3, 5, 9] and the references cited therein. Recent research works reveal that there are systems which are stable with some nonzero delays but unstable without delay. This kind of the delay falls into a category of interval time-varying delay which varies in an interval and the lower bound is not restricted to be zero [2, 10, 11]. Recently, the stability analysis of neural networks with interval time-varying delay has attracted considerable interest [5, 13, 14, 15]. In this regard, the stability problem for neural networks with interval time-varying delay was investigated by considering the relationship between the time-varying delay and its lower and upper bounds when constructing the Lyapunov functional and calculating the upper bound of its derivative. It is noted that the results in these works were obtained under the assumption that both the time-varying delay functions and the upper bound of the derivatives of the delays are known. In [15], a piecewise delay method was used in studying asymptotic stability of neural networks with interval time-varying delay, where the delay is divided into two subintervals with equal length by introducing its central point. However, the stability conditions reported in this work not only require the delay function to be boundedly differentiable, but the system are constrained by the only activation function and without the distributed delay. In addition, to the best of our knowledge, there does not seem to be much, if any, study so far on the exponential stability and stabilization of neural networks with mixed interval time-varying non-differentiable delays.

In this paper we consider stability problem for a general class of delayed neural networks. The novel features here are that the neural networks in consideration are time-varying with interval time-varying delays and with various activation functions. The time delays are assumed to be any continuous function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, but the delay functions are bounded but not necessary differentiable. By constructing a set of new augmented Lyapunov-Krasovskii functionals combined with Newton-Leibniz formula, new delay-dependent criteria for exponential stability and stabilization of the system are established. The achieved stability criteria are expressed by solving LMIs which can be easily checked by Matlab LMI toolbox [6]. Some numerical examples are provided to show the effectiveness and applicability of the proposed method.

The outline of the paper is as follows. Section 2 presents definitions and some well-known technical propositions needed for the proof of the main result. In Section 3, delay-dependent sufficient conditions are derived for exponential stability and stabilization of the system. Illustrative examples are given in Section 4.

2. Preliminaries

The following notations will be used throughout this paper:

- $\mathbb{R}^+$ denotes the set of all real non-negative numbers;
- $\mathbb{R}^n$ denotes the n-dimensional space with the scalar product $\langle \cdot, \cdot \rangle$ and the vector norm $\| \cdot \|$;
- $\mathbb{R}^{n\times r}$ denotes the space of all matrices of $(n \times r)$-dimensions; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix;
- $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\max}(A) = \max\{\mathrm{Re}\lambda; \lambda \in \lambda(A)\}$;
- $x_t := \{x(t+s) : s \in [-h, 0]\}, \|x_t\|_h = \sup_{s\in[-h,0]} \|x(t+s)\|$;
- $C^1([0,t], \mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuously differentiable functions on $[0,t]$;
- $L_2([0,t], \mathbb{R}^m)$ denotes the set of all $\mathbb{R}^m$-valued square integrable functions on $[0,t]$;
- Matrix A is called semi-positive definite $(A \geq 0)$ if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$; A is positive definite $(A > 0)$ if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$.
- The notation diag$\{\ldots\}$ stands for a block-diagonal matrix. The symmetric term in a matrix is denoted by $*$.

Consider the following class of mixed delayed neural networks:

$$\begin{aligned}
&\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 g(x(t - h(t))) + W_2 \int_{t-k(t)}^{t} c(x(s))\, ds + Bu(t)\\
&x(t) = \phi(t), t \in [-d, 0], \quad d = \max\{h_2, k\},
\end{aligned} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$ is the state of the neural networks, $u(\cdot) \in L_2([0,t], \mathbb{R}^m)$ is the control, n is the number of neurals, and

$$\begin{aligned}
f(x(t)) &= [f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t))]^T,\\
g(x(t)) &= [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T,\\
c(x(t)) &= [c_1(x_1(t)), c_2(x_2(t)), \ldots, c_n(x_n(t))]^T
\end{aligned}$$

are the activation functions; $A = \mathrm{diag}\{\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_n\}, \overline{a}_i > 0$ represents the self-feedback term and W_0, W_1, W_2, B denote the connection weight matrix, the discretely delayed connection weight matrix, the distributively delayed connection weight matrix and the control input matrix, respectively. The time-varying delay functions $h(t), k(t)$ satisfy the condition

$$0 \leq h_1 \leq h(t) \leq h_2, \tag{2}$$

$$0 \leq k(t) \leq k. \tag{3}$$

The initial function $\phi(t) \in C^1([-d, 0], \mathbb{R}^n), d = \max\{h_2, k\}$, with the norm

$$\|\phi\|_C = \sup_{t\in[-d,0]}\sqrt{\|\phi(t)\|^2 + \|\dot{\phi}(t)\|^2}.$$

In this paper we consider various activation functions and assume that the activation functions $f(\cdot), g(\cdot), c(\cdot)$ satisfy the following growth conditions with the growth constants $a_i, b_i, c_i > 0$

$$\begin{aligned} |f_i(\xi)| &\le a_i|\xi|, \quad i = 1,2,\ldots,n, \forall\xi \in \mathbb{R},\\ |g_i(\xi)| &\le b_i|\xi|, \quad i = 1,2,\ldots,n, \forall\xi \in \mathbb{R},\\ |c_i(\xi)| &\le c_i|\xi|, \quad i = 1,2,\ldots,n, \forall\xi \in \mathbb{R}. \end{aligned} \tag{4}$$

