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Positive Periodic Solutions of n-Species Lotka-Volterra Cooperative Systems with Delays*

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Abstract. In this paper, a class of periodic n species cooperative Lotka-Volterra population systems with distributed delays is discussed. Based on the continuation theorem of the coincidence degree theory developed by Gaines and Mawhin, some new sufficient conditions on the existence of positive periodic solutions are established.

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1. Introduction

As well known, in recent years the nonautonomous and periodic population dynamical systems have been extensively studied. The basic and important studied questions for these systems are the persistence, permanence and extinction of species, global stability of systems and the existence of positive periodic solutions, positive almost periodic solutions and strictly positive solutions, etc. Many

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important and influential results have been established and can be found in many articles and books. Particularly, the existence of positive periodic solutions for various Lotka-Volterra-type population dynamical systems has been extensively studied in [1, 2, 4–16] and the references cited therein.

In [14], the authors studied the following single-species periodic logistic equation with infinite distributed delay

$$\dot{N}(t) = N(t)\Big(a(t) - \sum_{i=1}^{\infty} b_i(t)N(t-\tau_i(t)) - \int_{-\infty}^{t} c(s-t)N(s)ds\Big).$$

By using the Mawhin's continuation theorem, the sufficient conditions on the existence of positive periodic solutions are established. In [12], the following periodic n species Lotka-Volterra competitive system with feedback controls and finite distributed delays are discussed

$$\begin{aligned}\dot{y}_i(t) &= y_i(t)\Big[r_i(t) - a_{ii}(t)y_i(t) - \sum_{j=1,j\neq i}^{n} a_{ij}(t)\int_0^{\omega} K_{ij}(s)y_j(t-s)ds \\ &\quad -\alpha_i(t)\int_0^{\omega} H_i(s)u_i(t-s)ds\Big], \\ \dot{u}_i(t) &= -\eta_i(t) + a_i(t)\int_0^{\omega} K_i(s)y_i(t-s)ds,\end{aligned}$$

where $i = 1, 2, \ldots, n$. By using the technique of coincidence degree and Liapunov functionals method, the sufficient conditions for the existence and global stability of positive periodic solutions are obtained. In [13], the authors considered the following periodic single-species logistic system with feedback regulation and infinite distributed delay

$$\begin{aligned}\dot{N}(t) &= r(t)N(t)\left(1 - \frac{1}{K(t)}\int_0^{\infty} H(s)N(t-s)ds - c(t)u(t)\right), \\ \dot{u}(t) &= -a(t)u(t) + b(t)\int_0^{\infty} H(s)N^2(t-s)ds.\end{aligned}$$

The sufficient conditions for the existence of positive periodic solutions are established, based on the Mawhin's continuation theorem.

Motivated by the above works, in this paper, we investigate the following periodic n-species Lotka-Volterra cooperative system with pure finite distributed delays

$$\begin{aligned}\dot{x}_i(t) &= x_i(t)\Big[r_i(t) - \sum_{l=1}^{m} a_{iil}(t)\int_{-\tau}^{0} k_{iil}(s)x_i(t+s)ds \\ &\quad + \sum_{j\neq i}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)x_j(t+s)ds\Big], \quad i = 1, 2, \ldots, n. \qquad (1)\end{aligned}$$

By using the technique of coincidence degree developed by Gains and Mawhin in [3], we will establish some new sufficient conditions which guarantee the system to have at least one positive periodic solution.

2. Preliminaries

In system (1), we have that $x_i(t)$ $(i = 1, 2, \ldots, n)$ represent the density of n cooperative species x_i $(i = 1, 2, \ldots, n)$ at time t, respectively; $r_i(t)$ $(i = 1, 2, \ldots, n)$ represent the intrinsic growth rate of species x_i $(i = 1, 2, \ldots, n)$ at time t, respectively; $a_{iil}(t)$ $(i = 1, 2, \ldots, n, l = 1, 2, \ldots, m)$ represent the intrapatch restriction density of species x_i $(i = 1, 2, \ldots, n)$ at time t, respectively; $a_{ijl}(t)$ $(l = 1, 2, \ldots, m, i \neq j, i, j = 1, 2, \ldots, n)$ represent the cooperative coefficients between n species x_i $(i = 1, 2, \ldots, n)$ at time t, respectively. $\tau \geq 0$ is a constant and τ may be ∞ . In this paper, we always assume that

(H1) $r_i(t)$ $(i = 1, 2, \ldots, n)$ are ω-periodic continuous functions with $\int_0^\omega r_i(t)dt > 0$; $a_{ijl}(t)$ $(i, j = 1, 2, \ldots, n\ ; l = 1, 2, \ldots, m)$ are positive ω-periodic continuous functions; $k_{ijl}(s)$ $(i, j = 1, 2, \ldots, n\ ; l = 1, 2, \ldots, m)$ are nonnegative integrable functions on $[-\tau, 0]$ satisfying $\int_{-\tau}^0 k_{ijl}(s)ds = 1$.

