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Notes on n-Strongly Gorenstein Projective, Injective and Flat Modules

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Abstract. In this paper, we study some properties of n-strongly Gorenstein projective, injective and flat modules, and we consider these properties under change of rings.

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Key words. n-Strongly Gorenstein projective module, n-strongly Gorenstein injective module, n-strongly Gorenstein flat module.

1. Introduction

Throughout the paper all rings are associative with identity, and all modules are unitary. Let R be a ring and M an R-module. We denote by R-Mod (Mod-R) the category of left (right) R-module respectively. The Gorenstein projective (resp., injective and flat) dimension of M is denoted by $\mathrm{Gpd}_R(M)$ (resp., $\mathrm{Gid}_R(M)$ and $\mathrm{Gfd}_R(M)$).

When R is two-side noetherian, Auslander and Bridger introduced the G-dimension, $G\text{-}\dim_R(M)$ for every finitely generated R-module M. They proved the inequality $G\text{-}\dim_R(M) \leq \mathrm{pd}_R(M)$, with equality $G\text{-}\dim_R(M) = \mathrm{pd}_R(M)$ when $\mathrm{pd}_R(M)$ is finite. Several decades later, Enochs and Jenda extended the ideas of Auslander and Bridger introduced three homological dimensions, called the Gorenstein projective, injective and flat dimensions. These have been studied extensively by their founders and by Avramov, Christensen, Foxby, Frankild, Holm, and Martsinkovsky among others over arbitrary associative rings. They proved that these dimensions are similar to the classical homological dimensions; i.e., projective, injective and flat dimensions respectively. Bennis and Mahdou

[3] studied a particular case of Gorenstein projective, injective and flat modules, which they call respectively strongly Gorenstein projective, injective and flat modules. They proved that every Gorenstein projective, (resp. Gorenstein injective, Gorenstein flat) module is a direct summand of a strongly Gorenstein projective, (resp. strongly Gorenstein injective, strongly Gorenstein flat) module. Zhao and Huang [13] studied a generalization of strongly Gorenstein projective, injective and flat modules, which they call respectively n-strongly Gorenstein projective, injective and flat modules. In this paper, we continue the study of n-strongly Gorenstein projective, injective and flat modules. We consider these properties under change of rings.

We firstly recall some concepts. An R-module M is said to be n-strongly Gorenstein projective (n-SG-projective for short) if there exists an exact sequence:

$$0 \to M \to P_n \to \cdots \to P_1 \to M \to 0$$

in R-module with P_i projective for any $1 \leq i \leq n$, such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in R$-Mod is projective. The class of all n-strongly Gorenstein projective R-modules is denoted by n-$SGP(R)$. The n-strongly Gorenstein injective (n-SG-injective for short) modules are defined dually. The class of all n-strongly Gorenstein injective R-modules is denoted by n-$SGI(R)$. An R-module M is said to be n-strongly Gorenstein flat (n-SG-flat for short) if there exists an exact sequence:

$$0 \to M \to F_n \to \cdots \to F_1 \to M \to 0$$

in R-Mod with F_i projective for any $1 \leq i \leq n$, such that $I \otimes -$ leaves the sequence exact whenever $I \in$ Mod-R is injective. The class of all n-strongly Gorenstein flat R-modules is denoted by n-$SGF(R)$.

2. Main results

In this section, we will study the properties of n-strongly Gorenstein projective, injective and flat modules.

Proposition 2.1. *Let R be a commutative ring and P a finitely generated projective R-module. If M is an n-SG-projective R-module, then* $\mathrm{Hom}_R(P, M)$ *is an n-SG-projective R-module.*

Proof. Let Q be a projective R-module and $B \to C \to 0$ be an exact sequence. Consider the commutative diagram:

$$\begin{array}{ccccc} \mathrm{Hom}_R(\mathrm{Hom}_R(P,Q),B) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_R(P,Q),C) & & \\ \simeq\downarrow & & \simeq\downarrow & & \\ P \otimes \mathrm{Hom}_R(Q,B) & \longrightarrow & P \otimes \mathrm{Hom}_R(Q,C) & \longrightarrow & 0. \end{array}$$

