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Nadler's Fixed Point Theorem in Cone Metric Spaces

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Abstract. In this note we establish an analogue of Nadler's fixed point theorem for multivalued contractions in cone metric spaces.

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1. Introduction and preliminaries

The notion of cone metric was introduced in 2007 by Huang and Zhang [9]. Since then a lot of papers are devoted to this subject, see [3–5, 10, 13] and the references therein. In [10] the authors established some fixed point theorems for multivalued contractions in cone metric spaces but the conditions are very complicated and no relation to the well-known Nadler's fixed point theorem is indicated. Recently, in [4] Du gave a simple proof for Banach contraction principle in cone metric spaces. In this note, following the idea of Du we give a simple proof for Nadler's fixed point theorem, a multivalued version of Banach contraction principle in cone metric spaces.

Given a metric space (X, d), $CB(X)$ denotes the collection of all nonempty closed bounded subsets of X. For $x \in X, A \in CB(X)$ and $r > 0$ we shall use the following notations:

$$B(x,r) = \{y \in X : \ d(x,y) < r\},$$

$$d(x,A) = \inf\{d(x,y) : \ y \in A\} = \inf\{r > 0 : \ B(x,r) \cap A \neq \emptyset\},$$

$$N_r(A) = \bigcup_{x \in A} B(x,r) = \{y \in X : \ d(y,A) < r\},$$

$$\begin{aligned} H(A,B) &= \max\{\sup_{x \in A} d(x,B), \ \sup_{y \in B} d(y,A)\} \\ &= \inf\{r > 0 : \ A \subset N_r(B), B \subset N_r(A)\} \end{aligned}$$

(the Hausdorff distance between A and B).

The following results are well-known.

Theorem 1.1 (Nadler's theorem). [11] *Let (X,d) be a complete metric space, $T : X \longrightarrow CB(X)$ a multivalued mapping such that*

$$H(Tx,Ty) \leq ad(x,y) \quad \forall x,y \in X$$

for some $a \in [0,1)$. Then T has a fixed point, i. e., there is $x^* \in X$ such that $x^* \in Tx^*$.

Theorem 1.2 (Generalized Nadler's theorem). [1, 12] *Let (X,d) be a complete metric space, $T : X \longrightarrow CB(X)$ such that*

$$H(Tx,Ty) \leq ad(x,y) + b[d(x,Tx) + d(y,Ty)] + c[d(x,Ty) + d(y,Tx)]$$

for all $x,y \in X$ and for some $a,b,c \in [0,1)$ with $a + 2b + 2c < 1$. Then T has a fixed point.

Let $(E, \|\cdot\|)$ be a Banach space, K be a convex closed pointed cone in E with $\operatorname{int} K \neq \emptyset$. Define a partial ordering $\leq_K$ in E by

$$x \leq_K y \Leftrightarrow y - x \in K.$$

If $y - x \in \operatorname{int} K$, we write $x \ll_K y$.

Fixing an $e \in \operatorname{int} K$ we define a functional $\xi_e : E \longrightarrow \mathbb{R}$ as follows

$$\xi_e(u) = \inf\{r \in \mathbb{R} : \ u \in re - K\}, \ u \in E.$$

The functional ξ_e has the following properties [2, 3, 5–8]:

(i) $\xi_e(u) \leq r \Leftrightarrow u \in re - K, \quad u \in E, \ r \in \mathbb{R}$,

(ii) $\xi_e(u) < r \Leftrightarrow u \in re - \operatorname{int} K$,

(iii) $\xi_e(\lambda u) = \lambda \xi_e(u), \quad \lambda \geq 0, \ u \in E$,

(iv) $\xi_e(u + v) \leq \xi_e(u) + \xi_e(v), \quad u, v \in E$,

(v) $u \leq_K v \Rightarrow \xi_e(u) \leq \xi_e(v)$,

(vi) $\xi_e(e) = 1$.

In the following we always suppose that E is a Banach space, K is a convex closed pointed cone in E with $\operatorname{int} K \neq \emptyset$ and $\leq_K$ is partial ordering with respect to K.

