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Qualification Conditions and Farkas-Type Results for Systems Involving Composite Functions*

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. We are interested in establishing necessary and sufficient conditions for the validity of the functional inequality:

$$f(x) + g(x) + (k \circ H)(x) \geq h(x) \quad \forall x \in X.$$

Necessary and sufficient conditions for such an inequality give rise to Farkas-type results. However, non-asymptotic Farkas-type results often hold under some kind of qualification conditions. In this paper, we firstly propose variants of such conditions associated to the mentioned inequality in the absence of convexity and lower semi-continuity. Secondly, variants of necessary and sufficient conditions of this inequality under our new qualification conditions are established, which lead to new Farkas-type results in general setting (without convexity nor lower semi-continuity). It turns out that these qualification conditions are necessary and sufficient conditions for these Farkas-type results. The results extend or cover many known results of Farkas-type for convex systems or systems involving DC functions in the literature. Alternative-type theorems, set containment results, and generalized Fenchel-Rockafellar duality formula are obtained as consequences of the main results.

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1. Introduction

Let X, Z be locally convex Hausdorff topological spaces, $f, g, h : X \to \mathbb{R} \cup \{+\infty\}$ be proper functions. Let further $H : \operatorname{dom} H \subset X \to Z$ be a mapping, and $k : Z \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Assume that $\operatorname{dom}(f + g + k \circ H) \neq \emptyset$.

We are interested in variant necessary and sufficient conditions in dual forms (i.e., the forms concerning the conjugates of the involved functions) of the inequality

$$f(x) + g(x) + (k \circ H)(x) \geq h(x) \quad \forall x \in X. \tag{1}$$

Such necessary and sufficient conditions are also called *transcriptions* or *characterizations* of (1) (see, e.g., [11, 13]).

To see the role of necessary and sufficient conditions of such an inequality (i.e., (1)), let us consider some simple examples. Assume that C is a closed subset of X, S is a closed convex cone in Z. Assume further that $H : \operatorname{dom} H \subset X \to Z$ is a continuous mapping and convex with respect to the cone S. Let $f \in \Gamma(X)$, $h \equiv 0$, $g := i_C$, and $k := i_{-S}$, where i_C, i_{-S} are the indicator functions of the sets C and $-S$, respectively. Then (1) is equivalent to:

$$x \in C,\ H(x) \in -S \quad \Longrightarrow \quad f(x) \geq 0. \tag{2}$$

Under some qualification condition, such as f *is continuous at a point of* $C \cap H^{-1}(-S)$ and that $\operatorname{int} S \neq \emptyset$ and *there is a point* $\bar{x} \in C$ *such that* $H(\bar{x}) \in -\operatorname{int} S$ (Slater condition), one gets a necessary and sufficient condition which states that (2) is equivalent to: there exists $\lambda \in S^+$ satisfying

$$f(x) + \lambda H(x) \geq 0 \quad \forall x \in C$$

(see Section 6). This is a version of Farkas lemma for convex cone-constrained system (see [12, 32]). In addition, with the presence of the function $h \in \Gamma(X)$, where $\Gamma(X)$ is the set of all proper convex and lower semi-continuous functions on X, (1) becomes

$$x \in C,\ H(x) \in -S \quad \Longrightarrow \quad f(x) - h(x) \geq 0, \tag{3}$$

and each necessary and sufficient condition of (3) gives a version of Farkas lemma (or a Farkas-type result) for systems involving DC functions (see [10, 14, 15]). Special cases of necessary and sufficient conditions of (1) will give rise to Fenchel duality, Fenchel-Rockafellar duality formula (Proposition 6.1), alternative theorems, set containment results as well. These special cases (and more) will be given in Section 6.

It is well-known that Farkas lemma (and its generalized versions) plays an important role in optimization and other fields as well. It is an equivalent form of Hahn-Banch theorem and is a "mathematical version" of the First Fundamental Principle of financial markets [17]. The generalization of this lemma has attracted attentions of many mathematicians in the last decades (see [4, 9, 11, 13, 14, 15, 20, 23, 26, 27, 32] and the references therein). In [11] characterizations of (1) without any qualification conditions yield asymptotic forms of Farkas-type results for systems involving composite functions in convex setting while the ones without qualification conditions nor convexity were given in [13]. Recently, in [16], the authors proposed a kind of dual qualification conditions that guarantee non-asymptotic characterization of inequalities of the form (1) in purely algebraic setting. The present paper can be considered as a continuation of the last mentioned paper. Here, more kinds of dual conditions are introduced under which, variants of Farkas-type results are established. We also give several characterizations of these dual conditions which give rise to generalizations of Moreau-Rockafellar results involving composite functions (see Section 4). The results established in this paper may pay the way to approach nonconvex optimization problems. This needs a further research and will be published elsewhere.

The structure of this paper is as follows: Some notations and preliminaries are given in Section 2. In Section 3 we introduce dual qualification conditions, called (CA), (CB), (CC), (CD), (CE), (CF) and the relations between them. Section 4 is devoted to the characterizations (necessary and sufficient conditions) of each of these conditions. As by products, we get some explicit formulas for calculation of approximate subdifferentials of sum of functions and composite functions which are not necessarily convex. Moreover, the characterizations give rise also to several generalized versions of the well-known Moreau-Rockafellar results in convex analysis to nonconvex cases. General Farkas-type results associated to the inequality of the form (1) are given in Section 5. The results in this section are new in two features: firstly, non-asymptotic Farkas-type results are extended to nonconvex cases and with composite functions; secondly, the dual qualifications we introduced are the weakest ones (necessary and sufficient conditions) for such Farkas-type results. The only result exhibits this second feature, due to the authors' knowledge, was introduced for the first time recently in [23] for a special case associated to a simple convex system. In the last section, Section 6, we show some consequences and applications of the main results in Section 5 specified to special cases. Here, in addition to the Farkas-type results for some special cases which extend or improve the ones in the literature for nonconvex, convex systems or systems involving DC functions, Fenchel-Rockafellar duality formula, several alternative-type theorems, and characterizations of set containments of a convex set in a reverse convex set or in a DC set are also derived.

2. Preliminaries

We now fix some definitions and preliminaries that will be used later on.

Let X, Z be locally convex Hausdorff topological spaces. X^* and Z^* denote their topological dual (respectively), endowed with the weak*-topology. For a set $D \subset X^*$, the **weak*-closure** of D will be denoted simply by $\operatorname{cl} D$. The **indicator function** of D, i_D, is defined by: $i_D(x) := 0$ if $x \in D$ and $i_D(x) := +\infty$ if $x \notin D$.

For any cone S in Z, S^+ denotes the **dual cone** of S, defined by

$$S^+ := \{\theta \in Z^* \mid \langle \theta, s\rangle \geq 0 \quad \forall s \in S\}.$$

In the sequel of the paper, S is always assumed to be convex.

Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$. The **domain** of f, denoted by $\operatorname{dom} f$, is given by $\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}$. The function f is said to be **proper** if $\operatorname{dom} f \neq \emptyset$. The **epigraph** of f is the set

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} \mid x \in \operatorname{dom} f,\ f(x) \leq r\}.$$

For a proper function f, the **conjugate function** of f, $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{\langle v, x\rangle - f(x) \mid x \in \operatorname{dom} f\} \quad \forall v \in X^*.$$

The epigraph of the conjugate of f, f^*, will be defined in a similar way.

For any nonnegative number ϵ, the **ϵ-subdifferential** of a proper function f at a given a in $\operatorname{dom} f$ is defined as the convex set

$$\partial_\epsilon f(a) = \{v \in X^* \mid f(x) - f(a) \geq \langle v, x - a\rangle - \epsilon \quad \forall x \in \operatorname{dom} f\}.$$

It is clear that if $0 \leq \epsilon_1 < \epsilon_2$ then $\partial_{\epsilon_1} f(a) \subset \partial_{\epsilon_2} f(a)$. Note also that if f is convex and lower semi-continuous at $a \in \operatorname{dom} f$ and if $\epsilon > 0$ then $\partial_\epsilon f(a) \neq \emptyset$. Moreover, when f is convex and $\epsilon = 0$, ϵ-subdifferential of f collapses to the usual subdifferential in the sense of convex analysis and will be denoted as ∂f, instead of $\partial_0 f$ (see, e.g., [18, 19, 33]).

The **infimal convolution** of two proper functions $\phi, \psi : X^* \to \overline{\mathbb{R}}$ is the function $\phi \Box \psi$ defined by

$$(\phi \Box \psi)(x^*) = \inf_{u^* \in X^*} \{\phi(u^*) + \psi(x^* - u^*)\} \quad \forall x^* \in X^*.$$

Moreover, if ϕ and ψ are proper and convex then $\phi \Box \psi$ is convex.

Obviously, one also can define the infimum convolution of $\phi, \psi : X \to \overline{\mathbb{R}}$ in the similar way. Moreover, for any $x^*, u^* \in X^*$,

$$(\phi + \psi)^*(x^*) = \sup_{x \in X} \big\{\langle x^*, x\rangle - \phi(x) - \psi(x)\big\}$$

$$= \sup_{x \in X} \left\{ \langle u^*, x \rangle - \phi(x) + \langle x^* - u^*, x \rangle - \psi(x) \right\}$$
$$\leq \sup_{x \in X} \left\{ \langle u^*, x \rangle - \phi(x) \right\} + \sup_{x \in X} \left\{ \langle x^* - u^*, x \rangle - \psi(x) \right\}$$
$$= \phi^*(u^*) + \psi^*(x^* - u^*),$$

which leads to (by taking the infimum over $u^* \in X^*$),

$$(\phi + \psi)^*(x^*) \leq (\phi^* \Box \psi^*)(x^*) \quad \forall x^* \in X^*$$

and, consequently, $\text{epi}(\phi^* \Box \psi^*) \subset \text{epi}(\phi + \psi)^*$. It follows also from the previous simple calculation that for any $x^* \in X^*$, any $(u^*, s) \in \text{epi}\phi^*$, $(v^*, t) \in \text{epi}\psi^*$, and $r \in \mathbb{R}$ such that $x^* = u^* + v^*$, and $r = s + t$, then

$$r = s + t \geq \phi^*(u) + \psi^*(x^* - u) \geq (\phi^* \Box \psi^*)(x^*),$$

which means that $(x^*, r) \in \text{epi}(\phi^* \Box \psi^*)$, and therefore, $\text{epi}\phi^* + \text{epi}\psi^* \subset \text{epi}(\phi^* \Box \psi^*)$. At last, we get

$$\text{epi}\phi^* + \text{epi}\psi^* \subset \text{epi}(\phi^* \Box \psi^*) \subset \text{epi}(\phi + \psi)^*. \tag{4}$$

Now, let S be a convex cone in Z. The mapping $H : X \to Z$ is said to be **S-convex**, if for all $x, y \in X$ and $t \in [0, 1]$ one has

$$H(tx + (1 - t)y) \leq_S tH(x) + (1 - t)H(y),$$

or, equivalently, $H(tx + (1 - t)y) - tH(x) - (1 - t)H(y) \in -S$. Here, "$\leq_S$" is the pre-order defined on Z by S, i.e., for $z, t \in Z$, $z \leq_S t$ means $t - s \in S$.

For $\lambda \in Z^*$, we denote λH the function defined by $(\lambda H)(x) = \langle \lambda, H(x) \rangle$, for all $x \in X$. Moreover, a function $k : Z \to \mathbb{R} \bigcup \{+\infty\}$ is called **S-increasing**, if for all $x, y \in Z$ such that $x \leq_S y$ one has $k(x) \leq k(y)$. Observe that if k is S-increasing, convex and H is S-convex then $k \circ H$ is a convex function.

The set of all proper convex and lower semi-continuous functions on X will be denoted by $\Gamma(X)$.

For the sake of convenience, the following convention will be used: $(+\infty) + (-\infty) = (+\infty) + (+\infty) = +\infty$.