From the viewpoint of mathematical biology, in this paper for system (1) we only consider the solution with the following initial condition

$$x_i(s) = \phi_i(t) \quad \text{for all} \quad s \in [-\tau, 0],\ i = 1, 2, \ldots, n, \tag{2}$$

where $\phi_i(s)$ $(i = 1, 2, \ldots, n)$ are nonnegative continuous functions defined on $[-\tau, 0]$ satisfying $\phi_i(0) > 0$ $(i = 1, 2, \ldots, n)$.

In this paper, for any ω-periodic continuous function $f(t)$ we denote

$$f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t), \quad \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)dt.$$

In order to obtain the existence of positive ω-periodic solutions of system (1), we will use the continuation theorem developed by Gaines and Mawhin in [3]. For the reader's convenience, we will introduce the continuation theorem in the following.

Let X and Z be two normed vetor spaces. Let $L : \operatorname{Dom} L \subset X \to Z$ be a linear operator and $N : X \to Z$ be a continuous operator. The operator L is called a Fredholm operator of index zero, if $\dim\operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is a closed set in Z. If L is a Fredholm operator of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P : \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_P and denote by $J : \operatorname{Im} Q \to \operatorname{Ker} L$ an isomorphism of $\operatorname{Im} Q$ onto $\operatorname{Ker} L$. Let Ω be a bounded open subset of

X, we say that the operator N is L-compact on $\overline{\Omega}$, where $\overline{\Omega}$ denotes the closure of Ω in X, if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1. (see [12–14]) *Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. If*

(a) *for each $\lambda \in (0,1)$ and $x \in \partial\Omega \cap \operatorname{Dom} L,\ Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L,\ QNx \neq 0$;*
(c) $\deg\{JQN,\ \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0$,

then the operator equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

3. Main results

Now, for convenience of statements we denote the functions

$$a_{ij}(t) = \sum_{l=1}^{m} a_{ijl}(t), \quad i,j = 1,2,\ldots,n.$$

On the existence of positive periodic solutions of system (1) we have the following theorem.

Theorem 3.1. *Suppose that assumption* (**H1**) *holds and*

$$\min_{t\in[0,\omega]}\Big\{\sum_{l=1}^{m}\int_{-\tau}^{0}\Big[a_{iil}(t-s)k_{iil}(s)-\sum_{j\neq i}^{n}a_{jil}(t-s)k_{jil}(s)\Big]ds\Big\} =: \delta_i > 0,\ i = 1,2,\ldots,n$$

and the system of algebraic equations

$$\bar{r}_i - \bar{a}_{ii}v_i + \sum_{j\neq i}^{n}\bar{a}_{ij}v_j = 0, \quad i = 1,2,\ldots,n$$

has a unique positive solution. Then system (1) *has at least one positive ω-periodic solution.*

Proof. For system (1) we introduce new variables $u_i(t)$ $(i = 1,2,\ldots,n)$ such that

$$x_i(t) = \exp\{u_i(t)\}, \quad i = 1,2,\ldots,n. \tag{3}$$

Then system (1) is rewritten in the following form

$$\begin{aligned}\dot{u}_i(t) = r_i(t) &- \sum_{l=1}^{m} a_{iil}(t)\int_{-\tau}^{0} k_{iil}(s)\exp\{u_i(t+s)\}ds \\ &+ \sum_{j\neq i}^{n}\sum_{l=1}^{m} a_{ijl}(t)\int_{-\tau}^{0} k_{ijl}(s)\exp\{u_j(t+s))\}ds, \quad i = 1,2,\ldots,n.\end{aligned} \tag{4}$$

In order to apply Lemma 2.1 to system (4), we introduce the normed vector spaces X and Z as follows. Let $C(R, R^n)$ denote the space of all continuous functions $u(t) = (u_1(t), u_2(t), \ldots, u_n(t)) : R \to R^n$. We take

$$X = Z = \{u(t) \in C(R, R^n) : u(t) \text{ is an } \omega\text{-periodic function}\}$$

with norm

$$\|u\| = \sum_{i=1}^{n} \max_{t\in[0,\omega]} |u_i(t)|.$$

It is obvious that X and Z are Banach spaces.

We define a linear operator $L : \operatorname{Dom} L \subset X \to Z$ and a continuous operator $N : X \to Z$ as follows.

$$Lu(t) = \dot{u}(t)$$

and

$$Nu(t) = (Nu_1(t), Nu_2(t), \ldots, Nu_n(t)),$$

where

$$\begin{aligned} Nu_i(t) = r_i(t) &- \sum_{l=1}^{m} a_{iil}(t) \int_{-\tau}^{0} k_{iil}(s) \exp\{u_i(t+s)\} ds \\ &+ \sum_{j\neq i}^{n} \sum_{l=1}^{m} a_{ijl}(t) \int_{-\tau}^{0} k_{ijl}(s) \exp\{u_j(t+s))\} ds, \quad i = 1, 2, \ldots, n. \quad (5) \end{aligned}$$

Further, we define continuous projectors $P : X \to X$ and $Q : Z \to Z$ as follows.

$$Pu(t) = \frac{1}{\omega} \int_0^{\omega} u(t)dt, \quad Qv(t) = \frac{1}{\omega} \int_0^{\omega} v(t)dt.$$