Let X be a nonempty set. A mapping $p : X \times X \longrightarrow E$ is called a cone metric if

$\theta \leq_K p(x, y)\ \forall x, y \in X$ and $p(x, y) = \theta \Leftrightarrow x = y$ (where θ is the origin of E),

$p(x, y) = p(y, x)\ \forall x, y \in X$,

$p(x, z) \leq_K p(x, y) + p(y, z)\ \forall x, y, z \in X$.

The pair (X, p) is then called a cone metric space.

The convergence in (X, p) is defined as follows. Let $\{x_n\}$ be a sequence in (X, p). $\{x_n\}$ cone-converges to x whenever for every $c \in E$ with $\theta \ll_K c$ there is a natural number n_0 such that $p(x_n, x) \ll_K c$ for all $n \geq n_0$ and we write $x_n \xrightarrow{p} x$. $\{x_n\}$ is called a cone-Cauchy sequence if for every $c \in E$ with $\theta \ll_K c$ there is n_0 such that $p(x_n, x_m) \ll_K c$ for all $n, m \geq n_0$. A cone metric space is said to be cone-complete if every cone-Cauchy sequence in X is cone-convergent.

Fixing an $e \in \operatorname{int} K$ and defining $d_e = \xi_e \circ p :\ X \times X \longrightarrow \mathbb{R}$, Du established in [4] the following interesting results:

Theorem 1.3. *Let (X, p) be a cone metric space. Then $d_e = \xi_e \circ p$ is a metric in X.*

Theorem 1.4. *Let (X, p) and d_e be as in Theorem 1.3. Then the following statements hold:*

(i) *If $x_n \xrightarrow{p} x$ then $d_e(x_n, x) \to 0$ as $n \to \infty$.*

(ii) *If $\{x_n\}$ is a cone-Cauchy sequence in (X, p) then it is a Cauchy sequence (in the usual sense) in (X, d_e).*

(iii) *If (X, p) is cone-complete then (X, d_e) is complete (in the usual sense).*

Theorem 1.5. *Let (X, p) be a cone-complete metric space and $T :\ X \longrightarrow X$ satisfy the contractive condition*

$$p(Tx, Ty) \leq_K \lambda p(x, y) \quad \forall x, y \in X$$

for some $\lambda \in [0, 1)$. Then T has a unique fixed point and $\{T^n x\}$ cone-converges to the fixed point for every $x \in X$.

Our aim is to generalize Theorem 1.5 for multivalued contractions.

2. Main results

First of all we introduce the following notions.

Definition 2.1. A subset $A \subset (X, p)$ is said to be bounded in direction $e \in \operatorname{int} K$ if there exists $M < \infty$ such that $p(x, y) \leq_K Me$ for all $x, y \in A$.

From property (i) of ξ_e we have

$$p(x,y) \leq Me \Leftrightarrow d_e(x,y) \leq M. \tag{1}$$

So, if A is bounded in direction e then it is bounded in the metric space (X, d_e). We say that A is cone-bounded if it is bounded in every direction $e \in \operatorname{int} K$.

Definition 2.2. A sequence $\{x_n\} \subset (X,p)$ converges to x in direction $e \in \operatorname{int} K$ if for every $\varepsilon > 0$ there is n_0 such that $p(x_n, x) \ll_K \varepsilon e$ for all $n \geq n_0$.

A subset $A \subset (X,p)$ is said to be closed in direction $e \in \operatorname{int} K$ if every sequence $\{x_n\}$ in A converging in direction e to a point x then $x \in A$.

From property (i) of ξ_e we have

$$p(x_n, x) \ll_K \varepsilon e \Leftrightarrow d_e(x_n, x) < \varepsilon. \tag{2}$$

So, it is clear that if $A \subset (X,p)$ is closed in direction $e \in \operatorname{int}K$ then it is closed in the metric space (X, d_e).

We say that A is cone-closed if it is closed in every direction $e \in \operatorname{int} K$.

