Nadler’s Fixed Point Theorem in Cone Metric Spaces

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Abstract. In this note we establish an analogue of Nadler’s fixed point theorem for multivalued contractions in cone metric spaces.

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1. Introduction and preliminaries

The notion of cone metric was introduced in 2007 by Huang and Zhang [9]. Since then a lot of papers are devoted to this subject, see [3–5, 10, 13] and the references therein. In [10] the authors established some fixed point theorems for multivalued contractions in cone metric spaces but the conditions are very complicated and no relation to the well-known Nadler’s fixed point theorem is indicated. Recently, in [4] Du gave a simple proof for Banach contraction principle in cone metric spaces. In this note, following the idea of Du we give a simple proof for Nadler’s fixed point theorem, a multivalued version of Banach contraction principle in cone metric spaces.

Given a metric space $(X, d)$, $CB(X)$ denotes the collection of all nonempty closed bounded subsets of $X$. For $x \in X$, $A \in CB(X)$ and $r > 0$ we shall use the following notations:
\( B(x, r) = \{ y \in X : d(x, y) < r \} \),

\[ d(x, A) = \inf \{ d(x, y) : y \in A \} = \inf \{ r > 0 : B(x, r) \cap A \neq \emptyset \}, \]

\[ N_r(A) = \bigcup_{x \in A} B(x, r) = \{ y \in X : d(y, A) < r \}, \]

\[ H(A, B) = \max \{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \} \]

\[ = \inf \{ r > 0 : A \subset N_r(B), B \subset N_r(A) \} \]

(the Hausdorff distance between \( A \) and \( B \)).

The following results are well-known.

**Theorem 1.1 (Nadler’s theorem).** [11] Let \((X, d)\) be a complete metric space, \( T : X \rightarrow CB(X) \) a multivalued mapping such that

\[ H(Tx, Ty) \leq ad(x, y) \quad \forall x, y \in X \]

for some \( a \in [0, 1) \). Then \( T \) has a fixed point, i. e., there is \( x^* \in X \) such that \( x^* \in Tx^* \).

**Theorem 1.2 (Generalized Nadler’s theorem).** [1, 12] Let \((X, d)\) be a complete metric space, \( T : X \rightarrow CB(X) \) such that

\[ H(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Tx) + d(y, Ty)] \]

for all \( x, y \in X \) and for some \( a, b, c \in [0, 1) \) with \( a + 2b + 2c < 1 \). Then \( T \) has a fixed point.

Let \((E, \| \cdot \|)\) be a Banach space, \( K \) be a convex closed pointed cone in \( E \) with \( \text{int} K \neq \emptyset \). Define a partial ordering \( \leq_K \) in \( E \) by

\[ x \leq_K y \iff y - x \in K. \]

If \( y - x \in \text{int} K \), we write \( x \ll_K y \).

Fixing an \( e \in \text{int} K \) we define a functional \( \xi_e : E \rightarrow \mathbb{R} \) as follows

\[ \xi_e(u) = \inf \{ r \in \mathbb{R} : u \in re - K \}, \quad u \in E. \]

The functional \( \xi_e \) has the following properties [2, 3, 5–8]:

(i) \( \xi_e(u) \leq r \iff u \in re - K, \quad u \in E, \quad r \in \mathbb{R}, \)

(ii) \( \xi_e(u) < r \iff u \in re - \text{int} K, \)

(iii) \( \xi_e(\lambda u) = \lambda \xi_e(u), \quad \lambda \geq 0, \quad u \in E, \)

(iv) \( \xi_e(u + v) \leq \xi_e(u) + \xi_e(v), \quad u, v \in E, \)

(v) \( u \leq_K v \Rightarrow \xi_e(u) \leq \xi_e(v), \)

(vi) \( \xi_e(e) = 1. \)
In the following we always suppose that $E$ is a Banach space, $K$ is a convex closed pointed cone in $E$ with $\text{int } K \neq \emptyset$ and $\leq_K$ is partial ordering with respect to $K$.