To conclude this section, we recall a result which will play a role in the sequel. The result was proved in [24] where f is assumed to be convex and lower semi-continuous. However, it still holds without these assumptions. For the sake of convenience, its proof will be included.

Lemma 2.1. *Let X be a locally convex Hausdorff topological vector space and let f be a proper function on X. If $a \in \text{dom } f$, then*

$$\text{epi } f^* = \bigcup_{\epsilon \geq 0} \left\{ (v^*, \langle v^*, a \rangle + \epsilon - f(a)) \mid v^* \in \partial_\epsilon f(a) \right\}.$$

Proof. Let $(u^*, r) \in \operatorname{epi} f^*$. Then

$$r \geq f^*(u^*) \geq \langle u^*, x\rangle - f(x) \quad \forall x \in X,$$

which entails $f(x) \geq \langle u^*, x\rangle - r$ for all $x \in X$, and particularly, $f(a) \geq \langle u^*, a\rangle - r$. Let $\epsilon_0 := r + f(a) - \langle u^*, a\rangle$. Then $\epsilon_0 \geq 0$ and $r = \epsilon_0 - f(a) + \langle u^*, a\rangle$. Now, for each $x \in X$,

$$f(x) - f(a) \geq \langle u^*, x\rangle - r - f(a) = \langle u^*, x\rangle - \epsilon_0 + f(a) - \langle u^*, a\rangle - f(a) = \langle u^*, x-a\rangle - \epsilon_0,$$

which means that $u \in \partial_{\epsilon_0} f(a)$. Hence,

$$\text{epi } f^* \subset K := \bigcup_{\epsilon \geq 0} \{(v^*,\ \langle v^*, a\rangle + \epsilon - f(a)) \mid v^* \in \partial_\epsilon f(a)\}.$$

Conversely, let $(u^*, r) \in K$. Then, there exists $\varepsilon_0 \geq 0$ such that $r = \langle u^*, a\rangle + \epsilon_0 - f(a)$ and $u^* \in \partial_{\epsilon_0} f(a)$. By the definition of ϵ-subdifferential, the last inclusion means that

$$\langle u^*, x\rangle - f(x) \leq \epsilon_0 + \langle u^*, a\rangle - f(a) = r \quad \forall x \in X,$$

which entails $f^*(u^*) \leq r$, and so, $(u^*, r) \in \operatorname{epi} f^*$. ■

3. Dual qualification conditions and their relations

In this section we introduce several dual conditions in purely algebraic setting and establish their relations. Some special cases are considered.

3.1. Dual qualification conditions in purely algebraic setting

Let X, Z be Hausdorff topological spaces with its topological dual X^*, Z^*, respectively. The functions f, g, k, and λH $(\lambda \in Z^*)$ are proper functions but not necessarily convex nor lower semicontinuous. Let

$$\mathcal{A} := \text{epi } f^* + \text{epi } g^* + \bigcup_{\lambda \in \text{dom} k^*} \text{epi}(\lambda H - k^*(\lambda))^*,$$
$$\mathcal{B} := \text{epi } f^* + \bigcup_{\lambda \in \text{dom} k^*} \text{epi}(g + \lambda H - k^*(\lambda))^*,$$
$$\mathcal{C} := \text{epi } (f+g)^* + \bigcup_{\lambda \in \text{dom} k^*} \text{epi}(\lambda H - k^*(\lambda))^*,$$
$$\mathcal{D} := \text{epi } g^* + \bigcup_{\lambda \in \text{dom} k^*} \text{epi}(f + \lambda H - k^*(\lambda))^*,$$

$$\mathcal{E} := \bigcup_{\lambda\in \mathrm{dom} k^*} \mathrm{epi}(f+g+\lambda H - k^*(\lambda))^*,$$
$$\mathcal{F} := \mathrm{epi}\ f^* + \mathrm{epi}\ (g + k\circ H)^*.$$

We start with the relations between the sets in consideration.

Lemma 3.1. *With the previous notions, one gets*

$$\mathcal{A}\subset\mathcal{B}\subset\mathcal{E},\quad \mathcal{A}\subset\mathcal{C}\subset\mathcal{E},\quad \mathcal{A}\subset\mathcal{D}\subset\mathcal{E},\quad \mathcal{A}\subset\mathcal{B}\subset\mathcal{F},$$

and

$$\mathcal{E}\subset \mathrm{epi}(f+g+k\circ H)^*,\quad \mathcal{F}\subset \mathrm{epi}(f+g+k\circ H)^*.$$

In particular, if $\mathcal{A} = \mathrm{epi}(f+g+k\circ H)^*$ *then* $\mathcal{A}=\mathcal{B}=\mathcal{C}=\mathcal{D}=\mathcal{E}=\mathcal{F}$.

Proof. The proof is easy, using mainly (4). We first observe that

$$\mathrm{epi}\ g^* + \bigcup_{\lambda\in \mathrm{dom} k^*} \mathrm{epi}(\lambda H - k^*(\lambda))^* = \bigcup_{\lambda\in \mathrm{dom} k^*} \left(\mathrm{epi}\ g^* + \mathrm{epi}(\lambda H - k^*(\lambda))^*\right)$$

and, by (4) with $\phi = g$ and $\psi = \lambda H - k^*(\lambda)$, $\lambda\in \mathrm{dom}\ k^*$, we have $\mathcal{A}\subset\mathcal{B}$. Applying (4) one more time for $\phi = f$ and $\psi = g + \lambda H - k^*(\lambda)$, $(\lambda \in \mathrm{dom}\ k^*)$, we get $\mathcal{B}\subset\mathcal{E}$.

Now, for any $\lambda\in \mathrm{dom}\, k^*$, $x\in \mathrm{dom}\, H$, we get

$$-k^*(\lambda) \le k\circ H(x) - (\lambda H)(x),$$

and so, $\lambda H - k^*(\lambda)\le k\circ H$, which yields $f+g+\lambda H - k^*(\lambda) \le f+g+k\circ H$. We thus have, for all $\lambda\in \mathrm{dom}\ k^*$,

$$\big(f+g+\lambda H - k^*(\lambda)\big)^* \ge \big(f+g+k\circ H\big)^*,\ \big(g+\lambda H - k^*(\lambda)\big)^*\ge \big(g+k\circ H\big)^*,$$

and therefore, it holds for all $\lambda\in \mathrm{dom}\, k^*$,

$$\mathrm{epi}\ \big(f+g+\lambda H - k^*(\lambda)\big)^* \subset \mathrm{epi}\ \big(f+g+k\circ H\big)^*,\ \mathrm{epi}\ \big(g+\lambda H - k^*(\lambda)\big)^* \subset \mathrm{epi}\ \big(g+k\circ H\big)^*,$$

which, in turn, shows that

$$\mathcal{E} = \bigcup_{\lambda\in \mathrm{dom}\, k^*} \mathrm{epi}\ \big(f+g+\lambda H - k^*(\lambda)\big)^* \subset \mathrm{epi}\,(f+g+k\circ H)^*,$$

and

$$\mathrm{epi}\ f^* + \bigcup_{\lambda\in \mathrm{dom} k^*} \mathrm{epi}(g+\lambda H - k^*(\lambda))^* \subset \mathrm{epi}\ f^* + \mathrm{epi}\ (g+k\circ H)^*.$$

We have just proved that

$$\mathcal{A}\subset\mathcal{B}\subset\mathcal{E}\subset \mathrm{epi}\,(f+g+k\circ H)^*$$

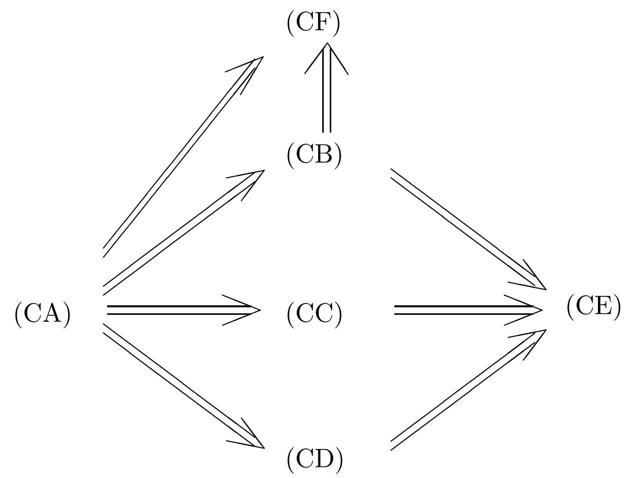
and $\mathcal{B} \subset \mathcal{F}$. The inclusion $\mathcal{F} \subset$ epi $(f + g + k \circ H)^*$ follows from (4). Other inclusions can be proved in the same way. The last assertion is obvious. ∎

We now introduce the following dual conditions:

$$
\begin{array}{ll}
\text{(CA)} & \mathcal{A} = \text{epi}(f + g + k \circ H)^*, \\
\text{(CB)} & \mathcal{B} = \text{epi}(f + g + k \circ H)^*, \\
\text{(CC)} & \mathcal{C} = \text{epi}(f + g + k \circ H)^*, \\
\text{(CD)} & \mathcal{D} = \text{epi}(f + g + k \circ H)^*, \\
\text{(CE)} & \mathcal{E} = \text{epi}(f + g + k \circ H)^*, \\
\text{(CF)} & \mathcal{F} = \text{epi}(f + g + k \circ H)^*.
\end{array}
$$

The relations between these conditions are given in the next theorem.

Theorem 3.2. *The following implications hold.*



Here (A) $\Longrightarrow$ (B) *means that condition* (A) *implies condition* (B).

Proof. We observe that if (CA) holds, i.e.,

$$\mathcal{A} = \text{epi}(f + g + k \circ H)^*,$$

then by Lemma 3.1, we get $\mathcal{A} = \mathcal{B} = \mathcal{E} = \text{epi}(f+g+k\circ H)^*$, $\mathcal{A} = \mathcal{C} = \mathcal{E} = \text{epi}(f+g+k\circ H)^*$, $\mathcal{A} = \mathcal{D} = \mathcal{E} = \text{epi}(f+g+k\circ H)^*$, and $\mathcal{A} = \mathcal{B} = \mathcal{F} = \text{epi}(f+g+k\circ H)^*$. This means that (CA) holds then (CB), (CC), (CD), (CE), and (CF) do, too. Also, by the same argument using Lemma 3.1 we see that if one of (CB), (CC), (CD) holds then (CE) holds, and if (CB) holds then (CF) holds. The proof is complete. ∎

The conditions (CA) and (CB) were introduced in [16], where some Farkas-type results were established under (CA).