Consider the unforced system $(u(t) = 0)$ of (1)

$$\dot{x}(t) = -Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2 \int\limits_{t-k(t)}^{t} c(x(s))ds, \ t \in \mathbb{R}^+. \tag{5}$$

Definition 2.1. Given $\alpha > 0$. The zero solution of unforced system (5) is α-stable if there exists a positive number $\beta > 0$ such that every solution $x(t,\phi)$ satisfies the condition

$$\|x(t,\phi)\| \le \beta e^{-\alpha t}\|\phi\|_C \quad \forall t \ge 0.$$

Definition 2.2. The system (1) is exponentially stabilizable if there exists a feedback control $u(t) = Kx(t), K \in \mathbb{R}^{m\times n}$ such that the zero solution of the resulting closed-loop system

$$\dot{x}(t) = -[A - BK]x(t) + W_0 f(x(t)) + W_1 g(x(t-\bar{h}(t))) + W_2 \int_{t-k(t)}^{t} c(x(s))\, ds$$

is α-stable.

We introduce the following technical well-known propositions, which will be used in the proof of our results.

Proposition 2.3. *For any $x, y \in \mathbb{R}^n$ and positive definite matrix $N \in \mathbb{R}^{n\times n}$, we have*

$$\pm 2y^T x \le x^T N^{-1} x + y^T N y.$$

Proposition 2.4. [4] *For any symmetric positive definite matrix $M > 0$, scalar $\gamma > 0$ and vector function $\omega : [0,\gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, we have*

$$\left(\int_0^\gamma \omega(s)\, ds\right)^T M \left(\int_0^\gamma \omega(s)\, ds\right) \le \gamma \left(\int_0^\gamma \omega^T(s) M \omega(s) ds\right).$$

Proposition 2.5. (Schur complement lemma [1]) *Given constant matrices* X, Y, Z *with appropriate dimensions satisfying* $X = X^T, Y = Y^T > 0$. *Then* $X + Z^T Y^{-1} Z < 0$ *if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0.$$

3. Main result

In this section, we present exponential stability and stabilization criteria for neural networks (1). For simplicity of matrix expression, we denote

$$\begin{aligned}
&G = \operatorname{diag}\{b_i, i = 1, \ldots, n\}, \quad H = \operatorname{diag}\{c_i, i = 1, \ldots, n\} \\
&F = \operatorname{diag}\{a_i, i = 1, \ldots, n\}, \quad c^2 = \max\{c_i^2, i = 1, 2, \ldots, n\}, \\
&\lambda = \lambda_{\min}(P), \\
&\Lambda = \lambda_{\max}(P) + h_1 \lambda_{\max}(Q) + \frac{1}{2} h_2^3 \lambda_{\max}(R) + \frac{1}{2}(h_2 - h_1)^2 (h_2 \\
&\qquad + h_1)\lambda_{\max}(S) + \frac{1}{2} c^2 k^2 \lambda_{\max}(D_2), \\
&\Xi_{11} = Q + 2\alpha P - e^{-2\alpha h_2} R + F D_0 F + k H D_2 H - A^T N_1 - N_1^T A, \\
&\Xi_{12} = -A^T N_2 + e^{-2\alpha h_2} R, \\
&\Xi_{14} = -A^T N_4 + P - N_1^T, \\
&\Xi_{22} = -e^{-2\alpha h_2} R - e^{-2\alpha h_2} S + G D_1 G, \\
&\Xi_{33} = -e^{-2\alpha h_1} Q - e^{-2\alpha h_2} S, \\
&\Xi_{44} = h_2^2 R + (h_2 - h_1)^2 S - N_4 - N_4^T.
\end{aligned}$$

Theorem 3.1. *Given* $\alpha > 0$. *Assume that there exist symmetric positive definite matrices* P, Q, R, S, *three diagonal positive matrices* $D_i, i = 0, 1, 2$, *and matrices* $N_j, j = 1, \ldots, 4$, *satisfying the following LMI*

$$\Xi = \begin{bmatrix}
\Xi_{11} & \Xi_{12} & -A^T N_3 & \Xi_{14} & N_1^T W_0 & N_1^T W_1 & k N_1^T W_2 \\
* & \Xi_{22} & e^{-2\alpha h_2} S & -N_2^T & N_2^T W_0 & N_2^T W_1 & k N_2^T W_2 \\
* & * & \Xi_{33} & -N_3^T & N_3^T W_0 & N_3^T W_1 & k N_3^T W_2 \\
* & * & * & \Xi_{44} & N_4^T W_0 & N_4^T W_1 & k N_4^T W_2 \\
* & * & * & * & -D_0 & 0 & 0 \\
* & * & * & * & * & -D_1 & 0 \\
* & * & * & * & * & * & -k e^{-2\alpha k} D_2
\end{bmatrix} < 0. \tag{6}$$