Let $X$ be a nonempty set. A mapping $p : X \times X \to E$ is called a cone metric if

\begin{align*}
\theta \leq_K p(x, y) \forall x, y \in X & \quad \text{and} \quad p(x, y) = \theta \iff x = y \quad \text{(where $\theta$ is the origin of $E$)}, \\
p(x, y) = p(y, x) \forall x, y \in X & \\
p(x, z) \leq_K p(x, y) + p(y, z) \forall x, y, z \in X.
\end{align*}

The pair $(X, p)$ is then called a cone metric space.

The convergence in $(X, p)$ is defined as follows. Let $\{x_n\}$ be a sequence in $(X, p)$. $\{x_n\}$ cone-converges to $x$ whenever for every $c \in E$ with $\theta \ll_K c$ there is a natural number $n_0$ such that $p(x_n, x) \ll_K c$ for all $n \geq n_0$ and we write $x_n \xrightarrow{p} x$. $\{x_n\}$ is called a cone-Cauchy sequence if for every $c \in E$ with $\theta \ll_K c$ there is $n_0$ such that $p(x_n, x_m) \ll_K c$ for all $n, m \geq n_0$. A cone metric space is said to be cone-complete if every cone-Cauchy sequence in $X$ is cone-convergent.

Fixing an $e \in \text{int } K$ and defining $d_e = \xi_e \circ p : X \times X \to \mathbb{R}$, Du established in [4] the following interesting results:

**Theorem 1.3.** Let $(X, p)$ be a cone metric space. Then $d_e = \xi_e \circ p$ is a metric in $X$.

**Theorem 1.4.** Let $(X, p)$ and $d_e$ be as in Theorem 1.3. Then the following statements hold:

1. If $x_n \xrightarrow{p} x$ then $d_e(x_n, x) \to 0$ as $n \to \infty$.
2. If $\{x_n\}$ is a cone-Cauchy sequence in $(X, p)$ then it is a Cauchy sequence (in the usual sense) in $(X, d_e)$.
3. If $(X, p)$ is cone-complete then $(X, d_e)$ is complete (in the usual sense).

**Theorem 1.5.** Let $(X, p)$ be a cone-complete metric space and $T : X \to X$ satisfy the contractive condition

\[ p(Tx, Ty) \leq_K \lambda p(x, y) \quad \forall x, y \in X \]

for some $\lambda \in [0, 1)$. Then $T$ has a unique fixed point and $\{T^n x\}$ cone-converges to the fixed point for every $x \in X$.

Our aim is to generalize Theorem 1.5 for multivalued contractions.

### 2. Main results

First of all we introduce the following notions.

**Definition 2.1.** A subset $A \subset (X, p)$ is said to be bounded in direction $e \in \text{int } K$ if there exists $M < \infty$ such that $p(x, y) \leq_K Me$ for all $x, y \in A$. 


From property (i) of \( \xi_e \) we have
\[
p(x, y) \leq M e \Leftrightarrow d_e(x, y) \leq M.
\] (1)

So, if \( A \) is bounded in direction \( e \) then it is bounded in the metric space \( (X, d_e) \). We say that \( A \) is cone-bounded if it is bounded in every direction \( e \in \text{int } K \).

**Definition 2.2.** A subset \( A \subset (X, p) \) is said to be closed in direction \( e \in \text{int } K \) if every sequence \( \{x_n\} \) in \( A \) converging in direction \( e \) to a point \( x \) then \( x \in A \).

So, if \( A \subset (X, p) \) is closed in direction \( e \in \text{int } K \) then it is closed in the metric space \( (X, d_e) \).

We say that \( A \) is cone-closed if it is closed in every direction \( e \in \text{int } K \).