We now consider a special case. Let C be a subset of X, S be a convex cone in Z. Let $g := i_C$, $k := i_{-S}$ and $A := C \cap H^{-1}(-S)$. Then it is easily seen that $k^* = i_{S^+}$, and hence, $\operatorname{dom} k^* = S^+$. Moreover, $i_A = i_C + i_{-S} \circ H$, where

$$k \circ H(x) = i_{-S} \circ H(x) = \begin{cases} 0 & \text{if } H(x) \in -S, \\ +\infty & \text{otherwise.} \end{cases}$$

Thus, the condition (CA) becomes

$$\operatorname{epi} f^* + \bigcup_{\lambda \in S^+} \operatorname{epi} (\lambda H)^* + \operatorname{epi} i_C^* = \operatorname{epi} (f + i_A)^*.$$

By the same procedure, we get other modifications of other conditions (CB), (CD), (CE), and (CF). Concretely, we consider the following modifications of these conditions:

$$(\mathrm{CA}_1) \qquad \operatorname{epi} f^* + \bigcup_{\lambda \in S^+} \operatorname{epi} (\lambda H)^* + \operatorname{epi} i_C^* = \operatorname{epi} (f + i_A)^*,$$

$$(\mathrm{CB}_1) \qquad \operatorname{epi} f^* + \bigcup_{\lambda \in S^+} \operatorname{epi} (\lambda H + i_C)^* = \operatorname{epi} (f + i_A)^*,$$

$$(\mathrm{CC}_1) \qquad \operatorname{epi} (f + i_C)^* + \bigcup_{\lambda \in S^+} \operatorname{epi} (\lambda H)^* = \operatorname{epi} (f + i_A)^*,$$

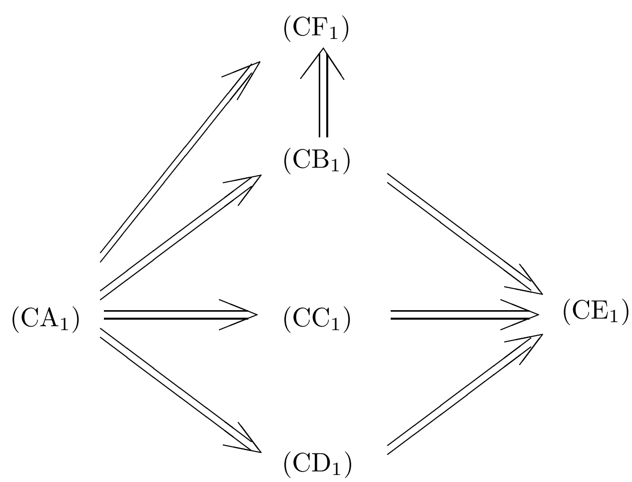
$$(\mathrm{CD}_1) \qquad \operatorname{epi} i_C^* + \bigcup_{\lambda \in S^+} \operatorname{epi} (f + \lambda H)^* = \operatorname{epi} (f + i_A)^*,$$

$$(\mathrm{CE}_1) \qquad \bigcup_{\lambda \in S^+} \operatorname{epi} (f + \lambda H + i_C)^* = \operatorname{epi} (f + i_A)^*,$$

$$(\mathrm{CF}_1) \qquad \operatorname{epi} f^* + \operatorname{epi} i_A^* = \operatorname{epi} (f + i_A)^*.$$

Then as a direct consequence of Theorem 3.2, we get

Corollary 3.3. *The following relations hold.*



3.2. Dual qualification conditions in convex setting

We now give necessary and sufficient for dual qualification conditions introduced in the previous section in the cases where the convexity and lower semi-continuity of functions involved are assumed. Some sufficient conditions (also, necessary and sufficient conditions) for the validity of (CA) were given in [16]. We quote some of them for the sake of convenience.

Assume that Z will be preordered by a non-negative convex cone $\emptyset \neq S \subset Z$ and $\mathrm{dom}\,(f+g+k\circ H) \neq \emptyset$. Assume that $f, g \in \Gamma(X)$, $k \in \Gamma(Z)$, $\lambda H \in \Gamma(X)$ for all $\lambda \in \mathrm{dom}\,k^*$, H is S-convex, k is S-increasing on $H(\mathrm{dom}\,H) + S$.

Proposition 3.4. [16] *If k is finite and continuous at a point of $H(\mathrm{dom}\,H)$, and f, g are finite and continuous at the same point of $H^{-1}(\mathrm{dom}\,k)$. Then*

$$\mathrm{epi}\,(f+g+k\circ H)^* = \mathcal{A}.$$

Proposition 3.5. [16] $\mathrm{epi}\,(f+g+k\circ H)^* = \mathcal{A}$ *(resp. $\mathcal{B}$) if and only if $\mathcal{A}$ (resp. $\mathcal{B}$) is w^*-closed.*

Proposition 3.6. $\mathrm{epi}\,(f+g+k\circ H)^* = \mathcal{C}$ *if and only if $\mathcal{C}$ is w^*-closed.*

Proof. Let us introduce the extended real-valued function ϕ defined on X^* by

$$\phi = (f+g)^* \Box \Big(\inf_{\lambda \in \mathrm{dom} k^*} (\lambda H - k^*(\lambda))^* \Big).$$

We firstly observe that ϕ is convex. Indeed, let $\psi : \mathrm{dom}\ k^* \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$\psi(\lambda, u) := (\lambda H - k^*(\lambda))^*.$$

It is easy to verify that ψ is convex. By Theorem 2.1.3(v) in [33], the function

$$\inf_{\lambda \in \mathrm{dom} k^*} (\lambda H - k^*(\lambda))^* = \inf_{\lambda \in \mathrm{dom} k^*} \psi(\lambda, u)$$

is convex and hence, ϕ is convex.

Now, from the assumption that $f, g \in \Gamma(X)$, $k \in \Gamma(Z)$, and $\lambda H \in \Gamma(X)$ for all $\lambda \in \mathrm{dom} k^*$, one gets, for any $x \in X$,

$$\begin{aligned}
\phi^*(x) &= \sup_{x^* \in X^*} \Big\{ \langle x^*, x\rangle - (f+g)^* \Box \big(\inf_{\lambda \in \mathrm{dom} k^*} (\lambda H - k^*(\lambda))^* \big) \Big\} \\
&= \sup_{x^* \in X^*} \Big\{ \langle x^*, x\rangle - \inf_{x^* = u+v} \big\{ (f+g)^*(u) + \inf_{\lambda \in \mathrm{dom} k^*} (\lambda H - k^*(\lambda))^*(v) \Big\} \\
&= \sup_{x^* \in X^*} \Big\{ \langle x^*, x\rangle - \inf_{\lambda \in \mathrm{dom} k^*} \inf_{x^* = u+v} \big\{ (f+g)^*(u) + (\lambda H - k^*(\lambda))^*(v) \Big\} \\
&= \sup_{\lambda \in \mathrm{dom} k^*} \sup_{x^* \in X^*} \Big\{ \langle x^*, x\rangle - \inf_{x^* = u+v} \big\{ (f+g)^*(u) + (\lambda H - k^*(\lambda))^*(v) \big\} \Big\}
\end{aligned}$$

$$= \sup_{\lambda\in\text{dom}k^*} \sup_{u,v\in X^*} \Big\{\langle u,x\rangle + \langle v,x\rangle - (f+g)^*(u) - (\lambda H)^*(v) - k^*(\lambda)\Big\}$$
$$= \sup_{\lambda\in\text{dom}k^*} \Big\{(f+g)(x) + (\lambda H)(x) - k^*(\lambda)\Big\}$$
$$= f(x) + g(x) + \sup_{\lambda\in\text{dom}k^*} \big\{(\lambda H)(x) - k^*(\lambda)\big\}$$
$$= f(x) + g(x) + (k\circ H)(x)$$

and thus,

$$\text{epi}\,(f+g+k\circ H)^* = \text{epi}\,\phi^{**} = \text{epi}\,\overline{\phi}, \tag{5}$$

where $\overline{\phi}$ is the lower semi-continuous regularization of ϕ (which is defined via the equality $\text{epi}\,\overline{\phi} = \text{clepi}\,\phi$).

On the other hand, By Theorem 2.2(e) in [30], we get

$$\begin{aligned}\text{epi}\,\overline{\phi} &= \text{epi}\,\big(\overline{(f+g)^*\Box \inf_{\lambda\in\text{dom}k^*}(\lambda H - k^*(\lambda))^*}\big)\\ &= \text{cl}\,\big(\text{epi}\,(f+g)^* + \text{epi}\,[\inf_{\lambda\in\text{dom}k^*}(\lambda H - k^*(\lambda))^*]\big)\\ &= \text{cl}\,\big(\text{epi}\,(f+g)^* + \bigcup_{\lambda\in\text{dom}k^*} \text{epi}\,[\lambda H - k^*(\lambda)]^*\big)\\ &= \text{cl}\,\mathcal{C}.\end{aligned} \tag{6}$$

Combining (5) and (6), we get $\text{epi}\,(f+g+k\circ H)^* = \text{cl}\,\mathcal{C}$. The proof is complete. ■

In the same way, we get

Proposition 3.7. *The following statements are true:*

(i) $\text{epi}\,(f+g+k\circ H)^* = \mathcal{D}$ *if and only if* $\mathcal{D}$ *is* w^**-closed,*

(ii) $\text{epi}\,(f+g+k\circ H)^* = \mathcal{E}$ *if and only if* $\mathcal{E}$ *is* w^**-closed,*

(iii) $\text{epi}\,(f+g+k\circ H)^* = \mathcal{F}$ *if and only if* $\mathcal{F}$ *is* w^**-closed.*

Proof. Similar to the proof of Proposition 3.6, using Theorem 2.2 in [30] and consider the following functions ϕ_1, ϕ_2, and ϕ_3 for the case (i), (ii), and (iii), respectively: $\phi_1 = g^*\Box\big(\inf_{\lambda\in\text{dom}k^*}(f+\lambda H - k^*(\lambda))^*\big)$, $\phi_2 = \inf_{\lambda\in\text{dom}k^*}(f+g+\lambda H - k^*(\lambda))^*$, and $\phi_3 = f^*\Box(g+k\circ H)^*$. ■

4. Characterizations of dual conditions - Generalized Moreau-Rockafellar results

In this section we shall establish characterizations of the dual conditions introduced in Section 3, namely, (CA)–(CF). These characterizations at the same time are variant versions of generalization of Moreau-Rockafellar results on the conjugates and subdifferentials of a sum of a convex function with a composition of convex functions in Hausdorff locally convex spaces (see e.g., [1, 3, 33] for more details) to nonconvex cases and with approximate subdifferentials. The results cover some recent ones from [3] and generalized many other results of this type with the presence of convexity in the literature. It is worth mentioning that these results generalize classical results in three aspects: The class of functions is broaden (to nonconvex ones), the assumptions are weaken, and they supply necessary and sufficient conditions while in most of the cases (even for the convex one) only the sufficient conditions are given in the literature. In particular, they cover Fenchel duality, Moreau-Rockafellar theorems concerning the sum of convex functions or of a convex function with a composite of a convex function and a linear mapping (Corollary 4.6).

Assume that the functions f, g, k, and λH $(\lambda \in Z^*)$ are proper functions but not necessarily convex nor lower semicontinuous. We start by establishing the characterizations of (CA).

Theorem 4.1. *The following statements are equivalent:*

(a) (CA) *holds,*

(b) *for all* $x^* \in \operatorname{dom}(f+g+k\circ H)^*$,

$$(f+g+k\circ H)^*(x^*) = \min_{\substack{\lambda\in\operatorname{dom}k^* \\ u\in\operatorname{dom}f^*,\ v\in\operatorname{dom}g^*}} \Big\{ f^*(u)+g^*(v)+(\lambda H)^*(x^*-u-v)+k^*(\lambda)\Big\},$$

(c) *for all* $\bar{x} \in \operatorname{dom}(f+g+k\circ H)$ *and all* $\epsilon \geq 0$,

$$\partial_\epsilon(f+g+k\circ H)(\bar{x}) = \bigcup_{\lambda\in\operatorname{dom}k^*} \bigcup_{\substack{\epsilon_1,\epsilon_2,\epsilon_3\geq 0 \\ \epsilon_1+\epsilon_2+\epsilon_3+k^*(\lambda)+(kH)(\bar{x})=\epsilon+(\lambda H)(\bar{x})}} \Big\{ \partial_{\epsilon_1} f(\bar{x}) + \partial_{\epsilon_2} g(\bar{x}) + \partial_{\epsilon_3}(\lambda H)(\bar{x})\Big\}.$$

Proof. $[(a) \Rightarrow (b)]$ Applying the Fenchel inequality to the conjugate functions f^*, g^*, $(\lambda H)^*$, and k^*, it is easy to see that for all $x^* \in X^*$,

$$(f+g+k\circ H)^*(x^*) \leq \min_{\substack{\lambda\in\operatorname{dom}k^* \\ u\in\operatorname{dom}f^*,\ v\in\operatorname{dom}g^*}} \Big\{ f^*(u)+g^*(v)+(\lambda H)^*(x^*-u-v)+k^*(\lambda)\Big\}. \tag{7}$$

To proceed, let $x^* \in \operatorname{dom}(f+g+k\circ H)^*$. Then $\big(x^*, (f+g+k\circ H)^*(x^*)\big) \in \operatorname{epi}(f+g+k\circ H)^*$. By (a) there exist $\overline{\lambda} \in \operatorname{dom}k^*$, $(\overline{u}, r) \in \operatorname{epi}f^*$, $(\overline{v}, s) \in \operatorname{epi}g^*$, and $(\overline{w}, t) \in \operatorname{epi}(\overline{\lambda}H - k^*(\overline{\lambda}))^*$ such that

$$\big(x^*, (f+g+k\circ H)^*(x^*)\big) = (\overline{u}, r) + (\overline{v}, s) + (\overline{w}, t),$$

which yields $x^* = \overline{u} + \overline{v} + \overline{w}$ and

$$(f+g+k\circ H)^*(x^*) = r+s+t \geq f^*(\overline{u}) + g^*(\overline{v}) + (\overline{\lambda}H)^*(x^* - \overline{u} - \overline{v}) + k^*(\overline{\lambda}).$$

Combining the last inequality and (7) we get

$$\begin{aligned}(f+g+k\circ H)^*(x^*) &= f^*(\overline{u}) + g^*(\overline{v}) + (\overline{\lambda}H)^*(x^* - \overline{u} - \overline{v}) + k^*(\overline{\lambda}) \\ &= \min_{\substack{\lambda\in \mathrm{dom} k^* \\ u\in \mathrm{dom} f^*, v\in \mathrm{dom} g^*}} \Big\{ f^*(u)+g^*(v)+(\lambda H)^*(x^*-u-v) + k^*(\lambda)\Big\},\end{aligned}$$

which shows that the equality in (b) holds and moreover, the minimum is attained.