Since the second row is exact, then

$$\mathrm{Hom}_R(\mathrm{Hom}_R(P,Q),B) \to \mathrm{Hom}_R(\mathrm{Hom}_R(P,Q),C) \to 0$$

is exact, so $\mathrm{Hom}_R(P,Q)$ is a projective R-module. Since M is an n-SG-projective R-module, then there is an exact sequence

$$\mathbb{P}: \ 0 \to M \to P_n \to \cdots \to P_1 \to M \to 0$$

with P_i projective for any $1 \leq i \leq n$. Then $\mathrm{Hom}_R(P,\mathbb{P})$:

$$0 \to \mathrm{Hom}_R(P,M) \to \mathrm{Hom}_R(P,P_n) \to \cdots \to \mathrm{Hom}_R(P,P_1) \to \mathrm{Hom}_R(P,M) \to 0$$

is exact, where each $\mathrm{Hom}_R(P,P_i)$ is a projective R-module for any $1 \leq i \leq n$. Let Q be any projective R-module. By [1, Proposition 20.11],

$$\mathrm{Hom}_R(\mathrm{Hom}_R(P,\mathbb{P}),Q) \cong P \otimes \mathrm{Hom}_R(\mathbb{P},Q)$$

is exact. Then $\mathrm{Hom}_R(P,M)$ is an n-SG-projective R-module. ■

A ring R is said to be an IF-ring if every injective R-module is flat.

Proposition 2.2. *Let R be a commutative IF-ring and E an injective R-module. If M is an n-SG-injective R-module, then $\mathrm{Hom}_R(M,E)$ is an n-SG-injective R-module.*

Proof. Let Q be an injective R-module and $0 \to B \to C$ be an exact sequence. Consider the commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}_R(C,(\mathrm{Hom}_R(E,Q)) & \longrightarrow & \mathrm{Hom}_R(B,(\mathrm{Hom}_R(E,Q)) \\ \simeq\downarrow & & \simeq\downarrow \\ \mathrm{Hom}_R(E \otimes C,Q) & \longrightarrow & \mathrm{Hom}_R(E \otimes B,Q) \longrightarrow 0. \end{array}$$

Since the second row is exact, then

$$\mathrm{Hom}_R(C,(\mathrm{Hom}_R(E,Q)) \to \mathrm{Hom}_R(B,(\mathrm{Hom}_R(E,Q)) \to 0$$

is exact, so $\mathrm{Hom}_R(E,Q)$ is an injective R-module. Since M is an n-SG-injective R-module, then there is an exact sequence

$$\mathbb{E}: \ 0 \to M \to E_n \to \cdots \to E_1 \to M \to 0$$

with E_i injective for any $1 \leq i \leq n$. Then $\mathrm{Hom}_R(\mathbb{E},E)$:

$$0 \to \mathrm{Hom}_R(M,E) \to \mathrm{Hom}_R(E_n,E) \to \cdots \to \mathrm{Hom}_R(E_1,E) \to \mathrm{Hom}_R(M,E) \to 0$$

is exact, where each $\mathrm{Hom}_R(E_i,E)$ is an injective R-module for any $1 \leq i \leq n$. Let I be any injective R-module. By [1, Proposition 20.7],

$$\mathrm{Hom}_R(I, (\mathrm{Hom}_R(\mathbb{E}, E)) \cong \mathrm{Hom}_R(\mathbb{E}, (\mathrm{Hom}_R(I, E))$$

is exact. Then $\mathrm{Hom}_R(M, E)$ is an n-SG-injective R-module. ■

Proposition 2.3. *Let R be a commutative Noetherian ring. If M is an n-SG-flat R-module and Q is a flat R-module, then $M \otimes_R Q$ is an n-SG-flat R-module.*

Proof. There exists an exact sequence

$$0 \to M \to F_n \to \cdots \to F_1 \to M \to 0,$$

where each F_i is a flat R-module for all $1 \leq i \leq n$. Then

$$0 \to M \otimes Q \to F_n \otimes Q \to \cdots \to F_1 \otimes Q \to M \otimes Q \to 0$$

in R-Mod with $F_i \otimes_R Q$ flat for any $1 \leq i \leq n$. Let I be an injective R-module and let $\mathbb{E}$ be a flat resolution of I. Then for all $i \geq 1$, since $Q \otimes_R I$ is an injective R-module by [8, Theorem 3.2.16]

$$\begin{aligned} \mathrm{Tor}_i^R(M \otimes_R Q, I) &= H_i(M \otimes_R Q \otimes \mathbb{E}) \\ &\cong H_i(M \otimes_R (Q \otimes \mathbb{E})) \\ &= \mathrm{Tor}_i^R(M, Q \otimes_R I) = 0. \end{aligned}$$

Hence $M \otimes Q$ is an n-SG-flat R-module. ■

Theorem 2.4. *Let R and S be equivalent rings via equivalences $F : R\text{-Mod} \to S\text{-Mod}$ and $G : S\text{-Mod} \to R\text{-Mod}$. Then*

(1) *$M \in n\text{-}SGP(R)$ if and only if $F(M) \in n\text{-}SGP(S)$ for all $M \in R\text{-Mod}$;*
(2) *$M \in n\text{-}SGI(R)$ if and only if $F(M) \in n\text{-}SGI(S)$ for all $M \in R\text{-Mod}$;*
(3) *$M \in n\text{-}SGF(R)$ if and only if $F(M) \in n\text{-}SGF(S)$ for all $M \in R\text{-Mod}$.*