We easily see $\operatorname{Im} L = \{v \in Z : \int_0^{\omega} v(t)dt = 0\}$ and $\operatorname{Ker} L = R^n$. It is obvious that $\operatorname{Im} L$ is closed in Z and $\dim\operatorname{Ker} L = n$. Since for any $v \in Z$ there are unique $v_1 \in R^n$ and $v_2 \in \operatorname{Im} L$ with

$$v_1 = \frac{1}{\omega} \int_0^{\omega} v(t)dt, \quad v_2(t) = v(t) - v_1$$

such that $v(t) = v_1 + v_2(t)$, we have $\operatorname{codim}\operatorname{Im} L = n$. Therefore, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ is given in the following form

$$K_p v(t) = \int_0^t v(s)ds - \frac{1}{\omega} \int_0^{\omega} \int_0^t v(s)dsdt.$$

For convenience, we denote $F(t) = (F_1(t), F_2(t), \ldots, F_n(t))$, where

$$F_i(t) = r_i(t) - \sum_{l=1}^{m} a_{iil}(t) \int_{-\tau}^{0} k_{iil}(s) \exp\{u_i(t+s)\} ds$$

$$+\sum_{j\neq i}^{n}\sum_{l=1}^{m}a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{u_j(t+s)\}ds, \quad i=1,2,\dots,n. \quad (6)$$

Thus, we have

$$QNu(t)=\frac{1}{\omega}\int_0^\omega F(t)dt \tag{7}$$

and

$$\begin{aligned}K_p(I-Q)Nu(t)&=K_pINu(t)-K_pQNu(t)\\&=\int_0^t F(s)ds-\frac{1}{\omega}\int_0^\omega\int_0^t F(s)dsdt\\&\quad+\Big(\frac{1}{2}-\frac{t}{\omega}\Big)\int_0^\omega F(s)ds.\end{aligned} \tag{8}$$

From formulas (7) and (8), we easily see that QN and $K_p(I-Q)N$ are continuous operators. Furthermore, it can be verified that $\overline{K_p(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega\subset X$ by using Arzela-Ascoli theorem and $QN(\overline{\Omega})$ is bounded. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded subset $\Omega\subset X$.

Now, we are able to find an appropriate open bounded subset Ω for the application of the continuation theorem (Lemma 2.1) to system (1).

Corresponding to the operator equation $Lu(t)=\lambda Nu(t)$ with parameter $\lambda\in(0,1)$, we have

$$\dot{u}_i(t)=\lambda F_i(t), \quad i=1,2,\dots,n, \tag{9}$$

where $F_i(t)$ $(i=1,2,\dots,n)$ are given in Eqs. (6).

Assume that $u(t)=(u_1(t),u_2(t),\dots,u_n(t))\in X$ is a solution of system (9) for some parameter $\lambda\in(0,1)$. By integrating system (9) over the interval $[0,\omega]$, we obtain

$$\begin{aligned}&\int_0^\omega\Big[r_i(t)-\sum_{l=1}^{m}a_{iil}(t)\int_{-\tau}^{0}k_{iil}(s)\exp\{u_i(t+s)\}ds\\&+\sum_{j\neq i}^{n}\sum_{l=1}^{m}a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{u_j(t+s)\}ds\Big]dt=0, \quad i=1,2,\dots,n.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_0^\omega\Big[\sum_{l=1}^{m}a_{iil}(t)\int_{-\tau}^{0}k_{iil}(s)\exp\{u_i(t+s)\}ds\\&-\sum_{j\neq i}^{n}\sum_{l=1}^{m}a_{ijl}(t)\int_{-\tau}^{0}k_{ijl}(s)\exp\{u_j(t+s)\}ds\Big]dt=\bar{r}_i\omega, \ i=1,2,\dots,n.\end{aligned} \tag{10}$$

From the continuity of $u(t)=(u_1(t),u_2(t),\dots,u_n(t))$, there exist constants $\xi_i,\eta_i\in[0,\omega]$ $(i=1,2,\dots,n)$ such that

$$u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t), \quad u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t), \quad i = 1, 2, \ldots, n. \tag{11}$$

From (10) and (11), we obtain

$$\int_0^\omega a_{ii}(t) \exp\{u_i(\xi_i)\} dt \geq \bar{r}_i \omega, \quad i = 1, 2, \ldots, n.$$

Therefore, we further have

$$u_i(\xi_i) \geq \ln(\frac{\bar{r}_i}{\bar{a}_{ii}}), \quad i = 1, 2, \ldots, n. \tag{12}$$

For each $i, j = 1, 2, \ldots, n$ and $l = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\int_0^\omega a_{ijl}(t) \int_{-\tau}^0 k_{ijl}(s) \exp\{u_j(t+s)\} ds dt \\
&= \int_{-\tau}^0 \int_0^\omega a_{ijl}(t) k_{ijl}(s) \exp\{u_j(t+s)\} dt ds \\
&= \int_{-\tau}^0 \int_s^{s+\omega} a_{ijl}(v-s) k_{ijl}(s) \exp\{u_j(v)\} dv ds \\
&= \int_{-\tau}^0 \int_0^\omega a_{ijl}(v-s) k_{ijl}(s) \exp\{u_j(v)\} dv ds \\
&= \int_0^\omega \int_{-\tau}^0 a_{ijl}(v-s) k_{ijl}(s) \exp\{u_j(v)\} ds dv \\
&= \int_0^\omega \Big(\int_{-\tau}^0 a_{ijl}(t-s) k_{ijl}(s) ds \Big) \exp\{u_j(t)\} dt.
\end{aligned} \tag{13}$$