$[(b) \Rightarrow (c)]$ The inclusion " $\supset$ " in (c) follows easily from the definition of the ϵ-subdifferential. For the converse inclusion, let us take $x^* \in \partial_\epsilon(f+g+k\circ H)(\bar{x})$. Since $\bar{x} \in \mathrm{dom}(f+g+k\circ H)$, Lemma 2.1 yields

$$(f+g+k\circ H)^*(x^*)) \leq \epsilon + \langle x^*, \bar{x}\rangle - (f+g+k\circ H)(\bar{x}). \tag{8}$$

By (b) there exist $\lambda \in \mathrm{dom}\, k^*$, $u \in \mathrm{dom} f^*, v \in \mathrm{dom} g^*, w \in \mathrm{dom}(\lambda H)^*$ such that $u+v+w = x^*$ and

$$(f+g+k\circ H)^*(x^*) = f^*(u) + g^*(v) + (\lambda H)^*(w) + k^*(\lambda).$$

Combining this and (8), we get

$$\epsilon + \langle x^*, \bar{x}\rangle - (f+g+k\circ H)(\bar{x}) \geq f^*(u) + g^*(v) + (\lambda H)^*(w) + k^*(\lambda). \tag{9}$$

Note that $(u, f^*(u)) \in \mathrm{epi} f^*$, and so, if we set $\epsilon_1 = f^*(u) - \langle u, \bar{x}\rangle + f(\bar{x}) \geq 0$ then by Lemma 2.1, $u \in \partial_{\epsilon_1} f(\bar{x})$. Similarly, if we set $\epsilon_2 = g^*(v) - \langle v, \bar{x}\rangle + g(\bar{x}) \geq 0$ and $\epsilon_3' = (\lambda H)^*(w) - \langle w, \bar{x}\rangle + (\lambda H)(\bar{x}) \geq 0$ then $v \in \partial_{\epsilon_2} g(\bar{x})$ and $w \in \partial_{\epsilon_3'}(\lambda H)(\bar{x})$.

Now, (9) together with the fact that $x^* = u+v+w$ yields

$$\begin{aligned}\epsilon \geq \big\{f^*(u) - \langle u, \bar{x}\rangle + f(\bar{x})\big\} + \big\{g^*(v) - \langle v, \bar{x}\rangle + g(\bar{x})\big\} + \big\{(\lambda H)^*(w) - \langle w, \bar{x}\rangle + (\lambda H)(\bar{x})\big\} \\ + (k\circ H)(\bar{x}) - (\lambda H)(\bar{x}) + k^*(\lambda),\end{aligned}$$

or, equivalently,

$$\epsilon \geq \epsilon_1 + \epsilon_2 + \epsilon_3' + (k\circ H)(\bar{x}) - (\lambda H)(\bar{x}) + k^*(\lambda).$$

Set

$$\epsilon_3 := \epsilon - \epsilon_1 - \epsilon_2 + (\lambda H)(\bar{x}) - (k\circ H)(\bar{x}) - k^*(\lambda) \geq \epsilon_3'.$$

Then $\epsilon_3 \geq 0$ and hence, $w \in \partial_{\epsilon_3'}(\lambda H)(\bar{x}) \subset \partial_{\epsilon_3}(\lambda H)(\bar{x})$. Moreover,

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + k^*(\lambda) + (kH)(\bar{x}) = \epsilon + (\lambda H)(\bar{x}),$$

which shows that $x^* = u+v+w$ belongs to the right-hand side of the relation in (c).

$[(c) \Rightarrow (a)]$ By Lemma 3.1, it is sufficient to show that

$$\text{epi}(f+g+k\circ H)^* \subset \text{epi } f^* + \text{epi } g^* + \bigcup_{\lambda\in\text{dom}k^*} \text{epi}(\lambda H - k^*(\lambda))^*.$$

Take $(x^*, r) \in \text{epi}(f+g+k\circ H)^*$ and $\bar{x} \in \text{dom}(f+g+k\circ H)$. By Lemma 2.1 there exists $\epsilon \geq 0$ such that $x^* \in \partial_\epsilon (f+g+k\circ H)(\bar{x})$ and $r = \epsilon + \langle x^*, \bar{x}\rangle - (f+g+k\circ H)(\bar{x})$.

It now follows from (c) that there are $\lambda \in \text{dom}k^*, \epsilon_1, \epsilon_2, \epsilon_3 \geq 0$, $u \in \partial_{\epsilon_1} f(\bar{x})$, $v \in \partial_{\epsilon_2} g(\bar{x})$, and $w \in \partial_{\epsilon_3}(\lambda H)(\bar{x})$ such that $x^* = u + v + w$ and

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + k^*(\lambda) + (k\circ H)(\bar{x}) = \epsilon + (\lambda H)(\bar{x}).$$

Let $s = \epsilon_1 + \langle u, \bar{x}\rangle - f(\bar{x})$, $t = \epsilon_2 + \langle v, \bar{x}\rangle - g(\bar{x})$, $q = \epsilon_3 + \langle w, \bar{x}\rangle - (\lambda H)(\bar{x})$. Again, by Lemma 2.1, $(u, s) \in \text{epi}f^*$, $(v, t) \in \text{epi}g^*$, and $(w, q) \in \text{epi}(\lambda H)^*$. Moreover,

$$\begin{aligned} s + t + q &= \epsilon_1 + \epsilon_2 + \epsilon_3 + \langle x^*, \bar{x}\rangle - f(\bar{x}) - g(\bar{x}) - (\lambda H)(\bar{x}) \\ &= \epsilon - (k\circ H)(\bar{x}) - k^*(\lambda) + \langle x^*, \bar{x}\rangle - f(\bar{x}) - g(\bar{x}) \\ &= \epsilon + \langle x^*, \bar{x}\rangle - (f+g+k\circ H)(\bar{x}) - k^*(\lambda) \\ &= r - k^*(\lambda), \end{aligned}$$

and hence,

$$\begin{aligned} (x^*, r) &= (x^*, s + t + q + k^*(\lambda)) \\ &= (u, s) + (v, t) + (w, q) + (0, k^*(\lambda)) \\ &\in \text{epi}f^* + \text{epi}g^* + \text{epi}(\lambda H)^* + (0, k^*(\lambda)) \\ &\subset \text{epi } f^* + \text{epi } g^* + \bigcup_{\lambda\in\text{dom}k^*} \text{epi}(\lambda H - k^*(\lambda))^*. \end{aligned}$$

The proof is complete. ■

In the same way, we get

Theorem 4.2. *The following statements are equivalent:*

(a) (CB) *holds,*

(b) *for each* $x^* \in \text{dom}\,(f+g+k\circ H)^*$,

$$(f+g+k\circ H)^*(x^*) = \min_{\substack{\lambda\in\text{dom}k^* \\ u\in\text{dom}f^*}} \Big\{ f^*(u) + (g+\lambda H)^*(x^* - u) + k^*(\lambda) \Big\},$$

(c) *for all* $\bar{x} \in \text{dom}(f+g+k\circ H)$ *and all* $\epsilon \geq 0$,

$$\begin{aligned} &\partial_\epsilon (f+g+k\circ H)(\bar{x}) \\ &= \bigcup_{\lambda\in\text{dom}k^*} \bigcup_{\substack{\epsilon_1,\epsilon_2\geq 0 \\ \epsilon_1+\epsilon_2+k^*(\lambda)+(kH)(\bar{x})=\epsilon+(\lambda H)(\bar{x})}} \Big\{ \partial_{\epsilon_1} f(\bar{x}) + \partial_{\epsilon_2}(g+\lambda H)(\bar{x}) \Big\}. \end{aligned}$$

Theorem 4.3. *The following statements are equivalent:*

(a) (CC) *holds,*

(b) *for all* $x^* \in \mathrm{dom}(f+g+k\circ H)^*$,

$$(f+g+k\circ H)^*(x^*) = \min_{\substack{\lambda\in \mathrm{dom}k^*\\ u\in\mathrm{dom}(f+g)^*}} \Big\{(f+g)^*(u)+(\lambda H)^*(x^*-u)+k^*(\lambda)\Big\},$$

(c) *for all* $\bar{x} \in \mathrm{dom}(f+g+k\circ H)$ *and all* $\epsilon \geq 0$,

$$\begin{aligned}&\partial_\epsilon(f+g+k\circ H)(\bar{x})\\ &= \bigcup_{\lambda\in\mathrm{dom}k^*} \bigcup_{\substack{\epsilon_1,\epsilon_2\geq 0\\ \epsilon_1+\epsilon_2+k^*(\lambda)+(kH)(\bar{x})=\epsilon+(\lambda H)(\bar{x})}} \Big\{\partial_{\epsilon_1}(f+g)(\bar{x})+\partial_{\epsilon_2}(\lambda H)(\bar{x})\Big\}.\end{aligned}$$

Characterizations for (CD), (CE), and (CF) are easy modifications and will be ignored.

Remark 4.4. By Proposition 3.3, if the condition (CA) holds then all the others (CB), (CC), (CD), (CE), and (CF) do, too, and so, Theorems 4.1–4.3 yield variants of representations of $(f+g+k\circ H)^*(x^*)$ which extend Moreau-Rockafellar-type formula (see [3]), and representations of $\partial_\epsilon(f+g+k\circ H)(\bar{x})$.

Theorems 4.1–4.3 also yield many extended versions of results known in the literature. Let us take an example, when $k\equiv 0$ and $H\equiv 0$, Theorem 4.1 turns back to a recent result in [7], which is a nonasymptotic version of [33, Corollary 2.6.7], and is also a relaxed version of a result in [18, Section II] where it is assumed that at least one of the functions f, g is continuous at a point in the domain of the other, which is stronger than the condition proposed in this corollary [7].

It is worth observing also that when f is convex and $\epsilon = 0$ then ϵ-subdifferential of f, $\partial_\epsilon f(\cdot)$ reduces to the $\partial f(\cdot)$ in the sense of convex analysis. The following corollary is a direct consequence of Theorem 4.1.