Then the zero solution of system (5) *is* α*-stable. Moreover, the solution* $x(t, \phi)$ *of the system satisfies*

$$\|x(t, \phi)\| \le \sqrt{\frac{\Lambda}{\lambda}} \|\phi\|_C e^{-\alpha t} \quad \forall t \ge 0.$$

Proof. Consider the following Lyapunov-Krasovskii functional for system (5)

$$V(t,x_t)=\sum_{i=1}^{5}V_i(t,x_t),$$

where

$$\begin{aligned}
V_1(t,x_t)&=x^T(t)Px(t),\\
V_2(t,x_t)&=\int_{t-h_1}^{t}e^{2\alpha(s-t)}x^T(s)Qx(s)\,ds,\\
V_3(t,x_t)&=h_2\int_{-h_2}^{0}\int_{t+s}^{t}e^{2\alpha(\theta-t)}\dot{x}^T(\theta)R\dot{x}(\theta)\,d\theta\,ds,\\
V_4(t,x_t)&=(h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t}e^{2\alpha(\theta-t)}\dot{x}^T(\theta)S\dot{x}(\theta)\,d\theta\,ds,\\
V_5(t,x_t)&=\int_{-k}^{0}\int_{t+s}^{t}e^{2\alpha(\theta-t)}c^T(x(\theta))D_2c(x(\theta))\,d\theta\,ds.
\end{aligned}$$

It is easy to check that

$$\lambda\|x(t)\|^2\le V(t,x_t)\le \Lambda\|x_t\|_h^2,\quad t\ge 0. \tag{7}$$

Taking the time derivative of $V_i(t,x_t), i=1,\dots,5$, in t along the solution we obtain

$$\begin{aligned}
\dot{V}_1(t,x_t)&=2x^T(t)P\dot{x}(t),\\
\dot{V}_2(t,x_t)&=x^T(t)Qx(t)-e^{-2\alpha h_1}x^T(t-h_1)Qx(t-h_1)-2\alpha V_2(t,x_t),\\
\dot{V}_3(t,x_t)&\le h_2^2\dot{x}^T(t)R\dot{x}(t)-h_2e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot{x}^T(s)R\dot{x}(s)\,ds-2\alpha V_3(t,x_t),\\
\dot{V}_4(t,x_t)&\le (h_2-h_1)^2\dot{x}^T(t)S\dot{x}(t)-2\alpha V_4(t,x_t)\\
&\quad-(h_2-h_1)e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}\dot{x}^T(s)S\dot{x}(s)\,ds,\\
\dot{V}_5(t,x_t)&\le kc^T(x(t))D_2c(x(t))-2\alpha V_5(t,x_t)\\
&\quad-e^{-2\alpha k}\int_{t-k}^{t}c^T(x(s))D_2c(x(s))\,ds.
\end{aligned} \tag{8}$$

Using condition (4) and since the matrix D_2 is diagonal, we have

$$kc^T(x(t))D_2c(x(t))\le kx^T(t)HD_2Hx(t). \tag{9}$$

From (8), (9) we obtain

$$\dot{V}(t,x_t)+2\alpha V(t,x_t)\le 2x^T(t)P\dot{x}(t)+x^T(t)[Q+2\alpha P+kHD_2H]x(t)$$

$$
\begin{aligned}
&+ \dot{x}^T(t)[h_2^2 R + (h_2-h_1)^2 S]\dot{x}(t)\\
&- e^{-2\alpha h_1} x^T(t-h_1)Qx(t-h_1)\\
&- h_2 e^{-2\alpha h_2}\int_{t-h_2}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds \qquad (10)\\
&- (h_2-h_1)e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1} \dot{x}^T(s)S\dot{x}(s)\,ds\\
&- e^{-2\alpha k}\int_{t-k}^{t} c^T(x(s))D_2 c(x(s))\,ds.
\end{aligned}
$$