Hence, from (10) we further obtain

$$\begin{aligned}
&\int_0^\omega \Big[\sum_{l=1}^m \Big(\int_{-\tau}^0 a_{iil}(t-s) k_{iil}(s) ds \Big) \exp\{u_i(t)\} \\
&- \sum_{j\neq i}^n \sum_{l=1}^m \Big(\int_{-\tau}^0 a_{ijl}(t-s) k_{ijl}(s) ds \Big) \exp\{u_j(t)\} \Big] dt = \bar{r}_i \omega, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\int_0^\omega \Big[\sum_{l=1}^m \Big(\int_{-\tau}^0 a_{11l}(t-s) k_{11l}(s) ds \Big) \exp\{u_1(t)\} \\
&- \sum_{j\neq 1}^n \sum_{l=1}^m \Big(\int_{-\tau}^0 a_{1jl}(t-s) k_{1jl}(s) ds \Big) \exp\{u_j(t)\} \Big] dt \\
&+ \int_0^\omega \Big[\sum_{l=1}^m \Big(\int_{-\tau}^0 a_{22l}(t-s) k_{22l}(s) ds \Big) \exp\{u_2(t)\}
\end{aligned}$$

$$-\sum_{j\neq 2}^{n}\sum_{l=1}^{m}\Big(\int_{-\tau}^{0}a_{2jl}(t-s)k_{2jl}(s)ds\Big)\exp\{u_j(t)\}\Big]dt$$

$$+\cdots+\int_0^{\omega}\Big[\sum_{l=1}^{m}\Big(\int_{-\tau}^{0}a_{nnl}(t-s)k_{nnl}(s)ds\Big)\exp\{u_n(t)\}$$

$$-\sum_{j\neq n}^{n}\sum_{l=1}^{m}\Big(\int_{-\tau}^{0}a_{njl}(t-s)k_{njl}(s)ds\Big)\exp\{u_j(t)\}\Big]dt$$

$$=\int_0^{\omega}\Big[\sum_{l=1}^{m}\Big(\int_{-\tau}^{0}a_{11l}(t-s)k_{11l}(s)ds$$

$$-\sum_{j\neq 1}^{n}\Big(\int_{-\tau}^{0}a_{j1l}(t-s)k_{j1l}(s)ds)\Big)\exp\{u_1(t)\}\Big]dt$$

$$+\int_0^{\omega}\Big[\sum_{l=1}^{m}\Big(\int_{-\tau}^{0}a_{22l}(t-s)k_{22l}(s)ds$$

$$-\sum_{j\neq 2}^{n}\Big(\int_{-\tau}^{0}a_{j2l}(t-s)k_{j2l}(s)ds)\Big)\exp\{u_2(t)\}\Big]dt$$

$$+\cdots+\int_0^{\omega}\Big[\sum_{l=1}^{m}\Big(\int_{-\tau}^{0}a_{nnl}(t-s)k_{nnl}(s)ds$$

$$-\sum_{j\neq n}^{n}\Big(\int_{-\tau}^{0}a_{jnl}(t-s)k_{jnl}(s)ds)\Big)\exp\{u_n(t)\}\Big]dt$$

$$=\sum_{l=1}^{m}\int_0^{\omega}\Big(\int_{-\tau}^{0}[a_{11l}(t-s)k_{11l}(s)-\sum_{j\neq 1}^{n}a_{j1l}(t-s)k_{j1l}(s)]ds\Big)\exp\{u_1(t)\}dt$$

$$+\sum_{l=1}^{m}\int_0^{\omega}\Big(\int_{-\tau}^{0}[a_{22l}(t-s)k_{22l}(s)-\sum_{j\neq 2}^{n}a_{j2l}(t-s)k_{j2l}(s)]ds\Big)\exp\{u_2(t)\}dt$$

$$+\cdots+\sum_{l=1}^{m}\int_0^{\omega}\Big(\int_{-\tau}^{0}\Big[a_{nnl}(t-s)k_{nnl}(s)-\sum_{j\neq n}^{n}a_{jnl}(t-s)k_{jnl}(s)\Big]ds\Big)\exp\{u_n(t)\}dt$$

$$=\sum_{i=1}^{n}\bar{r}_i\omega.$$

From the assumptions of Theorem 3.1, we can obtain

$$\sum_{l=1}^{m}\int_0^{\omega}\Big(\int_{-\tau}^{0}\Big[a_{iil}(t-s)k_{iil}(s)-\sum_{j\neq i}^{n}a_{jil}(t-s)k_{jil}(s)\Big]ds\Big)\exp\{u_i(t)\}dt$$

$$\leq\sum_{i=1}^{n}\bar{r}_i\omega,\quad i=1,2,\ldots,n.$$

Hence

$$\delta_i \int_0^\omega \exp\{u_i(t)\}dt \le \sum_{i=1}^n \bar{r}_i\omega, \quad i = 1, 2, \dots, n.$$