Corollary 4.5. *Assume that* $f, g \in \Gamma(X)$ *and* $\lambda H \in \Gamma(X)$ *for all* $\lambda \in \mathrm{dom}\, k^*$*. If* (CA) *holds then for each* $\bar{x} \in \mathrm{dom}(f+g+k\circ H)$*, one has*

$$\partial(f+g+k\circ H)(\bar{x}) = \partial f(\bar{x}) + \partial g(\bar{x}) + \bigcup_{\lambda\in\partial k(H(\bar{x}))} \partial(\lambda H)(\bar{x}).$$

In particular, if $\mathrm{epi}\,(f+k\circ H)^* = \mathrm{epi}\, f^* + \bigcup_{\lambda\in\mathrm{dom}k^*} \mathrm{epi}(\lambda H - k^*(\lambda))^*$ *then*

$$\partial(f+k\circ H)(\bar{x}) = \partial f(\bar{x}) + \bigcup_{\lambda\in\partial k(H(\bar{x}))} \partial(\lambda H)(\bar{x}).$$

Proof. This is a direct consequence of Theorem 4.1 by taking $\epsilon = 0$ in (c). Indeed, if $\epsilon = 0$ then from the equality

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + k^*(\lambda) + (kH)(\bar{x}) = \epsilon + (\lambda H)(\bar{x}),$$

the Fenchel inequality $k^*(\lambda) \geq \lambda H(\bar{x}) - k(H(\bar{x}))$, and the fact that $\epsilon_1, \epsilon_2, \epsilon_3 \geq 0$ we get $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$ and $k^*(\lambda) = \lambda H(\bar{x}) - k(H(\bar{x}))$. The last equality yields $\lambda \in \partial k(H(\bar{x}))$. The second assertion is obvious by taking $g \equiv 0$. ■

We now consider the case where H is a continuous linear operator $H = A \in \mathcal{L}(X, Z)$, where $\mathcal{L}(X, Z)$ denotes the set of all continuous linear operators from X to Z, we then find again the results in convex analysis under a weaker qualification condition which was established recently in [2] and [4]. Here A^* denotes the adjoint operator of A and $(A^* \times Id_{\mathbb{R}})(\operatorname{epi} k^*)$ denotes the image of the set $\operatorname{epi} k^*$ through the function $A^* \times Id_{\mathbb{R}} : Z^* \times \mathbb{R} \longrightarrow X^* \times \mathbb{R}$, defined by $(A^* \times Id_{\mathbb{R}})(z^*, r) = (A^* z^*, r)$.

Corollary 4.6. [4, Theorem 5.4] *Assume that $A \in \mathcal{L}(X, Z)$. Assume further that*

$$\operatorname{epi}(f + k \circ A)^* = \operatorname{epi} f^* + (A^* \times Id_{\mathbb{R}})(\operatorname{epi} k^*). \tag{10}$$

Then

(a) *For any $x^* \in \operatorname{dom}(f + k \circ A)^*$,*

$$(f + k \circ A)^*(x^*) = \min_{\lambda \in \operatorname{dom} k^*} [k^*(\lambda) + f^*(x^* - A^*\lambda)],$$

(b) *If in addition, $f \in \Gamma(X)$ and $k \in \Gamma(Z)$ then for each $\bar{x} \in \operatorname{dom} f \cap A^{-1}(\operatorname{dom} k)$,*

$$\partial(f + k \circ A)(\bar{x}) = \partial f(\bar{x}) + A^* \partial k(A\bar{x}).$$

Proof. We firstly observe that if $\lambda \in \operatorname{dom} k^*$ then

$$\begin{aligned}
\big(\lambda A - k^*(\lambda)\big)^*(x^*) &= \sup_{x \in X} \{\langle x^*, x\rangle - \langle \lambda, Ax\rangle + k^*(\lambda)\} \\
&= \sup_{x \in X} \{\langle x^* - A^*\lambda, x\rangle\} + k^*(\lambda) \\
&= \begin{cases} k^*(\lambda) & \text{if } \ x^* = A^*\lambda, \\ +\infty & \text{else.} \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
(x^*, r) \in (A^* \times Id_{\mathbb{R}})(\operatorname{epi} k^*) &\iff (\exists \lambda \in \operatorname{dom} k^* : \ A^*\lambda = x^*, \text{and } (\lambda, r) \in \operatorname{epi} k^*) \\
&\iff \begin{pmatrix} \exists \lambda \in \operatorname{dom} k^*, \ x^* = A^*\lambda, \\ (\lambda A - k^*(\lambda))^* (x^*) = k^*(\lambda) \leq r \end{pmatrix} \\
&\iff (x^*, r) \in \bigcup_{\lambda \in \operatorname{dom} k^*} \operatorname{epi}(\lambda A - k^*(\lambda))^*,
\end{aligned}$$

which, together with (10), shows that

$$\operatorname{epi}(f + k \circ A)^* = \operatorname{epi} f^* + (A^* \times Id_{\mathbb{R}})(\operatorname{epi} k^*) = \operatorname{epi} f^* + \bigcup_{\lambda \in \operatorname{dom} k^*} \operatorname{epi}(\lambda A - k^*(\lambda))^*,$$

and hence, (d) in Theorem 4.2 holds (i.e., (CB) holds) for the case where $g \equiv 0$ and $H = A \in \mathcal{L}(X, Z)$.

- *Proof of* (a). By Theorem 4.2,

$$(f + k \circ A)^*(x^*) = \min_{\substack{\lambda \in \mathrm{dom} k^* \\ u \in \mathrm{dom} f^*}} \Big\{ f^*(u) + (\lambda A)^*(x^* - u) + k^*(\lambda) \Big\}. \tag{11}$$

Observe that the left-hand side of (11) is finite and for $u \in \mathrm{dom}\, f^*$,

$$\begin{aligned} \big(\lambda A\big)^*(x^* - u) &= \sup_{x \in X} \{ \langle x^* - u, x \rangle - \langle \lambda, Ax \rangle \} \\ &= \sup_{x \in X} \{ \langle x^* - A^*\lambda - u, x \rangle \} \\ &= \begin{cases} 0 & \text{if } u = x^* - A^*\lambda, \\ +\infty & \text{else.} \end{cases} \end{aligned}$$

Combining this and (11), one gets

$$\begin{aligned} (f + k \circ A)^*(x^*) &= \min_{\substack{\lambda \in \mathrm{dom} k^* \\ u \in \mathrm{dom} k^*}} \Big\{ f^*(u) + (\lambda A)^*(x^* - u) + k^*(\lambda) \Big\} \\ &= \min_{\lambda \in \mathrm{dom} k^*} [k^*(\lambda) + f^*(x^* - A^*\lambda)], \end{aligned}$$

which is (a).

- *Proof of* (b). It is easy to see that $(\lambda A)^*(u) = 0$ when $u = A^*\lambda$ and that

$$\begin{aligned} u \in A^*\partial k(A(\bar{x})) &\iff (\exists \lambda \in \partial k(A\bar{x}),\ u = A^*\lambda) \\ &\iff (\exists \lambda \in \partial k(A\bar{x}),\ (\lambda A)^*(u) + \langle A^*\lambda, \bar{x} \rangle = \langle u, \bar{x} \rangle) \\ &\iff (\exists \lambda \in \partial k(A\bar{x}),\ u \in \partial(\lambda A)(\bar{x})) \\ &\iff u \in \bigcup_{\lambda \in \partial k(A(\bar{x}))} \partial(\lambda A)(\bar{x}), \end{aligned}$$

which means that $\bigcup_{\lambda \in \partial k(A(\bar{x}))} \partial(\lambda A)(\bar{x}) = A^*\partial k(A\bar{x})$. Assertion (b) now follows from Corollary 4.5. The proof is complete. ■

5. Characterizations of (1) - Nonconvex Farkas-type results

We are now in a position to establish the key results of this paper: variant characterizations (necessary and sufficient conditions) of (1) in pure algebraic setting (i.e., without any convexity nor topological assumptions). Results of this type are often known as Farkas-type results in the literature. We will show that the dual conditions introduced in Section 3 are both necessary and sufficient for the validity of these Farkas-type results. The results amount to two new features: Firstly, to the knowledge of the authors, this is the first time non-asymptotic Farkas lemma is extended to the case with composite functions which are not necessarily convex nor lower semicontinuous. Secondly, the results give also necessary conditions (not only the sufficient conditions) for such Farkas-type results, which mean that our dual qualification conditions are the weakest possible conditions under which these Farkas-type results remain true. One result of this type was found in the literature for the first time in [23] for a special case where $g \equiv 0$, $k = i_{-S}$ (S is a closed convex cone), H is a S-convex, continuous mapping, and f is a continuous and convex function.

As in the previous section, the functions f, g, k, and λH ($\lambda \in Z^*$) are proper functions but not necessarily convex nor lower semicontinuous provided that $f + g + k \circ H$ is proper. We first give different characterizations of the functional inequality (1).

Theorem 5.1. *Assume that* (CA) *holds and* $h \in \Gamma(X)$*. The following statements are equivalent:*

(I) $f(x) + g(x) + (k \circ H)(x) \geq h(x) \quad \forall x \in X$;

(II) *For any* $x^* \in \operatorname{dom} h^*$*, there exist* $\lambda \in \operatorname{dom} k^*$*,* $u \in \operatorname{dom} f^*$*,* $v \in \operatorname{dom} g^*$ *such that*

$$h^*(x^*) \geq f^*(u) + g^*(v) + (\lambda H)^*(x^* - u - v) + k^*(\lambda);$$

(III) *For any* $x^* \in \operatorname{dom} h^*$*, there exist* $\lambda \in \operatorname{dom} k^*$*,* $u \in \operatorname{dom} f^*$ *such that*

$$h^*(x^*) \geq f^*(u) + (g + \lambda H)^*(x^* - u) + k^*(\lambda);$$

(IV) *For any* $x^* \in \operatorname{dom} h^*$*, there exists* $\lambda \in \operatorname{dom} k^*$ *such that*

$$h^*(x^*) \geq (f + g + \lambda H)^*(x^*) + k^*(\lambda);$$

(V) *For any* $x^* \in \operatorname{dom} h^*$*, there exist* $\lambda \in \operatorname{dom} k^*$*,* $u \in \operatorname{dom} g^*$ *such that*

$$h^*(x^*) \geq g^*(u) + (f + \lambda H)^*(x^* - u) + k^*(\lambda);$$

(VI) *For any* $x^* \in \operatorname{dom} h^*$*, there exist* $\lambda \in \operatorname{dom} k^*$*,* $u \in \operatorname{dom}(f + g)^*$ *such that*

$$h^*(x^*) \geq (f + g)^*(u) + (\lambda H)^*(x^* - u) + k^*(\lambda);$$

(VII) *For any* $x^* \in \operatorname{dom} h^*$*, there exists* $u \in \operatorname{dom} f^*$ *such that*

$$h^*(x^*) \geq f^*(u) + (g + k \circ H)^*(x^* - u).$$

Proof. • [(I)$\Longrightarrow$(II)] Assume that (I) holds. Then $h^* \geq (f + g + k \circ H)^*$ which yields epi $h^* \subset \operatorname{epi}(f + g + k \circ H)^*$. For any $x^* \in \operatorname{dom} h^*$, it is clear that $\big(x^*, h^*(x^*)\big) \in$ epi $h^* \subset$ epi $(f + g + k \circ H)^*$. Since (CA) holds, there exist $\lambda \in \operatorname{dom} k^*$, $(u, r) \in \operatorname{epi} f^*$, $(v, s) \in \operatorname{epi} g^*$, $(w, t) \in \operatorname{epi}(\lambda H - k^*(\lambda))^*$ such that

$$\big(x^*, h^*(x^*)\big) = (u, r) + (v, s) + (w, t).$$

Thus $h^*(x^*) = r + s + t \geq f^*(u) + g^*(v) + (\lambda H)^*(x^* - u - v) + k^*(\lambda)$. Noting that here we have $u \in \operatorname{dom} f^*$, $v \in \operatorname{dom} g^*$, and $w = x^* - u - v \in \operatorname{dom}(\lambda H)^*$.