Applying Proposition 2.4 and the Leibniz-Newton formula, we have

$$
\begin{aligned}
-h_2\int_{t-h_2}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds &\le -h(t)\int_{t-h(t)}^{t} \dot{x}^T(s)R\dot{x}(s)\,ds\\
&\le -\Big(\int_{t-h(t)}^{t} \dot{x}(s)\,ds\Big)^T R\Big(\int_{t-h(t)}^{t} \dot{x}(s)\,ds\Big) \qquad (11)\\
&= -[x(t)-x(t-h(t))]^T R[x(t)-x(t-h(t))]\\
&= -x^T(t)Rx(t) + 2x^T(t)Rx(t-h(t))\\
&\quad - x^T(t-h(t))Rx(t-h(t));
\end{aligned}
$$

$$
\begin{aligned}
&- (h_2-h_1)\int_{t-h_2}^{t-h_1} \dot{x}^T(s)S\dot{x}(s)ds\\
\le &- (h(t)-h_1)\int_{t-h(t)}^{t-h_1} \dot{x}^T(s)S\dot{x}(s)\,ds\\
\le &- \Big(\int_{t-h(t)}^{t-h_1} \dot{x}(s)\,ds\Big)^T S\Big(\int_{t-h(t)}^{t-h_1} \dot{x}(s)\,ds\Big)\\
= &- [x(t-h_1)-x(t-h(t))]^T S[x(t-h_1)-x(t-h(t))]\\
= &- x^T(t-h_1)Sx(t-h_1) + 2x^T(t-h_1)Sx(t-h(t)) - x^T(t-h(t))Sx(t-h(t)).
\end{aligned}
\qquad (12)
$$

Setting

$$
\zeta^T(t) = \left[x^T(t)\ x^T(t-h(t))\ x^T(t-h_1)\ \dot{x}^T(t)\right],
$$

and auxiliary variables with appropriate dimensions

$$
\mathcal{N} = \left[N_1\ N_2\ N_3\ N_4\right],
$$

and using the following identity relation

$$
-\dot{x}(t) - Ax(t) + W_0 f(x(t)) + W_1 g(x(t-h(t))) + W_2\int_{t-k(t)}^{t} c(x(s))\,ds = 0,
$$

we obtain

$$2\zeta^T(t)\mathcal{N}^T\Big[-\dot{x}(t)-Ax(t)+W_0f(x(t)) + W_1g(x(t-h(t)))+W_2\int_{t-k(t)}^{t}c(x(s))\,ds\Big]=0. \tag{13}$$

From (10) - (13) we have

$$\begin{aligned}
&\dot{V}(t,x_t)+2\alpha V(t,x_t)\\
&\le 2x^T(t)P\dot{x}(t)+x^T(t)[Q+2\alpha P+kHD_2H-e^{-2\alpha h_2}R]x(t)\\
&\quad+\dot{x}^T(t)[h_2^2R+(h_2-h_1)^2S]\dot{x}(t)-x^T(t-h_1)[e^{-2\alpha h_1}Q+e^{-2\alpha h_2}S]x(t-h_1)\\
&\quad-x^T(t-h(t))[e^{-2\alpha h_2}R+e^{-2\alpha h_2}S]x(t-h(t))\\
&\quad+2e^{-2\alpha h_2}x^T(t)Rx(t-h(t))+2e^{-2\alpha h_2}x^T(t-h_1)Sx(t-h(t))\\
&\quad-e^{-2\alpha k}\int_{t-k}^{t}c^T(x(s))D_2c(x(s))\,ds+2\zeta^T(t)\mathcal{N}^T[-Ax(t)-\dot{x}(t)]\\
&\quad+2\zeta^T(t)\mathcal{N}^TW_0f(x(t))+2\zeta^T(t)\mathcal{N}^TW_1g(x(t-h(t)))\\
&\quad+2\zeta^T(t)\mathcal{N}^TW_2\int_{t-k(t)}^{t}c(x(s))\,ds.
\end{aligned} \tag{14}$$

By using Proposition 2.3 and Proposition 2.4, we have

$$2\zeta^T(t)\mathcal{N}^TW_0f(x(t))\le\zeta^T(t)\mathcal{N}^TW_0D_0^{-1}W_0^T\mathcal{N}\zeta(t)+f^T(x(t))D_0f(x(t)), \tag{15}$$

$$\begin{aligned}2\zeta^T(t)\mathcal{N}^TW_1g(x(t-h(t)))&\le\zeta^T(t)\mathcal{N}^TW_1D_1^{-1}W_1^T\mathcal{N}\zeta(t)\\&\quad+g^T(x(t-h(t)))D_1g(x(t-h(t))),\end{aligned} \tag{16}$$

$$\begin{aligned}2\zeta^T(t)\mathcal{N}^TW_2\int_{t-k(t)}^{t}c(x(s))ds&\le ke^{2\alpha k}\zeta^T(t)\mathcal{N}^TW_2D_2^{-1}W_2^T\mathcal{N}\zeta(t)\\
&\quad+k^{-1}e^{-2\alpha k}\Big(\int_{t-k(t)}^{t}c(x(s))ds\Big)^T\times\\
&\quad\times D_2\Big(\int_{t-k(t)}^{t}c(x(s))\,ds\Big)\\
&\le ke^{2\alpha k}\zeta^T(t)\mathcal{N}^TW_2D_2^{-1}W_2^T\mathcal{N}\zeta(t)\\
&\quad+e^{-2\alpha k}\int_{t-k}^{t}c^T(x(s))D_2c(x(s))\,ds.\end{aligned} \tag{17}$$