Consequently

$$\int_0^\omega \exp\{u_i(t)\}dt \le \frac{\sum_{i=1}^n \bar{r}_i\omega}{\delta_i}, \quad i = 1, 2, \dots, n. \tag{14}$$

From (11), we further obtain

$$u_i(\eta_i) \le \ln\Big(\frac{\sum_{i=1}^n \bar{r}_i}{\delta_i}\Big) \quad i = 1, 2, \dots, n. \tag{15}$$

On the other hand, directly from system (4) we have

$$\begin{aligned}
&\int_0^\omega |\dot{u}_i(t)|dt \\
\le& \int_0^\omega \Big[|r_i(t)| + \sum_{l=1}^m a_{iil}(t)\int_{-\tau}^0 k_{iil}(s)\exp\{u_i(t+s)\}ds \\
&+ \sum_{j\ne 1}^n \sum_{l=1}^m a_{ijl}(t)\int_{-\tau}^0 k_{ijl}(s)\exp\{u_j(t+s)\}ds\Big]dt \\
=& \int_0^\omega |r_i(t)|dt + \int_0^\omega \Big(\sum_{l=1}^m \int_{-\tau}^0 a_{iil}(t-s)k_{iil}(s)ds\Big)\exp\{u_i(t)\}dt \\
&+ \int_0^\omega \Big(\sum_{j\ne i}^n \sum_{l=1}^m \int_{-\tau}^0 a_{ijl}(t-s)k_{ijl}(s)ds\Big)\exp\{u_j(t)\}dt \\
\le& \int_0^\omega |r_i(t)|dt + \sum_{l=1}^m a_{iil}^M \int_0^\omega \exp\{u_i(t)\}dt \\
&+ \sum_{j\ne i}^n \sum_{l=1}^m a_{ijl}^M \int_0^\omega \exp\{u_j(t)\}dt \\
\le& \bar{|r_i|}\omega + \sum_{j=1}^n \sum_{l=1}^m a_{ijl}^M \frac{\sum_{i=1}^n \bar{r}_i\omega}{\delta_i}, \quad i = 1, 2, \dots, n.
\end{aligned} \tag{16}$$

From (15) and (16), we have for any $t \in [0, \omega]$

$$\begin{aligned}
u_i(t) &\le u_i(\eta_i) + \int_0^\omega |\dot{u}_i(t)|dt \\
&\le \ln\Big(\frac{\sum_{i=1}^n \bar{r}_i}{\delta_i}\Big) + \bar{|r_i|}\omega + \sum_{j=1}^n \sum_{l=1}^m a_{ijl}^M \frac{\sum_{i=1}^n \bar{r}_i\omega}{\delta_i} =: M_i, \; i = 1, 2, \dots, n.
\end{aligned} \tag{17}$$

Further from (12) and (16), we have for any $t \in [0, \omega]$

$$u_i(t) \geq u_i(\xi_i) - \int_0^\omega |\dot{u}_i(t)|dt$$
$$\geq \ln\left(\frac{\bar{r}_i}{\bar{a}_{ii}}\right) - |\bar{r_i}|\omega - \sum_{j=1}^{n}\sum_{l=1}^{m} a_{ijl}^M \frac{\sum_{i=1}^n \bar{r}_i\omega}{\delta_i} =: N_i, \quad i = 1, 2, \ldots, n. \quad (18)$$

Therefore, from (17), (18) we have

$$\max_{t\in[0,\omega]} |u_i(t)| \leq \max\big\{|M_i|, |N_i|\big\} =: B_i, \quad i = 1, 2, \ldots, n.$$

It can be seen that the constants $B_i\,(i = 1, 2, \ldots, n)$ are independent of parameter $\lambda \in (0, 1)$.

For any $u = (u_1, u_2, \ldots, u_n) \in R^n$, from (5) we can obtain

$$QNu = (QNu_1, QNu_2, \ldots, QNu_n),$$

where

$$QNu_i = \bar{r}_i - \bar{a}_{ii}\exp\{u_i\} + \sum_{j\neq i}^{n} \bar{a}_{ij}\exp\{u_j\}, \quad i = 1, 2, \ldots, n.$$

We consider the following system of algebraic equations

$$\bar{r}_i - \bar{a}_{ii}v_i + \sum_{j\neq i}^{n} \bar{a}_{ij}v_j = 0, \quad i = 1, 2, \ldots, n.$$

From the assumption of Theorem 3.1, the system has a unique positive solution $v^* = (v_1^*, v_2^*, \ldots, v_n^*)$. Hence, the equation $QNu = 0$ has a unique solution $u^* = (u_1^*, u_2^*, \ldots, u_n^*) = (\ln v_1^*, \ln v_2^*, \ldots, \ln v_n^*) \in R^n$.