• $\big[$(II) $\Longrightarrow$ (III)$\big]$ Let $x^* \in \operatorname{dom} h^*$. For $u \in \operatorname{dom} f^*$, $v \in \operatorname{dom} g^*$ and $\lambda \in \operatorname{dom} k^*$ whose existence ensure from (II), one has

$$(g + \lambda H)^*(x^* - u) \leq \big(g^* \square (\lambda H)^*\big)(x^* - u) \leq g^*(v) + (\lambda H)^*(x^* - u - v).$$

So, (III) follows from this and (II).

• $\big[$(III) $\Longrightarrow$ (IV)$\big]$ This assertion can be proved by the same argument as in that of $\big[$(II) $\Longrightarrow$ (III)$\big]$.

• $\big[$(IV) $\Longrightarrow$ (I)$\big]$ Take arbitrarily $x \in X$, $x^* \in \operatorname{dom} h^*$. By (IV), there exists $\lambda \in \operatorname{dom} k^*$ satisfying

$$h^*(x^*) \geq (f + g + \lambda H)^*(x^*) + k^*(\lambda),$$

which ensures

$$\begin{aligned} \langle x^*, x\rangle - h^*(x^*) &\leq \langle x^*, x\rangle - (f + g + \lambda H)^*(x^*) - k^*(\lambda) \\ &\leq \langle x^*, x\rangle - \langle x^*, x\rangle + (f + g + \lambda H)(x) - (\lambda H)(x) + (k \circ H)(x) \\ &= (f + g + k \circ H)(x). \end{aligned}$$

Since $h \in \Gamma(X)$, taking the supremum over all $x^* \in \operatorname{dom} h^*$, we get

$$h^{**}(x) = h(x) \leq (f + g + k \circ H)(x),$$

which is (I).

The proofs of the implications $\big[$(II) $\Longrightarrow$ (V)$\big]$ and $\big[$(V) $\Longrightarrow$ (IV)$\big]$ are similar to that of $\big[$(II) $\Longrightarrow$ (III)$\big]$, using the inequality concerning the conjugate of the sum and the convolution of two conjugates of functions (see (4)).

• $\big[$(II) $\Longrightarrow$ (V)$\big]$ For any $x^* \in \operatorname{dom} h^*$, $u \in \operatorname{dom} g^*$, $v \in \operatorname{dom} f^*$ and $\lambda \in \operatorname{dom} k^*$, one has

$$(f + \lambda H)^*(x^* - u) \leq \big(f^* \square (\lambda H)^*\big)(x^* - u) \leq f^*(v) + (\lambda H)^*(x^* - u - v).$$

So, (V) follows from this and (II).

• $\big[$(V) $\Longrightarrow$ (IV)$\big]$ For any $x^* \in \operatorname{dom} h^*$, $u \in \operatorname{dom} g^*$ and $\lambda \in \operatorname{dom} k^*$, one has

$$(f + g + \lambda H)^*(x^*) \leq \big((f + \lambda H)^* \square g^*\big)(x^*) \leq (f + \lambda H)^*(x^* - u) + g^*(u).$$

(IV) follows from this and (V).

•$\big[$(I) $\Longrightarrow$ (VI)$\big]$ Assume that (I) holds. Then $h^* \geq (f+g+k\circ H)^*$ and $\operatorname{epi} h^* \subset \operatorname{epi}(f+g+k\circ H)^*$. If $x^* \in \operatorname{dom} h^*$ then $\big(x^*, h^*(x^*)\big) \in \operatorname{epi} h^* \subset \operatorname{epi}(f+g+k\circ H)^*$. Since (CA) holds then (CC) also holds, there exist $\lambda \in \operatorname{dom} k^*$, $(u, r) \in \operatorname{epi}(f+g)^*$ and $(v, s) \in \operatorname{epi}(\lambda H - k^*(\lambda))^*$ such that

$$\big(x^*, h^*(x^*)\big) = (u, r) + (v, s).$$

Thus $h^*(x^*) = r + s \geq (f+g)^*(u) + (\lambda H)^*(x^* - u) + k^*(\lambda)$. Obviously, $u \in \operatorname{dom}(f+g)^*$ and $v = x^* - u \in \operatorname{dom}(\lambda H)^*$.

•$\big[$(VI) $\Longrightarrow$ (IV)$\big]$ Let $x^* \in \operatorname{dom} h^*$. For $u \in \operatorname{dom}(f+g)^*$ and $\lambda \in \operatorname{dom} k^*$, one has

$$(f + g + \lambda H)^*(x^*) \leq \big((f+g)^* \square (\lambda H)^*\big)(x^*) \leq (f+g)^*(u) + (\lambda H)^*(x^* - u).$$

So, (IV) follows from this and (VI).

•$\big[$(I) $\Longrightarrow$ (VII)$\big]$ Assume that (I) holds. From the fact that $h^* \geq (f+g+k\circ H)^*$ and epi $h^* \subset \operatorname{epi}(f+g+k\circ H)^*$, for any $x^* \in \operatorname{dom} h^*$, $\big(x^*, h^*(x^*)\big) \in$ epi $h^* \subset$ epi $(f+g+k\circ H)^*$. In this case, it follows from (CF) (since (CA) holds) that there exist $(u, r) \in \operatorname{epi} f^*$ and $(v, s) \in \operatorname{epi}(g + k \circ H)^*$ such that

$$\big(x^*, h^*(x^*)\big) = (u, r) + (v, s).$$

Thus $h^*(x^*) = r + s \geq f^*(u) + (g + k \circ H)^*(x^* - u)$.

•$\big[$(VII) $\Longrightarrow$ (I)$\big]$ Let $x \in X$. It follows from (VII) that for any $x^* \in \operatorname{dom} h^*$ (there is $u \in \operatorname{dom} f^*$),

$$\begin{aligned}\langle x^*, x\rangle - h^*(x^*) &\leq \langle x^*, x\rangle - f^*(u) - (g + k\circ H)^*(x^* - u)\\ &\leq \langle x^*, x\rangle - \langle u, x\rangle + f(x) - \langle x^* - u, x\rangle + (g + k\circ H)(x)\\ &= (f + g + k \circ H)(x).\end{aligned}$$

Since $h \in \Gamma(X)$, taking the supremum over $x^* \in \operatorname{dom} h^*$, we get

$$h^{**}(x) = h(x) \leq (f + g + k \circ H)(x)$$

which is (I). The proof is complete. ■

Remark 5.2. It is worth noting that one of the equivalent pair between (I) and one among the left (i.e., (II)–(VII)), is a transcription/characterization of (1) or a Frakas-type result. So, Theorem 5.1 includes six various versions of Farkas-type results for the inequality system in consideration. Although this theorem gives only sufficient condition for these types of results, it covers and extends several Farkas-type results in the literature (see e.g., [5, 12, 14, 15, 20, 32]).

We now consider in more details each of Farkas-type results mentioned in Theorem 5.1. Concretely, we will give in Theorem 5.3 variant necessary and sufficient conditions for each of the equivalent pairs mentioned above.

Theorem 5.3. *The following assertions hold:*

(i) [(CA) *holds*] $\iff$ [$\forall h \in \Gamma(X)$, (I) $\iff$ (II)],
(ii) [(CB) *holds*] $\iff$ [$\forall h \in \Gamma(X)$, (I) $\iff$ (III)],
(iii) [(CC) *holds*] $\iff$ [$\forall h \in \Gamma(X)$, (I) $\iff$ (VI)],

(iv) [(CD) *holds*] $\iff$ $[\forall h \in \Gamma(X),\quad \text{(I)} \iff \text{(V)}]$,
(v) [(CE) *holds*] $\iff$ $[\forall h \in \Gamma(X),\quad \text{(I)} \iff \text{(IV)}]$,
(vi) [(CF) *holds*] $\iff$ $[\forall h \in \Gamma(X),\quad \text{(I)} \iff \text{(VII)}]$.

Proof. Assertions in (i) and (ii) were proved in [16]. The proofs of the left base on the argument similar to that of Theorem 5.1. So, we give in details only the proof of (iii). The others are similar and will be omitted.

Proof of (iii).

• We first prove that (CC) condition implies $[\forall h \in \Gamma(X),\quad \text{(I)} \iff \text{(VI)}]$. The proof of (I) $\Longrightarrow$ (VI) under (CC) is the same as in that of Theorem 5.1. (VI)$\Longrightarrow$ (I) holds without any further condition. Indeed, take any $x^* \in \operatorname{dom} h^*$. By (VI), there exist $u \in \operatorname{dom}(f+g)^*$, and $\lambda \in \operatorname{dom} k^*$ such that

$$h^*(x^*) \geq (f+g)^*(u) + (\lambda H)^*(x^* - u) + k^*(\lambda),$$

and hence, for any $x \in X$,

$$\begin{aligned}\langle x^*, x\rangle - h^*(x^*) &\leq \langle x^*, x\rangle - \langle u, x\rangle + (f+g)(x) - \langle x^* - u, x\rangle + (\lambda H)(x) \\ &\quad -\langle \lambda, H(x)\rangle + k \circ H(x) \\ &= (f + g + k \circ H)(x).\end{aligned}$$

Now, taking the supremum over $x^* \in \operatorname{dom} h^*$ (note that $h \in \Gamma(X)$), we get

$$h^{**}(x) = h(x) \leq (f + g + k \circ H)(x) \quad \forall x \in X,$$

which shows that (I) holds.

• We now show that $[\forall h \in \Gamma(X),\quad \text{(I)} \iff \text{(VI)}]$ ensures (CC). By Lemma 3.1, it is sufficient to show that $\operatorname{epi}(f+g+k\circ H)^* \subset \mathcal{C}$.

Let $(x^*, r) \in \operatorname{epi}(f+g+k\circ H)^*$. Then $(f+g+k\circ H)(x) \geq \langle x^*, x\rangle - r$ for all $x \in X$. In other words, (I) holds with $h(\cdot) = \langle x^*, \cdot\rangle - r$ and hence, (VI) holds and consequently, there exist $\lambda \in \operatorname{dom}k^*$, $u \in \operatorname{dom}(f+g)^*$ such that

$$(f+g)^*(u) + (\lambda H)^*(x^* - u) + k^*(\lambda) \leq h^*(x^*) = r.$$

Note that here we have $x^* - u \in \operatorname{dom}(\lambda H)^* = \operatorname{dom}\big(\lambda H - k^*(\lambda)\big)^*$
Setting $\alpha := r - (f+g)^*(u) - (\lambda H - k^*(\lambda))^*(x^* - u) \geq 0$, we get

$$\begin{aligned}(x^*, r) &= \Big(x^*, \alpha + (f+g)^*(u) + (\lambda H - k^*(\lambda))^*(x^* - u)\Big) \\ &= \big(0, \alpha\big) + \big(u, (f+g)^*(u)\big) + \big(x^* - u, (\lambda H - k^*(\lambda))^*(x^* - u)\big) \\ &\in \big(0, \alpha\big) + \operatorname{epi}(f+g)^* + \operatorname{epi}(\lambda H - k^*(\lambda))^* \subset \mathcal{C}.\end{aligned}$$

The proof is complete. ■

Remark 5.4. Characterizations of Farkas-type results given in Theorem 5.3 is very general. In special cases, they produce new results, extend or cover many known Farkas-type results for systems involving convex and DC functions as will be shown in the next section.

6. Special cases and applications

In order to illustrate the significance of the general Farkas-type results established in Section 5, in this section, we will point out some special cases. Here, we get new versions, extensions or find again Farkas-type results for convex, nonconvex systems, or for systems involving DC functions. Necessary and sufficient conditions for strong stable Lagrange duality of a general convex optimization problem is also given. Moreover, several alternative theorems, characterizations of set containments of a convex set in a DC set or in a reverse convex set are also derived. In particular, a nonconvex version of Fenchel-Rockafellar duality formula is obtained as consequence of a Farkas-type result introduced in Section 5 as well. We maintain the assumptions on the spaces, sets, and functions as in Section 5.