Again using condition (4) and since the matrices $D_i>0, i=0,1$, are diagonal, we have

$$\begin{aligned}&f^T(x(t))D_0f(x(t))\le x^T(t)FD_0Fx(t),\\&g^T(x(t-h(t)))D_1g(x(t-h(t)))\le x^T(t-h(t))GD_1Gx(t-h(t)).\end{aligned} \tag{18}$$

From (14)-(18) we obtain

$$\begin{aligned}
&\dot{V}(t,x_t) + 2\alpha V(t,x_t) \\
&\leq 2x^T(t)P\dot{x}(t) + x^T(t)[Q + 2\alpha P - e^{-2\alpha h_2}R + kHD_2H + FD_0F]x(t) \\
&\quad + \dot{x}^T(t)[h_2^2 R + (h_2 - h_1)^2 S]\dot{x}(t) - x^T(t-h_1)[e^{-2\alpha h_1}Q + e^{-2\alpha h_2}S]x(t-h_1) \\
&\quad + x^T(t-h(t))[-e^{-2\alpha h_2}R - e^{-2\alpha h_2}S + GD_1G]x(t-h(t)) \\
&\quad + 2e^{-2\alpha h_2}x^T(t)Rx(t-h(t)) + 2e^{-2\alpha h_2}x^T(t-h_1)Sx(t-h(t)) \\
&\quad + 2\zeta^T(t)\mathcal{N}^T[-Ax(t) - \dot{x}(t)] \\
&\quad + \zeta^T(t)[\mathcal{N}^T W_0 D_0^{-1} W_0^T \mathcal{N} + \mathcal{N}^T W_1 D_1^{-1} W_1^T \mathcal{N} + ke^{2\alpha k}\mathcal{N}^T W_2 D_2^{-1} W_2^T \mathcal{N}]\zeta(t).
\end{aligned} \tag{19}$$

Therefore

$$\dot{V}(t,x_t) + 2\alpha V(t,x_t) \leq \zeta^T(t)\Omega\zeta(t),$$

where

$$\Omega = \begin{pmatrix} \Xi_{11} & \Xi_{12} & -A^T N_3 & \Xi_{14} \\ * & \Xi_{22} & e^{-2\alpha h_2}S & -N_2^T \\ * & * & \Xi_{33} & -N_3^T \\ * & * & * & \Xi_{44} \end{pmatrix} + \overline{\Omega}$$

with

$$\overline{\Omega} = \mathcal{N}^T W_0 D_0^{-1} W_0^T \mathcal{N} + \mathcal{N}^T W_1 D_1^{-1} W_1^T \mathcal{N} + ke^{2\alpha k}\mathcal{N}^T W_2 D_2^{-1} W_2^T \mathcal{N}.$$

By using Schur complement lemma (Proposition 2.5), we have $\Omega < 0$ if and only if $\Xi < 0$. Thus we have

$$\dot{V}(t,x_t) + 2\alpha V(t,x_t) \leq 0 \quad \forall t \geq 0.$$

Integrating both sides of the above differential inequality from 0 to t, we obtain

$$V(t,x_t) \leq V(0,x_0)e^{-2\alpha t} \quad \forall t \in \mathbb{R}^+.$$

Furthermore, taking condition (7) into account, we have

$$\lambda\|x(t,\phi)\|^2 \leq V(t,x_t) \leq V(0,x_0)e^{-2\alpha t} \leq \Lambda e^{-2\alpha t}\|\phi\|_C^2,$$

and hence

$$\|x(t,\phi)\| \leq \sqrt{\frac{\Lambda}{\lambda}}e^{-\alpha t}\|\phi\|_C, \quad t \geq 0,$$

which completes the proof of the theorem. ∎

Remark 3.2. Note that from the proof of Theorem 3.1, the LMI condition (6) holds for any self-feedback matrix A, which needs not to be diagonal.

We conclude this section with the problem of exponential stabilization of the neural networks (1). Let us denote

$$G = \mathrm{diag}\{b_i, i = 1, \ldots, n\}, \quad H = \mathrm{diag}\{c_i, i = 1, \ldots, n\},$$
$$F = \mathrm{diag}\{a_i, i = 1, \ldots, n\}, \quad c^2 = \max\{c_i^2, i = 1, 2, \ldots, n\},$$
$$\Lambda_1 = \lambda_{\min}(M^{-1}\overline{P}M^{-1}),$$
$$\Lambda_2 = \lambda_{\max}(M^{-1}\overline{P}M^{-1}) + h_1\lambda_{\max}(M^{-1}\overline{Q}M^{-1}) + \frac{1}{2}h_2^3\lambda_{\max}(M^{-1}\overline{R}M^{-1}) + \frac{1}{2}(h_2 - h_1)^2(h_2 + h_1)\lambda_{\max}(M^{-1}\overline{S}M^{-1}) + \frac{1}{2}c^2k^2\lambda_{\max}(X_2^{-1}),$$