Choosing a constant $B > 0$ large enough such that $|u_1^*| + |u_2^*| + \cdots + |u_n^*| < B$ and $B > B_1 + B_2 + \cdots + B_n$, we define a bounded open set $\Omega \subset X$ as follows

$$\Omega = \{u \in X : \|u\| < B\}.$$

It is clear that Ω satisfies conditions (a) and (b) of Lemma 2.1. On the other hand, by directly calculating we can obtain

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, (0, 0, \cdots, 0)\} = \operatorname{sgn}\begin{vmatrix} f_{u_1}^1 & f_{u_2}^1 & \cdots & f_{u_n}^1 \\ f_{u_1}^2 & f_{u_2}^2 & \cdots & f_{u_n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ f_{u_1}^n & f_{u_2}^n & \cdots & f_{u_n}^n \end{vmatrix},$$

where $f_{u_j}^i = \bar{a}_{ij}\exp\{u_j^*\}(i, j = 1, 2, \ldots, n,\ i \neq j)$ and $f_{u_i}^i = -\bar{a}_{ii}\exp\{u_i^*\}(i = 1, 2, \ldots, n)$. From the assumption of Theorem 3.1, we have

$$\begin{vmatrix} f^1_{u_1} & f^1_{u_2} & \cdots & f^1_{u_n} \\ f^2_{u_1} & f^2_{u_2} & \cdots & f^2_{u_n} \\ \cdots & \cdots & \cdots & \cdots \\ f^n_{u_1} & f^n_{u_2} & \cdots & f^n_{u_n} \end{vmatrix} \neq 0.$$

From this, we finally have

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, (0,0,\ldots,0)\} \neq 0.$$

This shows that Ω satisfies condition (c) of Lemma 2.1. Therefore, system (4) has at least one ω-periodic solution. Further from (3), system (1) has at least one positive ω-periodic solution. This completes the proof of Theorem 3.1. ∎

Remark 3.2. Mawhin's continuation theorem is a powerful tool for study the existence of periodic solutions of periodic high-dimensional time-delayed problems. When dealing with time-delayed problem, it is very convenient and the result is relatively simple. The most critical in the using of the theorem is calculation of topological degree, that is, the condition (c) of the theorem.

4. Two examples

Example 4.1. First, we consider the following delayed system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Big[1+\sin(t) - (6+\sin(t))\int_{-\tau}^{0} k_{111}(s)x_1(t+s)ds \\ &\quad + \Big(\frac{1+\sin(t)}{2}\Big)\int_{-\tau}^{0} k_{121}(s)x_2(t+s)ds + \Big(\frac{1+\sin(t)}{2}\Big)\int_{-\tau}^{0} k_{131}(s)x_3(t+s)ds\Big], \\ \dot{x}_2(t) &= x_2(t)\Big[1+\cos(t) - (8+\cos(t))\int_{-\tau}^{0} k_{221}(s)x_2(t+s)ds \\ &\quad + \Big(\frac{1+\cos(t)}{2}\Big)\int_{-\tau}^{0} k_{211}(s)x_1(t+s)ds + \Big(\frac{1+\cos(t)}{2}\Big)\int_{-\tau}^{0} k_{231}(s)x_3(t+s)ds\Big], \\ \dot{x}_3(t) &= x_3(t)\Big[1.5+\sin(t) - (7+\sin(t))\int_{-\tau}^{0} k_{331}(s)x_3(t+s)ds \\ &\quad + \Big(\frac{1+\sin(t)}{2}\Big)\int_{-\tau}^{0} k_{311}(s)x_1(t+s)ds + \Big(\frac{1+\sin(t)}{2}\Big)\int_{-\tau}^{0} k_{321}(s)x_2(t+s)ds\Big]. \end{aligned} \tag{19}$$

Corresponding to system (1), $n = 3, m = 1, \omega = 2\pi$, by direct calculation we can get

$$\delta_1 \approx 3, \quad \delta_2 \approx 5, \quad \delta_3 \approx 4$$

and the following system of equations has a unique solution

$$\bar{a}_{11}v_1 - \bar{a}_{12}v_2 - \bar{a}_{13}v_3 = \bar{r}_1,$$

$$\bar{a}_{22}v_1 - \bar{a}_{21}v_2 - \bar{a}_{23}v_3 = \bar{r}_2,$$
$$\bar{a}_{33}v_1 - \bar{a}_{31}v_2 - \bar{a}_{32}v_3 = \bar{r}_3,$$

where

$$v_1 \approx 0.199, \quad v_2 \approx 0.152, \quad v_3 \approx 0.239.$$

It is clear that all the conditions of Theorem 3.1 hold. Hence, system (19) has a positive periodic solution.

From Figure 1 we can see that, system (19) has a positive periodic solution.