6.1. The case when h is an affine function

Taking h as an affine function, $h(\cdot) = \langle x^*, \cdot\rangle + \alpha$, where $\alpha \in \mathbb{R}$ and $x^* \in X^*$, we get from Theorem 5.3 (v),

Proposition 6.1. *Let C be a nonempty subset of X. The following statements are equivalent:*

(a) $\operatorname{epi}(g + i_C + k \circ H)^* = \bigcup_{\lambda \in \operatorname{dom} k^*} \operatorname{epi}\left(g + i_C + \lambda H - k^*(\lambda)\right)^*$;

(b) *For any $\alpha \in \mathbb{R}$ and any $x^* \in X^*$,*

$$\Big(g(x)+k\circ H(x) \geq \langle x^*, x\rangle + \alpha \ \ \forall x \in C\Big) \iff \begin{cases} \exists \lambda \in \operatorname{dom} k^* \quad \forall x \in C \\ g(x) + \lambda H(x) \geq \langle x^*, x\rangle + \alpha + k^*(\lambda). \end{cases}$$

Proof. [(a) $\Rightarrow$ (b)] The implication follows from Theorem 5.3(v) when taking $f = i_C$ and $h(x) = \langle x^*, x\rangle + \alpha$ for all $x \in X$ (and hence, $\operatorname{dom} h^* = \{x^*\}$ and $h^*(x^*) = -\alpha$).

[(b) $\Rightarrow$ (a)] Assume (b) holds. In order to show that (b) holds, by Lemma 3.1, it is sufficient to show that

$$\operatorname{epi}(g + i_C + k \circ H)^* \subset \bigcup_{\lambda \in \operatorname{dom} k^*} \operatorname{epi}\left(g + i_C + \lambda H - k^*(\lambda)\right)^*.$$

If $(x^*, r) \in \operatorname{epi}(g + i_C + k \circ H)^*$ then

$$g(x) + k \circ H(x) \geq \langle x^*, x\rangle - r \ \ \forall x \in C.$$

It now follows from (b) that ($-r$ plays the role of α) there exists $\lambda \in \operatorname{dom} k^*$ such that

$$g(x) + \lambda H(x) \geq \langle x^*, x\rangle - r + k^*(\lambda) \quad \forall x \in C.$$

This is equivalent to

$$r \geq \left(g + i_C + \lambda H - k^*(\lambda)\right)^* (x^*),$$

which means that

$$(x^*, r) \in \text{epi } \left(g + i_C + \lambda H - k^*(\lambda)\right)^* \subset \bigcup_{\lambda \in \operatorname{dom} k^*} \text{epi } \left(g + i_C + \lambda H - k^*(\lambda)\right)^* .$$

■

We now consider the case where Z is preordered by a nonempty convex cone S. Letting $k = i_{-S}$, we get from Proposition 6.1,

Corollary 6.2. *Let C be a nonempty subset of X and let S be a nonempty convex cone in Z. Assume that $C \cap \operatorname{dom}(g + i_{-S} \circ H) \neq \emptyset$. The following assertions are equivalent:*

(a) $\text{epi}\,(g + i_C + i_{-S} \circ H)^* = \bigcup_{\lambda \in S^+} \text{epi}\,(g + i_C + \lambda H)^*$;

(b) *For any $x^* \in X^*$ and any $\alpha \in \mathbb{R}$,*

$$\Big(H(x) \in -S,\ x \in C \Longrightarrow g(x) \geq \langle x^*, x\rangle + \alpha\Big) \Longleftrightarrow \begin{cases} \exists \lambda \in S^+ \quad \forall x \in C \\ g(x) + \lambda H(x) \geq \langle x^*, x\rangle + \alpha. \end{cases}$$

Remark 6.3. Corollary 6.2 is an extension of generalized Farkas lemmas in [22, Theorem 3.1], [10, Corollary 5.2], [15, Corollary 6.2], and in [12, Theorem 2.2] (where with $C = X$) to nonconvex systems. The extension is twofold: Firstly, it is a nonconvex version of results mentioned in the previous papers. Namely, the closedness of C and the cone S, the convexity of the function g, S-convexity of the mapping H are all removed. Secondly, Corollary 6.2 gives necessary and sufficient conditions for this Farkas lemma version while only the sufficient condition was proposed in the mentioned papers.

6.2. Characterization of Farkas-type results with composed functions in convex setting

We now assume that C is a nonempty closed, convex subset of X, S is a closed, convex cone in Z, and $f \in \Gamma(X)$ and H is an S-convex mapping. Assume further that $\lambda H \in \Gamma(X)$ for all $\lambda \in S^+$ and $C \cap H^{-1}(-S) \neq \emptyset$.

Corollary 6.4. *The following statements* (a) *and* (b) *are equivalent:*

(a) $\bigcup_{\lambda \in S^+} \operatorname{epi}(f + i_C + \lambda H)^*$ *is weak*-closed;*

(b) *For any* $h \in \Gamma(X)$*, the following assertions are equivalent:*

(i) $H(x) \in -S,\ x \in C \implies f(x) \geq h(x)$;

(ii) $\forall x^* \in \operatorname{dom} h^*,\ \exists \lambda \in S^+,\ \ f(x) + \lambda H(x) \geq \langle x^*, x\rangle - h^*(x^*)\ \forall x \in C.$

Proof. Let $k := i_{-S}$ and $g := i_C$. Then $k \in \Gamma(Z)$ and $g \in \Gamma(X)$. It is also easy to see that $\operatorname{dom} k^* = S^+$ and $k^*(\lambda) = 0$ for all $\lambda \in S^+$. So, (a) means that (CE) (namely, (CE_1)) holds. On the other hand, (i) is exactly (I) in this concrete case while (IV), in this setting, reads $h^*(x^*) \geq (f + i_C + \lambda H)^*(x^*)$ which is equivalent to (ii). The conclusion of the theorem follows from Theorem 5.3(v). ■

Remark 6.5. Farkas-type result for systems involved DC function of the form (b) in the previous corollary was established in [10, 15]. However, Corollary 6.4 is stronger than the one in [10, 15] in the sense that Corollary 6.4 gives a necessary and sufficient condition for (b) while only the sufficient condition was given in the mentioned papers. Concretely, our condition (in (a)) is (CE_1) which is weaker than the condition in the mentioned references which is equivalent to (CA_1). The same comment applies to the next corollaries in comparison with the corresponding one in [22, 10, 15, 32].

In the case where h is an affine function we get the following from Corollary 6.4.

Corollary 6.6. *The following statements are equivalent:*

(a) $\bigcup_{\lambda \in S^+} \operatorname{epi}(f + i_C + \lambda H)^*$ *is weak*-closed;*

(b) *For any* $x^* \in X^*$ *and any* $\alpha \in \mathbb{R}$,

$$\Big(H(x) \in -S,\ x \in C \Longrightarrow f(x) - \langle x^*, x\rangle \geq \alpha\Big) \Longleftrightarrow \begin{cases} \exists \lambda \in S^+ \quad \forall x \in C \\ f(x) + \lambda H(x) - \langle x^*, x\rangle \geq \alpha. \end{cases}$$

Proof. It is clear that the implication [(a) ⇒ (b)] is a direct consequence of Corollary 6.4. The proof of [(b) ⇒ (a)] is similar to that of Proposition 6.1. ■

Remark 6.7. Consider the convex optimization problem (P):

$$\text{Minimize } f(x) \ \text{ subject to } \ H(x) \in -S,\, x \in C.$$

Then the equivalence in (b) of Corollary 6.6 is nothing but

$$\inf_{H(x) \in -S,\, x \in C} [f(x) - \langle x^*, x\rangle] = \sup_{\lambda \in S^+} \inf_{x \in C} [f(x) + (\lambda H)(x) - \langle x^*, x\rangle] \ \forall x^* \in X^*,$$

which is often known as stable strong Lagrange duality for (P) in convex optimization. Corollary 6.6 gives necessary and sufficient conditions for such a result.

This was established in [2, Theorem 8.3] under an assumption on H called "S-epi closed" which is a bit weaker than our assumption that $\lambda H \in \Gamma(X)$ for all $\lambda \in S^+$. However, for the sake of simplicity, we do not consider this notion in this paper.

The following corollary gives necessary and sufficient conditions for strong duality of (P). This was established in [16, Corollary 5.1]. However, we will repeat it with a direct and simple proof. Set $A := C \cap H^{-1}(-S)$ and

$$\mathcal{E}' := \bigcup_{\lambda \in S^+} \operatorname{epi}(f + i_C + \lambda H)^*.$$

Corollary 6.8. [16, Corollary 5.1] *The following statements are equivalent:*

(a) $\big(\{0_{X^*}\} \times \mathbb{R}\big) \cap \mathcal{E}' = \big(\{0_{X^*}\} \times \mathbb{R}\big) \cap \operatorname{cl} \mathcal{E}'$,

(b) *For any* $\alpha \in \mathbb{R}$,

$$\Big(H(x) \in -S,\ x \in C \implies f(x) \geq \alpha\Big) \iff \begin{cases} \exists \lambda \in S^+ \quad \forall x \in C \\ f(x) + \lambda H(x) \geq \alpha. \end{cases}$$

Proof. We observe firstly that

$$\begin{aligned} \operatorname{epi}(f + i_A)^* &= \operatorname{cl}(\operatorname{epi} f^* + \operatorname{epi} i_A^*) = \operatorname{cl}\Big(\operatorname{epi} f^* + \bigcup_{\lambda \in S^+} \operatorname{epi}(\lambda H)^* + \operatorname{epi} i_C^*\Big) \\ &\subset \operatorname{cl}\Big(\bigcup_{\lambda \in S^+} \operatorname{epi}(f + \lambda H + i_C)^*\Big) = \operatorname{cl} \mathcal{E}'. \end{aligned}$$

Moreover, for any $\lambda \in S^+$ one has $f + \lambda H + i_C \leq f + i_A$, or equivalently, $(f + \lambda H + i_C)^* \geq (f + i_A)^*$, and hence

$$\bigcup_{\lambda \in S^+} \operatorname{epi}(f + \lambda H + i_C)^* \subset \operatorname{epi}(f + i_A)^*.$$

Consequently,

$$\operatorname{epi}(f + i_A)^* = \operatorname{cl} \mathcal{E}'. \tag{12}$$

[(a) $\Longrightarrow$ (b)]. Assume (a) holds. Let $\alpha \in \mathbb{R}$. The left hand side of (b) means that $(f + i_A) \geq \alpha$, or equivalently, $-\alpha \geq (f + i_A)^*(0_{X^*})$, and hence (see (12)),

$$(0_{X^*}, -\alpha) \in \big(\{0_{X^*}\} \times \mathbb{R}\big) \cap \operatorname{epi}(f + i_A)^* = \big(\{0_{X^*}\} \times \mathbb{R}\big) \cap \operatorname{cl} \mathcal{E}'.$$

Since (a) holds, one has $(0_{X^*}, -\alpha) \in \mathcal{E}'$, which ensures the existence of $\lambda \in S^+$ such that $-\alpha \geq (f + \lambda H + i_C)^*(0_{X^*})$. This is nothing but the right hand side of (b). The converse implication in (b) is obvious.

[(b) $\Longrightarrow$ (a)]. Assume (b) holds. It is sufficient to show that if $(0_{X^*}, r) \in \operatorname{cl} \mathcal{E}'$ then $(0_{X^*}, r) \in \mathcal{E}'$. If $(0_{X^*}, r) \in \operatorname{cl} \mathcal{E}'$ then by (12), $(0_{X^*}, r) \in \operatorname{epi}(f + i_A)^*$, or equivalently, $(f + i_A)^*(0_{X^*}) \leq r$. In turn, the last inequality is equivalent to

$$(f + i_A)(x) \geq -r \quad \forall x \in X,$$

which is the left hand side of the equivalence in (b) (with $-r$ plays the role of α). Since (b) holds, there is $\lambda \in S^+$ such that $f(x)+\lambda H(x) \geq -r$ for all $x \in C$. This means that $(f+\lambda H+i_C)^*(0_{X^*}) \leq r$ which yields $(0_{X^*}, r) \in \operatorname{epi}(f+\lambda H+i_C)^* \subset \mathcal{E}'$. The proof is complete. ■

6.3. Alternative-type theorems

In this subsection, Theorem 5.3 (and its consequences such as Proposition 6.1, Corollaries 6.2, 6.4, 6.6) will be used to derive alternative-type theorems.