$$\Psi_{11} = \overline{Q} + 2\alpha\overline{P} - e^{-2\alpha h_2}\overline{R} - MA^T - AM + BY + Y^TB^T,$$
$$\Psi_{12} = -MA^T + Y^TB^T + e^{-2\alpha h_2}\overline{R},$$
$$\Psi_{14} = -MA^T + Y^TB^T + \overline{P} - M,$$
$$\Psi_{22} = -e^{-2\alpha h_2}\overline{R} - e^{-2\alpha h_2}\overline{S},$$
$$\Psi_{33} = -e^{-2\alpha h_1}\overline{Q} - e^{-2\alpha h_2}\overline{S},$$
$$\Psi_{44} = h_2^2\overline{R} + (h_2 - h_1)^2\overline{S} - 2M,$$
$$\Psi_{77} = ke^{-2\alpha k}(X_2 - 2M).$$

Theorem 3.3. *Given $\alpha > 0$. Assume that there exist symmetric positive definite matrices $M, \overline{P}, \overline{Q}, \overline{R}, \overline{S}$, three diagonal positive matrices $X_i, i = 0, 1, 2$, and a matrix Y satisfying the following LMI*

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & (-MA^T + Y^TB^T) & \Psi_{14} & W_0M & W_1M & kW_2M & MF & kMH & 0 \\ * & \Psi_{22} & e^{-2\alpha h_2}\overline{S} & -M & W_0M & W_1M & kW_2M & 0 & 0 & MG \\ * & * & \Psi_{33} & -M & W_0M & W_1M & kW_2M & 0 & 0 & 0 \\ * & * & * & \Psi_{44} & W_0M & W_1M & kW_2M & 0 & 0 & 0 \\ * & * & * & * & X_0 - 2M & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & X_1 - 2M & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Psi_{77} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -X_0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -kX_2 & 0 \\ * & * & * & * & * & * & * & * & * & -X_1 \end{bmatrix} < 0. \tag{20}$$

Then the zero solution of system (1) *is exponentially stabilizable with the stabilizing feedback control*

$$u(t) = YM^{-1}x(t), \quad t \in \mathbb{R}^+$$

and the solution $x(t, \phi)$ satisfies

$$\|x(t, \phi)\| \leq \sqrt{\frac{\Lambda_2}{\Lambda_1}}e^{-\alpha t}\|\phi\|_C, \quad t \in \mathbb{R}^+.$$

Proof. Define

$$N_1 = N_2 = N_3 = N_4 = M^{-1},$$

$$D_i = X_i^{-1}, i = 0, 1, 2, \quad K = YM^{-1}, \quad \overline{A} = A - BK.$$

Consider the Lyapunov-Krasovskii functional defined as in the proof of Theorem 3.1 for the closed-loop system

$$\dot{x}(t) = -\overline{A}x(t) + W_0 f(x(t)) + W_1 g(x(t - h(t))) + W_2 \int_{t-k(t)}^{t} c(x(s))\, ds.$$

By Theorem 3.1 and Remark 3.2, the closed-loop system is α-stable if $\Phi < 0$, where

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & -\overline{A}^T M^{-1} & \Phi_{14} & M^{-1}W_0 & M^{-1}W_1 & kM^{-1}W_2 \\ * & \Phi_{22} & e^{-2\alpha h_2} S & -M^{-1} & M^{-1}W_0 & M^{-1}W_1 & kM^{-1}W_2 \\ * & * & \Phi_{33} & -M^{-1} & M^{-1}W_0 & M^{-1}W_1 & kM^{-1}W_2 \\ * & * & * & \Phi_{44} & M^{-1}W_0 & M^{-1}W_1 & kM^{-1}W_2 \\ * & * & * & * & -X_0^{-1} & 0 & 0 \\ * & * & * & * & * & -X_1^{-1} & 0 \\ * & * & * & * & * & * & -ke^{-2\alpha k} X_2^{-1} \end{bmatrix},$$

$$\begin{aligned}
&\Phi_{11} = Q + 2\alpha P - e^{-2\alpha h_2} R + F X_0^{-1} F + kHX_2^{-1}H - \overline{A}^T M^{-1} - M^{-1}\overline{A}, \\
&\Phi_{12} = -\overline{A}^T M^{-1} + e^{-2\alpha h_2} R, \quad \Phi_{14} = -\overline{A}^T M^{-1} + P - M^{-1}, \\
&\Phi_{22} = -e^{-2\alpha h_2} R - e^{-2\alpha h_2} S + GX_1^{-1}G, \quad \Phi_{33} = -e^{-2\alpha h_1} Q - e^{-2\alpha h_2} S \\
&\Phi_{44} = h_2^2 R + (h_2 - h_1)^2 S - 2M^{-1}.
\end{aligned}$$