**Definition 2.3.** Let \( (X, p) \) be a cone metric space, \( e \in \text{int } K \), \( A \subset X \). The \( re \)-neighborhood of \( A \) in \( (X, p) \) is defined as follows:
\[
N_{re}(A) = \bigcup_{x \in A} B_p(x, re),
\]
where \( B_p(x, re) = \{ y \in X : p(x, y) \ll_K re \} \).

The Hausdorff distance in direction \( e \) between \( A, B \subset X \) is defined by
\[
P_e(A, B) = \inf \{ r > 0 : A \subset N_{re}(B), B \subset N_{re}(A) \} \cdot e.
\] (3)

From (2) it follows that \( N_{re}(A) = N_r(A) \), the \( r \)-neighborhood of \( A \) in the metric space \( (X, d_e) \). Hence
\[
\inf \{ r > 0 : A \subset N_{re}(B), B \subset N_{re}(A) \} = \inf \{ r > 0 : A \subset N_r(B), B \subset N_r(A) \} = H_e(A, B),
\]
the usual Hausdorff distance in \( (X, d_e) \).

Thus, from (3) we have
\[
P_e(A, B) = H_e(A, B) \cdot e.
\] (4)

Applying the functional \( \xi_e \) to both sides of (4) and using properties (iii) and (vi) of \( \xi_e \) we get
\[
\xi_e(P_e(A, B)) = H_e(A, B).
\]

Now we are in a position to state our first result.
Theorem 2.4. Let \((X, p)\) be a cone-complete metric space, \(T\) be a multivalued mapping in \(X\) with cone-closed, cone-bounded values. If there is an \(e \in \text{int}K\) such that
\[
P_e(Tx, Ty) \leq_K ap(x, y) \quad \forall x, y \in X
\] (5)
for some \(a \in [0, 1)\), then \(T\) has a fixed point.

Proof. Applying \(\xi_e\) to both sides of (5) and using properties (iii) and (v) of \(\xi_e\) we get
\[
H_e(Tx, Ty) \leq ad_e(x, y) \quad \forall x, y \in X.
\]
The result follows from Nadler’s theorem.

For further generalization we need the following.

Definition 2.5. Let \((X, p)\) be a cone metric space, \(x \in X, A \subset X, e \in \text{int} K\). The distance in direction \(e\) from \(x\) to \(A\) is defined as follows:
\[
p_e(x, A) = \inf \{r > 0 : B_p(x, re) \cap A \neq \emptyset\} \cdot e.
\]
From (2) we get
\[
\inf \{r > 0 : B_p(x, re) \cap A \neq \emptyset\} = \inf \{r > 0 : B_d(x, r) \cap A \neq \emptyset\} = \inf_{y \in A} d_e(x, y) = d_e(x, A),
\]
where \(B_d(x, r) = \{y \in X : d_e(x, y) < r\}\).

Hence
\[
p_e(x, A) = d_e(x, A) \cdot e,
\]
from this
\[
\xi_e(p_e(x, A)) = d_e(x, A).
\]

Theorem 2.4 can be generalized as follows.

Theorem 2.6. Theorem 2.4 is still valid if we replace (5) by
\[
P_e(Tx, Ty) \leq_K ap(x, y) + b[p_e(x, Tx) + p_e(y, Ty)] + c[p_e(x, Ty) + p_e(y, Tx)] (6)
\]
for all \(x, y \in X, \) with \(a, b, c \in [0, 1), a + 2b + 2c < 1\).

Proof. Applying \(\xi_e\) to both sides of (6) and using properties (iii), (iv), (v) of \(\xi_e\) we get
\[
H_e(Tx, Ty) \leq ad_e(x, y) + b[d_e(x, Tx) + d_e(y, Ty)] + c[d_e(x, Ty) + d_e(y, Tx)]
\]
\[
\forall x, y \in X.
\]
The result follows from generalized Nadler’s theorem.

Remark 2.7. When \(E = \mathbb{R}, K = \mathbb{R}^+, e = 1\), Theorems 2.4 and 2.6 coincide with Nadler’s and generalized Nadler’s theorems respectively.
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