Proof. (1) ($\Rightarrow$) Since $M \in n\text{-}SGP(R)$, there exists an exact sequence

$$\mathbb{P}: \ 0 \to M \to P_n \to \cdots \to P_1 \to M \to 0,$$

where each P_i is a projective R-module for any $1 \leq i \leq n$, such that $\mathrm{Hom}_R(-, P)$ leaves the sequence exact whenever $P \in R$-Mod is projective. Then

$$\mathbb{F}(\mathbb{P}): \ 0 \to F(M) \to F(P_n) \to \cdots \to F(P_1) \to F(M) \to 0,$$

where each $F(P_i)$ is a projective S-module for any $1 \leq i \leq n$. Let Q be any projective S-module. Then

$$\mathrm{Ext}_R^i(F(M), Q) \cong \mathrm{Ext}_R^i(M, G(Q)) = 0.$$

Hence $F(M) \in n\text{-}SGP(S)$.

($\Leftarrow$) $GF(M) \cong M$.

(2) and (3) by analogy with the proof of (1). ■

Corollary 2.5. *Let R and S be equivalent rings via equivalences $F : R-Mod \to$ S-Mod and $G : S$-Mod $\to R$-Mod. Then*

(1) $\mathrm{Gpd}_R(M) = \mathrm{Gpd}_S(F(M))$ *for every n-SG-projective left R-module M;*

(2) $\mathrm{Gid}_R(M) = \mathrm{Gid}_S(R(M))$ *for every n-SG-injective left R-module M;*

(3) $\mathrm{Gfd}_R(M) = \mathrm{Gfd}_S(F(M))$ *for every n-SG-flat left R-module M.*

Proof. (1) We only prove that ${}_RM$ has finite dimension if and only if ${}_SF(M)$ has finite dimension.

(⇒) Since $\mathrm{Gpd}_R(M) \leq n$, then there exists an exact sequence

$$0 \to M \to P_n \to \cdots \to P_1 \to M \to 0,$$

where each P_i is projective R-module for any $1 \leq i \leq n$, and M is an n-SG-projective R-module, by [4, Proposition 2.5 (2)], M is Gorenstein projective. Then we have

$$0 \to F(M) \to F(P_n) \to \cdots \to F(P_1) \to F(M) \to 0,$$

where each $F(P_i)$ is a projective S-module for any $1 \leq i \leq n$, and by Theorem 2.4, $F(M)$ is an n-SG-projective S-module, so is Gorenstein projective. Hence $\mathrm{Gpd}_S(F(M)) \leq n$.

(⇐) Since $\mathrm{Gpd}_S(F(M)) \leq n$, then there exists an exact sequence

$$0 \to F(M) \to \bar{P}_n \to \cdots \to \bar{P}_1 \to F(M) \to 0,$$

where each $\bar{P}_i$ is a projective S-module for any $1 \leq i \leq n$. Then we have

$$0 \to GF(M) = M \to G(\bar{P}_n) \to \cdots \to G(\bar{P}_1) \to GF(M) = M \to 0,$$

where each $G(\bar{P}_i)$ is a projective R-module for any $1 \leq i \leq n$. Hence $\mathrm{Gpd}_R(M) \leq n$.

(2) and (3) by analogy with the proof of (1). ■

(1) The ring S is called right R-projective in case for any right S-module M_S with an S-submodule N_S, $N_R|M_R$ implies $N_S|M_S$. For example, every $n \times n$ matrix ring $M_n(R)$ is right R-projective.

(2) The ring extension $S \geq R$ is called a finite normalization extension in case there is a finite subset $s_1, \ldots, s_n$ of S such that $S = \sum_{i=1}^{n} s_iR$ and $s_iR = Rs_i$ for $i = 1, \ldots, n$.

(3) A finite normalization extension $S \geq R$ is called an excellent extension in case condition (1) is satisfied and S_R, ${}_RS$ are free modules with a common basis $s_1, \ldots, s_n$.