Example 4.2. Next, we consider the following delayed system:

$$\begin{aligned}
\dot{x}_1(t) &= x_1(t)\Big[2+\sin(t) - (1+\sin(t))\int_{-\tau}^{0} k_{111}(s)x_1(t+s)ds \\
&\quad + \frac{2}{5}(1+\sin(t))\int_{-\tau}^{0} k_{121}(s)x_2(t+s)ds + \frac{2}{5}(1+\sin(t))\int_{-\tau}^{0} k_{131}(s)x_3(t+s)ds\Big] \\
\dot{x}_2(t) &= x_2(t)\Big[2+\cos(t) - (2+\cos(t))\int_{-\tau}^{0} k_{221}(s)x_2(t+s)ds \\
&\quad + \frac{7}{20}(1+\cos(t))\int_{-\tau}^{0} k_{211}(s)x_1(t+s)ds + \frac{2}{5}(1+\sin(t))\int_{-\tau}^{0} k_{231}(s)x_3(t+s)ds\Big] \\
\dot{x}_3(t) &= x_3(t)\Big[2+\sin(t) - (4+\sin(t))\int_{-\tau}^{0} k_{331}(s)x_3(t+s)ds \\
&\quad + \frac{9}{20}(1+\sin(t))\int_{-\tau}^{0} k_{311}(s)x_1(t+s)ds + \frac{9}{20}(1+\sin(t))\int_{-\tau}^{0} k_{321}(s)x_2(t+s)ds\Big]
\end{aligned} \tag{20}$$

Corresponding to system (1),$n = 3, m = 1, \omega = 2\pi$, by direct calculation we can get

$$\delta_1 \approx -1.6, \quad \delta_2 \approx -1.5, \quad \delta_3 \approx 1.2$$

and the following system of equations has a unique solution

$$\bar{a}_{11}v_1 - \bar{a}_{12}v_2 - \bar{a}_{13}v_3 = \bar{r}_1,$$
$$\bar{a}_{22}v_1 - \bar{a}_{21}v_2 - \bar{a}_{23}v_3 = \bar{r}_2,$$
$$\bar{a}_{33}v_1 - \bar{a}_{31}v_2 - \bar{a}_{32}v_3 = \bar{r}_3,$$

where

$$v_1 = 3.122, \quad v_2 = 1.756, \quad v_3 = 1.049.$$

Clearly, the conditions of Theorem 3.1 do not hold. From Figure 2 we can see that, system (20) has no positive periodic solution.

Remark 4.3. From these two examples we can see that if the conditions of Theorem 3.1 hold, then the system has a positive periodic solution. If the conditions of Theorem 3.1 do not hold, then the system has no positive periodic solution.