We start with the application of Theorem 5.3. Note that each item in this theorem gives an alternative theorem without convexity nor lower semi-continuity. As an illustration, we give just one of them, namely Theorem 5.3 (iii).

Theorem 6.9. *Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$, $k : Z \to \mathbb{R} \cup \{+\infty\}$ be proper and $H : \operatorname{dom} H \subset X \to Z$ such that $f+g+k\circ H$ is proper. Then,* $\operatorname{epi}(f+g+k\circ H)^* = \mathcal{C}$ *if and only if for any $h \in \Gamma(X)$, precisely one of the following statements is true*

(i) $\exists x \in X,\ f(x) + g(x) + k[H(x)] < h(x)$;

(ii) *For any $x^* \in \operatorname{dom} h^*$, there exist $\lambda \in \operatorname{dom} k^*$, $u \in \operatorname{dom}(f+g)^*$ such that*

$$h^*(x^*) \geq (f+g)^*(u) + (\lambda H)^*(x^* - u) + k^*(\lambda).$$

Proof. Observe that non-(i) is equivalent to (I) and (ii) is nothing but (VI). Then, the conclusion follows from Theorem 5.3 (iii) yields the result. ■

It is worth mentioning that the alternative result in Theorem 6.9 gives necessary and sufficient conditions for the alternative. Similar characterizations of the alternative in nonconvex setting under other types of qualification conditions can be found in [16].

In the same way, for a special case where h is a constant function, we get from Proposition 6.1,

Corollary 6.10. *Let g, k, H be as in Theorem 6.9. If* $\operatorname{epi}(g + k \circ H)^* = \bigcup_{\lambda \in \operatorname{dom} k^*} \operatorname{epi}\,(g + \lambda H - k^*(\lambda))^*$ *then for any $\alpha \in \mathbb{R}$, precisely one of the following statements is true*

(i) $\exists x \in X,\ g(x) + k[H(x)] < \alpha$,

(ii) $\exists \lambda \in \operatorname{dom} k^*,\ \forall x \in X,\ \ g(x) + \lambda H(x) \geq \alpha + k^*(\lambda)$.

Turning to the convex setting, Corollary 6.4 gives

Corollary 6.11. *Let C be a nonempty closed, convex subset of X, S be a closed, convex cone in Z, and let $f \in \Gamma(X)$. Assume that $\lambda H \in \Gamma(X)$ for all $\lambda \in S^+$ and $C \cap H^{-1}(-S) \neq \emptyset$. Then,* $\bigcup_{\lambda \in S^+} \operatorname{epi}(f + i_C + \lambda H)^*$ *is weak*-closed if and only if for any $h \in \Gamma(X)$, precisely one of the following statements is true*

(i) $H(x) \in -S, x \in C$ *and* $f(x) < h(x)$,

(ii) $\forall x^* \in \operatorname{dom} h^*, \exists \lambda \in S^+$ *such that*

$$f(x) + \lambda H(x) \geq \langle x^*, x\rangle - h^*(x^*) \quad \forall x \in C.$$

The following alternative result is a consequence of Corollary 6.6 which appeared in several earlier works (and was often known as generalized Farkas lemma for convex cone-constrained systems) under stronger qualification conditions (see, e.g., [15], [32]).

Corollary 6.12. *Let C be a nonempty closed, convex subset of X, S be a closed, convex cone in Z, and let $f \in \Gamma(X)$. Assume that $\lambda H \in \Gamma(X)$ for all $\lambda \in S^+$ and $C \cap H^{-1}(-S) \neq \emptyset$. If $\bigcup_{\lambda \in S^+} \operatorname{epi}(f + i_C + \lambda H)^*$ is weak*-closed then for any $\alpha \in \mathbb{R}$, precisely one of the following statements is true*

(i) $H(x) \in -S, x \in C$ *and* $f(x) < \alpha$,

(ii) $\exists \lambda \in S^+, \forall x \in C, \ \ f(x) + \lambda H(x) \geq \alpha$.

6.4. Set containments

Theorem 5.3 also can be used to derive characterizations (necessary and sufficient conditions) of set containment results in very general cases. As illustrative examples, we give here some characterization of containments of convex sets in a reverse convex set (see [31]) or the one of a convex set in a DC set (defined by a DC function). For other results of this type, see [21] and the references therein.

Assume that $C \subset X$ is closed and convex, $f \in \Gamma(X)$, and $H : \operatorname{dom} H \to Z$ is an S-convex mapping where $S \subset Z$ is a closed convex cone. Let $g = i_C$ (hence, $g \in \Gamma(X)$). Let $[H \leq_S 0]$, $[f \geq h]$ denote the sets $\{x \in X \mid H(x) \in -S\}$ and $\{x \in X \mid f(x) \geq h(x)\}$, respectively. Then for any $h \in \Gamma(X)$, the inequality

$$f + i_C + i_{-S} \circ H \geq h$$

is none other than

$$C \cap [H \leq_S 0] \subset [f \geq h],$$

which is a containment of a convex set in a DC set [31] (the set defined by a difference of two convex functions, here we mean $[f - h \geq 0]$). In case $h \equiv 0$, the mentioned containment collapses to the one of a convex set in a reverse convex set:

$$C \cap [H \leq_S 0] \subset [f \geq 0]. \tag{13}$$

This observation enables us to apply Theorem 5.3 to get set containments in a very general case. However, for the sake of simplicity, we focus only to the convex case mentioned above. Assume that $C \cap [H \leq_S 0] \neq \emptyset$. As a consequence of Corollary 6.4, we get

Corollary 6.13. *The set* $\bigcup_{\lambda\in S^+} \operatorname{epi}(f+i_C+\lambda H)^*$ *is weak***-closed if and only if for any* $h\in\Gamma(X)$ *the following statements are equivalent:*

(i) $C\cap[H\leq_S 0]\subset[f\geq h]$;

(ii) $\forall x^*\in\operatorname{dom} h^*, \exists\lambda\in S^+$,

$$f(x)+\lambda H(x)\geq\langle x^*,x\rangle-h^*(x^*)\quad\forall x\in C.$$

Taking h as a constant function in Corollary 6.13, we get the characterization of the containment of a convex set defined by a cone-constrained and a set in a reverse convex set which extends the ones in [21] and in [15].

Corollary 6.14. *Let* $\alpha\in\mathbb{R}$. *If the set* $\bigcup_{\lambda\in S^+} \operatorname{epi}(f+i_C+\lambda H)^*$ *is weak***-closed then the following statements are equivalent:*

(i) $C\cap[H\leq_S 0]\subset[f\geq\alpha]$,

(ii) $\exists\lambda\in S^+$,

$$f(x)+\lambda H(x)\geq\alpha\quad\forall x\in C.$$

6.5. Fenchel-Rockafellar duality formula

We now consider the case where f and k are proper functions and $H=A\in\mathcal{L}(X,Z)$, where $\mathcal{L}(X,Z)$ denotes the set of all continuous linear operators from X to Z. Assume that $\operatorname{dom}(f+k\circ A)\neq\emptyset$. Let A^* denote the adjoint operator of A.

Consider the problem (P)

$$\inf_{x\in X}\{f(x)+k(A(x))\}.$$

The classical Fenchel dual problem to (P) is defined as the problem (D) below

$$\sup_{z^*\in Z^*}\{-f^*(A^*z^*)-k^*(-z^*)\}.$$

As a consequence of Theorem 5.3, we get

Proposition 6.15. *Assume that* $\inf(P)=\alpha\in\mathbb{R}$. *If the qualification condition* (10) *holds, i.e,*

$$\operatorname{epi}(f+k\circ A)^*=\operatorname{epi} f^*+(A^*\times Id_{\mathbb{R}})(\operatorname{epi} k^*)$$

then the strong duality holds between (P) *and* (D) *and the dual problem* (D) *possesses at least one solution, i.e.,*

$$\inf_{x\in X}\{f(x)+k(A(x))\}=\max_{z^*\in X^*}\{-f^*(A^*z^*)-k^*(-z^*)\}.$$

Proof. As it is shown in Corollary 4.6 that if (10) holds then

$$\operatorname{epi}(f+k\circ A)^* = \operatorname{epi} f^* + (A^*\times Id_{\mathbb{R}})(\operatorname{epi} k^*) = \operatorname{epi} f^* + \bigcup_{\lambda\in \operatorname{dom} k^*} \operatorname{epi}(\lambda A - k^*(\lambda))^*,$$

which means that the qualification condition (CC) holds with $g \equiv 0$ and $H = A \in \mathcal{L}(X, Z)$. Now let $h(x) = \alpha$ for all $x \in X$. Then $g^*(0) = 0$ and $h^*(0) = -\alpha$ while $g^*(x^*) = h^*(x^*) = +\infty$ if $x^* \neq 0$.

On the other hand, the fact that $\inf(\mathrm{P}) = \alpha \in \mathbb{R}$ is equivalent to

$$f(x) + k(A(x)) \geq \alpha \quad \forall x \in X, \tag{14}$$

which is (I) in Theorem 5.3 in this concrete setting ($g \equiv 0$, $H = A$ and $h(x) := \alpha$).

Now if (10) holds then it follows from Theorem 5.3(iii) that (14) is equivalent to (VI), i.e., there exist $z^* \in \operatorname{dom} k^*$, $u \in \operatorname{dom} f^*$ such that

$$-\alpha = h^*(0) \geq f^*(u) + (z^*A)^*(-u) + k^*(z^*).$$

Since $(z^*A)^*(-u) = 0$ if $u = -A^*z^*$ and $(z^*A)^*(-u) = +\infty$ in all other cases (see the proof of Corollary 4.6), the last inequality is equivalent to $-\alpha \geq f^*(-A^*z^*) + k^*(z^*)$, which yields

$$\alpha = \inf(\mathrm{P}) \leq -f^*(A^*\tilde{z}^*) - k^*(-\tilde{z}^*),$$

where $\tilde{z}^* = -z^*$. Together with the weak duality (which holds trivially), the conclusion follows. ■

Remark 6.16. The Fenchel-Rockafellar duality formula (i.e., strong duality between (P) and (D)) in the case where f and k are convex and lower semicontinuous was given in [28, Theorem 31.2] (for finite dimensional), in [33, Corollary 2.8.5], [3], and in [6] under different kinds of qualification conditions. Concretely, interior-type condition was used in [28, 33] and the condition (10) was used in [3] and [6]. The mentioned Fenchel-Rockafellar duality formula was also established recently in [25] for the convex case (i.e., f, k are convex) without any assumptions on the lower semi-continuity, and under the same qualification condition (10). The same result was given in [2, Remark 7.1] in the absence of convexity but the lower semicontinuity of f and k is still needed. Proposition 6.15 extends the mentioned results to the nonconvex case or to the case where the assumption on the lower semicontinuity is removed.

It is also worth mentioning that if we set $Z = X$ and $A = Id_X$ then the condition (10) collapses to the requirement that $\operatorname{epi}(f + k)^* = \operatorname{epi} f^* + \operatorname{epi} k^*$ and the conclusion of Proposition 6.15 becomes

$$\inf_{x\in X}\{f(x) + k(x)\} = \max_{z^*\in X^*}\{-f^*(z^*) - k^*(-z^*)\},$$

which is the nonconvex version of Fenchel theorem. The same qualification condition but for convex, lower semi-continuous functions f and g was established recently in [8]. Our result for this special case also generalizes the ones in [1] and [29].

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