Definition 2.3. Let (X,p) be a cone metric space, $e \in \operatorname{int}K$, $A \subset X$. The re-neighborhood of A in (X,p) is defined as follows:

$$N_{re}(A) = \bigcup_{x \in A} B_p(x, re),$$

where $B_p(x, re) = \{y \in X : \ p(x,y) \ll_K re\}$.

The Hausdorff distance in direction e between $A, B \subset X$ is defined by

$$P_e(A,B) = \inf\{r > 0 : \ A \subset N_{re}(B), \ B \subset N_{re}(A)\} \cdot e. \tag{3}$$

From (2) it follows that $N_{re}(A) = N_r(A)$, the r-neighborhood of A in the metric space (X, d_e). Hence

$$\begin{aligned} &\inf\{r > 0 : \ A \subset N_{re}(B), \ B \subset N_{re}(A)\} \\ &= \inf\{r > 0 : \ A \subset N_r(B), \ B \subset N_r(A)\} = H_e(A,B), \end{aligned}$$

the usual Hausdorff distance in (X, d_e).

Thus, from (3) we have

$$P_e(A,B) = H_e(A,B) \cdot e. \tag{4}$$

Applying the functional ξ_e to both sides of (4) and using properties (iii) and (vi) of ξ_e we get

$$\xi_e\left(P_e(A,B)\right) = H_e(A,B).$$

Now we are in a position to state our first result.

Theorem 2.4. *Let (X,p) be a cone-complete metric space, T be a multivalued mapping in X with cone-closed, cone-bounded values. If there is an $e \in intK$ such that*

$$P_e(Tx,Ty) \leq_K ap(x,y) \quad \forall x,y \in X \tag{5}$$

for some $a \in [0,1)$, then T has a fixed point.

Proof. Applying ξ_e to both sides of (5) and using properties (iii) and (v) of ξ_e we get

$$H_e(Tx,Ty) \leq ad_e(x,y) \quad \forall x,y \in X.$$

The result follows from Nadler's theorem. ■

For further generalization we need the following.

Definition 2.5. Let (X,p) be a cone metric space, $x \in X, A \subset X, e \in \operatorname{int} K$. The distance in direction e from x to A is defined as follows:

$$p_e(x,A) = \inf\{r > 0 : \; B_p(x,re) \cap A \neq \emptyset\} \cdot e.$$

From (2) we get

$$\begin{aligned} &\inf\{r > 0 : \; B_p(x,re) \cap A \neq \emptyset\} \\ &= \inf\{r > 0 : \; B_d(x,r) \cap A \neq \emptyset\} = \inf_{y \in A} d_e(x,y) = d_e(x,A), \end{aligned}$$

where $B_d(x,r) = \{y \in X : \; d_e(x,y) < r\}$.

Hence

$$p_e(x,A) = d_e(x,A) \cdot e,$$

from this

$$\xi_e\left(p_e(x,A)\right) = d_e(x,A).$$

Theorem 2.4 can be generalized as follows.

Theorem 2.6. *Theorem 2.4 is still valid if we replace* (5) *by*

$$P_e(Tx,Ty) \leq_K ap(x,y) + b[p_e(x,Tx) + p_e(y,Ty)] + c[p_e(x,Ty) + p_e(y,Tx)] \tag{6}$$

for all $x,y \in X$, with $a,b,c \in [0,1), a + 2b + 2c < 1$.

Proof. Applying ξ_e to both sides of (6) and using properties (iii), (iv), (v) of ξ_e we get

$$H_e(Tx,Ty) \leq ad_e(x,y) + b[d_e(x,Tx) + d_e(y,Ty)] + c[d_e(x,Ty) + d_e(y,Tx)] \quad \forall x,y \in X.$$

The result follows from generalized Nadler's theorem. ■

Remark 2.7. When $E = \mathbb{R}, K = \mathbb{R}^+, e = 1$, Theorems 2.4 and 2.6 coincide with Nadler's and generalized Nadler's theorems respectively.

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