Now, pre- and post-multiply both sides of Φ with

$$\Theta = \mathrm{diag}\{M, M, M, M, M, M, M\},$$

and denote $\overline{P} = MPM, \overline{Q} = MQM, \overline{R} = MRM, \overline{S} = MSM$, and design feedback control $u(t) = Kx(t), K = YM^{-1}$, we obtain

$$\begin{aligned}
\mathcal{L} &= \Theta\Phi\Theta \\
&= \begin{bmatrix} \overline{\Psi}_{11} & \Psi_{12} & (-MA^T + Y^T B^T) & \Psi_{14} & W_0 M & W_1 M & kW_2 M \\ * & \overline{\Psi}_{22} & e^{-2\alpha h_2}\overline{S} & -M & W_0 M & W_1 M & kW_2 M \\ * & * & \Psi_{33} & -M & W_0 M & W_1 M & kW_2 M \\ * & * & * & \Psi_{44} & W_0 M & W_1 M & kW_2 M \\ * & * & * & * & -MX_0^{-1}M & 0 & 0 \\ * & * & * & * & * & -MX_1^{-1}M & 0 \\ * & * & * & * & * & * & -ke^{-2\alpha k} MX_2^{-1}M \end{bmatrix},
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
\overline{\Psi}_{11} &= \overline{Q} + 2\alpha\overline{P} - e^{-2\alpha h_2}\overline{R} - MA^T - AM + BY \\
&\quad + Y^T B^T + MFX_0^{-1}FM + kMHX_2^{-1}HM, \\
\overline{\Psi}_{22} &= -e^{-2\alpha h_2}\overline{R} - e^{-2\alpha h_2}\overline{S} + MGX_1^{-1}GM.
\end{aligned}$$

From the inequality $(M - X_i)X_i^{-1}(M - X_i) \geq 0, i = 0, 1, 2$, we obtain

$$-MX_i^{-1}M \leq X_i - 2M, \quad i = 0, 1, 2. \tag{22}$$

Note that $\Phi < 0$ is equivalent to $\mathcal{L} < 0$. Therefore, using the Schur complement lemma and condition (22), the condition $\mathcal{L} < 0$ is equivalent to (20), which completes the proof of the theorem. ■

Remark 3.4. It should be noted that the results in [5, 8, 12, 14, 15] can provide sufficient conditions for the exponential stability and stabilization for neural networks with time-varying delays, which require either differentiable assumption on the delay or the delay function does not belong to an interval. In our paper we construct a Lyapunov functional different from the ones and estimate the derivative $\dot{V}(\cdot)$ by LMI technique and Newton-Leibniz formula, which leads to a new LMI condition and reduces numerical complexity.

4. Numerical examples

Example 4.1. (Exponential stability). Consider the neural networks with mixed interval time-varying delays (5) where

$$\begin{cases} h(t) = 0.1 + 0.3\sin^2 t & \text{if} \quad t \in \mathcal{I} = \cup_{k\geq 0}[2k\pi, (2k+1)\pi] \\ h(t) = 0.1 & \text{if} \quad t \in \mathbb{R}^+ \setminus \mathcal{I}, \end{cases}$$

$$k(t) = 0.5|\sin t|,$$

$$A = \begin{bmatrix} 4.5633 & 0 & 0 \\ 0 & 8.0563 & 0 \\ 0 & 0 & 10.4616 \end{bmatrix}, W_0 = \begin{bmatrix} 0.2651 & -3.1608 & -2.0491 \\ 3.1859 & -0.1573 & -2.4687 \\ 2.0368 & -1.3633 & 0.5776 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -0.7727 & -0.8370 & 3.8019 \\ 0.1004 & 0.6677 & -2.4431 \\ -0.6622 & 1.3109 & -1.8407 \end{bmatrix}, W_2 = \begin{bmatrix} 0.4516 & -0.7623 & -2.9857 \\ 0.2978 & -0.2356 & 2.9846 \\ -0.0245 & 1.2345 & 0.3798 \end{bmatrix},$$

$$F = \text{diag}\{0.1019, 0.3419, 0.0633\}, \quad G = \text{diag}\{0.1892, 0.2678, 0.0988\},$$

$$H = \text{diag}\{0.0478, 0.4786, 0.0573\}.$$

It is worth noting that the delay functions $h(t), k(t)$ are non-differentiable. Therefore the methods used in [5, 10, 14, 15] are not applicable to this system. We have $h_1 = 0.1, h_2 = 0.4, k = 0.5$. Given $\alpha = 0.1$ and any initial function $\phi(t) \in C^1([-0.5, 0], \mathbb{R}^2)$. Using the Matlab LMI toolbox, we have LMI (6) is feasible with the following matrices:

$$P = \begin{bmatrix} 16.6615 & 15.6676 & 3.3538 \\ 15.6676 & 31.9317 & -11.1455 \\ 3.3538 & -11.1455 & 80.2501 \end{bmatrix}, Q = \begin{bmatrix} 34.3438 & 49.6489 & 15.2746 \\ 49.6489 & 95.7703 & -26.6354 \\ 15.2746 & -26.6354 & 279.2169 \end{bmatrix},$$

$$R = S = \mathrm{diag}\{682.8857; 682.8857; 682.8857\},$$

$$D_0 = \mathrm{diag}\{530.1764; 259.8161; 770.6729\},$$

$$D_1 = \mathrm{diag}\{60.7901; 38.5704; 292.5837\},$$

$$D_2 = \mathrm{diag}\{598.3266; 113.2884; 679.6303\},$$

$$N_1 = \begin{bmatrix} 17.5677 & 23.3543 & 7.6534 \\ 13.2316 & 32.3273 & -0.7707 \\ 2.8624 & -2.0052 & 40.1712 \end{bmatrix}, N_2 = \begin{bmatrix} 0.0888 & 0.1231 & 0.0411 \\ 0.0729 & 0.1619 & -0.0010 \\ 0.0184 & 0.0006 & 0.1996 \end{bmatrix},$$

$$N_3 = \begin{bmatrix} -0.0015 & 0.0018 & 0.0003 \\ -0.0022 & 0.0034 & 0.0005 \\ -0.0003 & 0.0002 & 0.0001 \end{bmatrix}, N_4 = \begin{bmatrix} 4.2338 & 3.6774 & 1.1835 \\ 3.3037 & 5.0558 & -0.5220 \\ 0.6553 & -0.7298 & 7.5336 \end{bmatrix}.$$

Moreover, the solution $x(t, \phi)$ of the system (5) satisfies

$$\|x(t, \phi)\| \leq 28.9806 e^{-0.1t} \|\phi\|_C.$$

Example 4.2. (Exponential stabilization). Consider the neural networks (1), where $h(t) = 0.2 + 0.3|\sin t|$ and

$$\begin{cases} k(t) = 0.4\cos^2 t & \text{if} \quad i \in \mathcal{I} = \cup_{l \geq 0}[2l\frac{\pi}{2}, (2l+1)\frac{\pi}{2}] \\ k(t) = 0 & \text{if} \quad t \in \mathbb{R}^+ \setminus \mathcal{I}, \end{cases}$$

and

$$A = \begin{bmatrix} 0.15 & 0 \\ 0 & 1 \end{bmatrix}, W_0 = \begin{bmatrix} 0.5 & 0.12 \\ 0.1 & -0.3 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, W_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.1 \end{bmatrix},$$

$$F = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, B = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}.$$

It is worth noting that, the delay functions $h(t), k(t)$ is non-differentiable. Therefore the methods used in [7, 8, 12] are not applicable to this system. We have $h_1 = 0.2, h_2 = 0.5, k = 0.4$. Given $\alpha = 0.1$ and any initial function $\phi(t) \in C^1([-0.5, 0], \mathbb{R}^2)$. Using the Matlab LMI toolbox, we have LMI (20) is feasible with the following matrices:

$$P = \begin{bmatrix} 2.4644 & -0.0050 \\ -0.0050 & 2.5779 \end{bmatrix}, Q = \begin{bmatrix} 2.7017 & -0.0024 \\ -0.0024 & 1.7527 \end{bmatrix},$$

$$R = S = \begin{bmatrix} 442.4320 & 0 \\ 0 & 442.4320 \end{bmatrix}, Y = \begin{bmatrix} -22.3723 & -1.4497 \end{bmatrix},$$

$$M = \begin{bmatrix} 1.2045 & -0.0048 \\ -0.0048 & 1.2261 \end{bmatrix}, X_0 = \begin{bmatrix} 1.3405 & 0 \\ 0 & 1.4784 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 1.5293 & 0 \\ 0 & 1.5730 \end{bmatrix}, X_2 = \begin{bmatrix} 627.8544 & 0 \\ 0 & 629.8994 \end{bmatrix}.$$

Thus the system is 0.1−stabilization and the solution of the closed-loop system satisfies

$$\|x(t,\phi)\| \leq 18.1868e^{-0.1t}\|\phi\|_C.$$

The stabilization feedback control is

$$u(t) = YM^{-1}x(t) = [-18.5790 \quad -1.2549]x(t).$$

5. Conclusions

In this paper, the problem of stability and stabilization of neural networks with mixed interval non-differentiable time-varying delays has been studied. By constructing a set of Lyapunov-Krasovskii functionals combined with Newton-Leibniz formula, new delay-dependent sufficient conditions for the exponential stability and stabilization of the system have been established in terms of LMIs, which allows to compute simultaneously the two bounds that characterize the exponential stability of the solution. Numerical examples illustrated the effectiveness of the obtained results have been given.

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