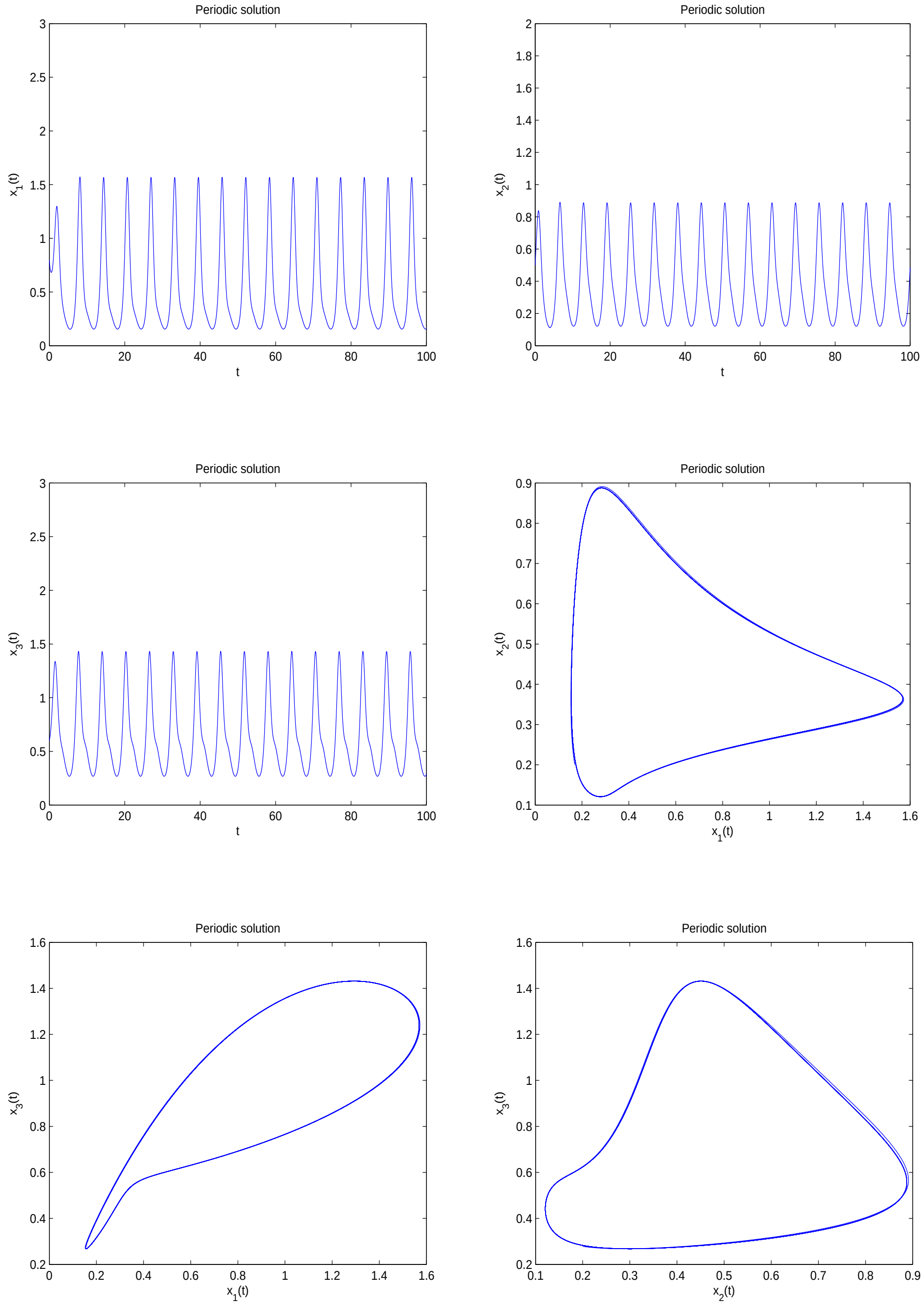

Fig. 1. Positive periodic solution of system (19). Here, we take the initial value $x_0 = (x_{10}, x_{20}, x_{30}) = (0.8, 0.5, 0.6)$.

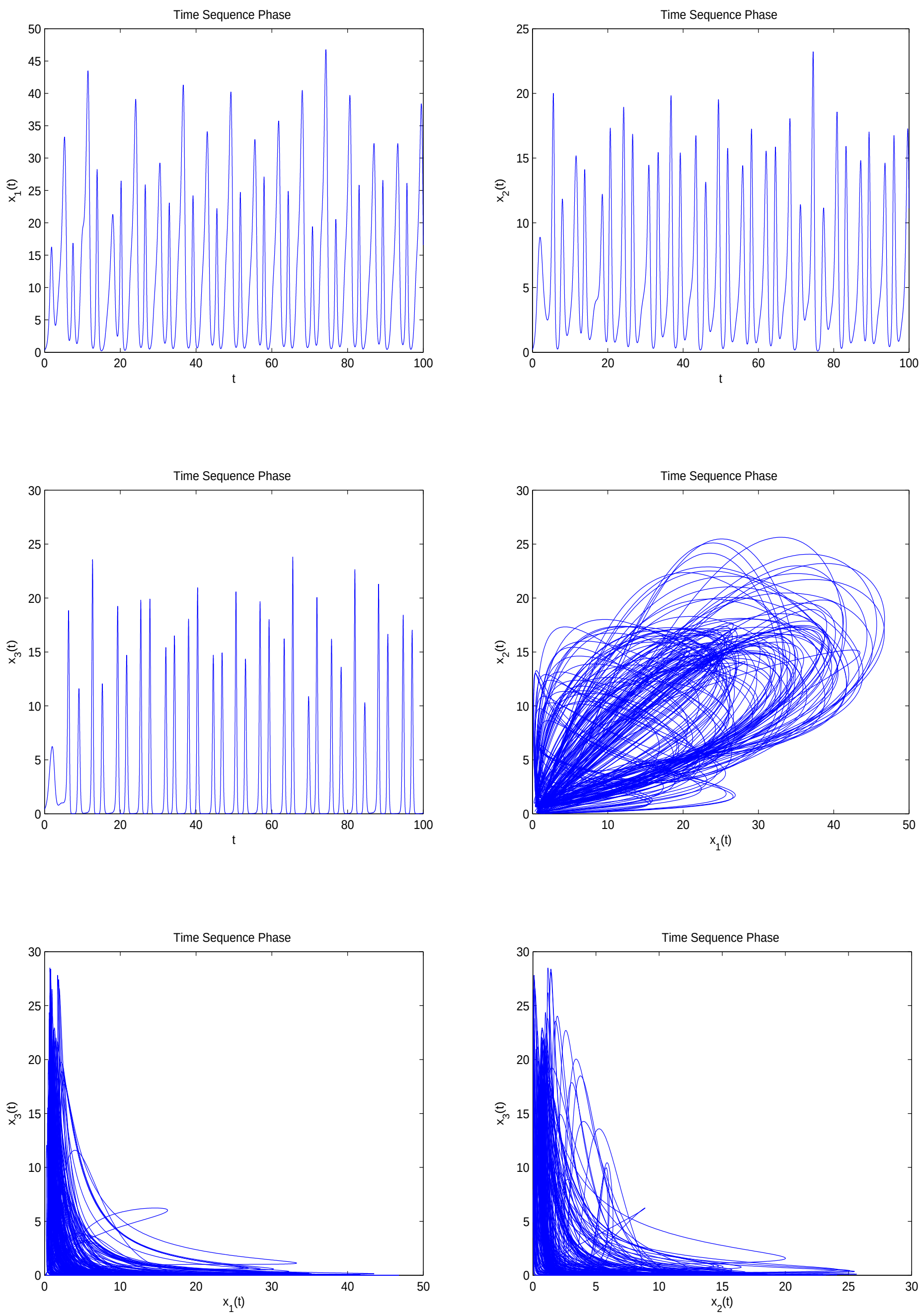

Fig. 2. Non-existence of positive periodic solution for system (4.2).Here, we take the initial value $x_0 = (x_{10}, x_{20}, x_{30}) = (0.3, 0.2, 0.4)$.

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