Theorem 2.6. *Assume that $S \geq R$ is an excellent extension. Then*

(1) *if M is an n-SG-projective R-module, then $S \otimes_R M$ is an n-SG-projective S-module;*

(2) *if M is an n-SG-injective R-module, then* $\mathrm{Hom}_R(S, M)$ *is an n-SG-injective S-module;*

(3) *if M is an n-SG-flat R-module, then $M \otimes_R S$ is an n-SG-flat S-module.*

Proof. (1) Since M is an n-SG-projective R-module, there exists an exact sequence:

$$\mathbb{P}: \ 0 \to M \to P_n \to \cdots \to P_1 \to M \to 0,$$

where each P_i is a projective R-module for all $1 \leq i \leq n$, such that $\mathrm{Ext}^i_R(M, P) = 0$ for any projective R-module P. Then

$$\mathbb{F}(\mathbb{P}): \ 0 \to S \otimes M \to S \otimes P_n \to \cdots \to S \otimes P_1 \to S \otimes M \to 0,$$

where each $S \otimes P_i$ is a projective S-module for all $1 \leq i \leq n$. Let Q be any projective S-module. Then by [11, p. 258, 9.21] for all $1 \leq i \leq n$

$$\mathrm{Ext}^i_S(S \otimes M, Q) \cong \mathrm{Ext}^i_R(M, Q) = 0.$$

Hence $S \otimes_R M$ is an n-SG-projective S-module.

(2) Since if M is an n-SG-injective R-module, there exists an exact sequence:

$$\mathbb{E}: \ 0 \to M \to E_n \to \cdots \to E_1 \to M \to 0,$$

where each E_i is an injective R-module for all $1 \leq i \leq n$, such that $\mathrm{Ext}^i_R(E, M) = 0$ for any injective R-module E. Then

$$\mathbb{F}(\mathbb{E}): \ 0 \to \mathrm{Hom}_R(S, M) \to \mathrm{Hom}_R(S, E_n) \to \ldots \to \mathrm{Hom}_R(S, E_1) \to \mathrm{Hom}_R(S, M) \to 0,$$

where each $\mathrm{Hom}_R(S, E_i)$ is an injective S-module for all $1 \leq i \leq n$. Let I be any injective S-module. Then by [11, p. 258, 9.21] for all $1 \leq i \leq n$

$$\mathrm{Ext}^i_S(I, \mathrm{Hom}_R(S, M)) \cong \mathrm{Ext}^i_R(I, M) = 0.$$

Hence $\mathrm{Hom}_R(S, M)$ is an n-SG-injective S-module.

(3) Since if M is an n-SG-flat R-module, there exists an exact sequence:

$$\mathbb{F}: \ 0 \to M \to F_n \to \cdots \to F_1 \to M \to 0,$$

where each F_i is a flat R-module for all $1 \leq i \leq n$, such that $\mathrm{Tor}^R_i(M, E) = 0$ for any injective R-module E. Then

$$\mathbb{F}(\mathbb{F}): \ 0 \to M \otimes S \to F_n \otimes S \to \cdots \to F_1 \otimes S \to M \otimes S \to 0,$$

where each $F_i \otimes S$ is a flat S-module for all $1 \leq i \leq n$. Let I be an injective S-module and let $\mathbb{E}$ be a flat resolution of I. Then

$$\begin{aligned}\mathrm{Tor}_i^R(M\otimes_R S, I) &= H_i(M\otimes_R S\otimes \mathbb{E})\\ &\cong H_i(M\otimes_R \mathbb{E})\\ &= \mathrm{Tor}_i^R(M, I)=0.\end{aligned}$$

Hence $M\otimes_R S$ is an n-SG-flat S-module. ■

Corollary 2.7. *Assume that $S\geq R$ is an excellent extension. Then*

(1) $\mathrm{Gpd}_R(M)=\mathrm{Gpd}_S(S\otimes M)$ *for every n-SG-projective left R-module M;*

(2) $\mathrm{Gid}_R(M)=\mathrm{Gid}_S(\mathrm{Hom}_R(S,M))$ *for every n-SG-injective left R-module M;*

(3) $\mathrm{Gfd}_R(M)=\mathrm{Gfd}_S(M\otimes S)$ *for every n-SG-flat right R-module M.*

Proposition 2.8. *Let R be a commutative ring. If M is an n-SG-projective R-module, then $M[x]$ is an n-SG-projective $R[x]$-module.*

Proof. There is an exact sequence

$$\mathbb{P}:\ 0\to M\to P_n\to\cdots\to P_1\to M\to 0,$$

where each P_i is a projective R-module for all $1\leq i\leq n$. Then

$$R[x]\otimes_R\mathbb{P}:\ 0\to M[x]\to P_n[x]\to\cdots\to P_1[x]\to M[x]\to 0,$$

where each $P_i[x]$ is a projective $R[x]$-module for all $1\leq i\leq n$. Let Q be any projective $R[x]$-module. Then $Q[x]\cong R[x]\otimes_R Q\cong R^{\mathbb{N}}\otimes_R Q\cong Q^{\mathbb{N}}$. Hence $Q[x]$ is a projective $R[x]$-module, and so Q is a projective R-module. Thus

$$\mathrm{Hom}_{R[x]}(R[x]\otimes_R\mathbb{P}, Q)\cong \mathrm{Hom}_R(\mathbb{P}, Q).$$

Hence $M[x]$ is an n-SG-projective $R[x]$-module. ■

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