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On the Regularity of Generalized Local Cohomology

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Abstract. Let $R = \oplus_{j\geq 0} R_j$ be a positively graded Noetherian ring with local base ring $(R_0, \mathfrak{m}_0)$ and the irrelevant ideal $R_+ = \oplus_{j\geq 1} R_j$. Let M, N be two finitely generated graded R-modules with $\mathrm{pd}(M) < \infty$. The vanishing of homogeneous components of generalized local cohomology modules with support in different graded ideals in a certain graded ring will be compared and using the result a bound for the regularity of the pair (M, N) relative to the graded ideal $\mathfrak{a} \supseteq R_+$ will be obtained.

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1. Introduction

Let $R = \oplus_{j\in\mathbb{Z}} R_j$ be a graded Noetherian ring, $\mathfrak{a}$ a graded ideal of R and M, N be two finitely generated graded R-modules. Let $H^i_{\mathfrak{a}}(M, N)$ be the i-th generalized local cohomology of M and N with support in $\mathfrak{a}$ [7]. Then, as it has been shown in [9], the R-module $H^i_{\mathfrak{a}}(M, N) = \oplus_{j\in\mathbb{Z}}[H^i_{\mathfrak{a}}(M, N)]_j$ has a natural graded structure in such a way that the connected sequence of functors $(H^i_{\mathfrak{a}}(M, -))_{i\geq 0}$ from the category of finitely generated R-modules to itself has the *restriction property in the sense of [2, Chapter 12] . Furthermore, whenever $R = \oplus_{j=0}^{\infty} R_j$ is positively graded, the R_0-module $[H^i_{R_+}(M, N)]_j$ is finitely generated for all j and is zero for all $j \gg 0$.

Now, assume that $\mathrm{pd}(M)$ (the projective dimension of M) is finite. Then, by [8, 2.5], $H^i_{\mathfrak{a}}(M, N) = 0$ for all $i > \mathrm{pd}(M) + \mathrm{ara}(\mathfrak{a})$, where $\mathrm{ara}(\mathfrak{a})$ is the arithmetic rank of the ideal $\mathfrak{a}$, i.e., the least number of elements of R required to generate an ideal of R with the same radical as $\mathfrak{a}$. In the light of the above properties of the generalized local cohomology, in [3], the authors defined the generalized regularity of the pair (M, N) of finitely generated graded $R = \oplus_{j\geq 0} R_j$-modules as

$$\operatorname{reg}(M, N) = \max\{\operatorname{end}(H^i_{R_+}(M, N)) + i| \quad 0 \leq i \leq \operatorname{pd}(M) + \operatorname{ara}(R_+)\},$$

and obtained interesting concerning results, whereas for a graded R-module X, we denote by $\operatorname{end}(X)$ the supremum of j-th such that $X_j \neq 0$, with the convention that $\operatorname{end}(0) = -\infty$. In the following, by abuse of terminology, for each graded ideal $\mathfrak{a}$ containing R_+, we shall define the regularity of the pair (M, N) relative to $\mathfrak{a}$ as

$$\operatorname{reg}_{\mathfrak{a}}(M, N) = \max\{\operatorname{end}(H^i_{\mathfrak{a}}(M, N)) + i| \quad 0 \leq i \leq \operatorname{pd}(M) + \operatorname{ara}(\mathfrak{a})\}.$$

Assume that R is a positively graded (Noetherian) ring with local base ring $(R_0, \mathfrak{m}_0)$ and M, N be two finitely generated graded R-modules with $\operatorname{pd}(M) < \infty$. We put $s(M, N) = \operatorname{pd}_R(M) + \dim_R(N)$, and for each prime ideal $\mathfrak{p}_0$ of R_0, $s(\mathfrak{p}_0, M, N) = \operatorname{pd}_{R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}) + \dim_{R_{\mathfrak{p}_0}}(N_{\mathfrak{p}_0})$. The aim of this paper is to prove the following theorem.

Theorem 1.1. *Let R be a positively graded Noetherian ring with a local base ring and let M, N be two finitely generated graded R-modules with $pd(M) < \infty$. Let n be a given integer such that $[H^i_{\mathfrak{p}_0 R_{\mathfrak{p}_0} + (R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})]_j = 0$ for all $j > n$, all $\mathfrak{p}_0 \in Spec(R_0)$ and all $0 \leq i \leq s(\mathfrak{p}_0, M, N)$. Then for each graded ideal $\mathfrak{a} \supseteq R_+$,*

$$reg_{\mathfrak{a}}(M, N) < s(M, N) + n + 1.$$

2. Proof of Theorem 1.1

For ease in access, we first quote some known observations about generalized local cohomology modules.

In the rest R is a commutative Noetherian ring and M, N are R-modules.

Remark 2.1. A) ([1, Lemma 5.1]) Assume that $\mathfrak{a}$ is an ideal of R, M a finitely generated R-module of finite projective dimension $\operatorname{pd}(M)$, and N an R-module of finite Krull dimension $\dim(N)$. Then $H^i_{\mathfrak{a}}(M, N) = 0$ for all $i > \operatorname{pd}(M) + \dim(N)$.

B) ([5, Lemma 2.1 (i)]) For ideal $\mathfrak{a}$ of R, $H^i_{\mathfrak{a}}(M, N) \cong H^i_{\sqrt{\mathfrak{a}}}(M, N)$.

C) ([4, Lemma 3.1]) Let $\mathfrak{a}$ be an ideal of R and $x \in R$. Let M, N be finitely generated R-modules. Then there is a natural long exact sequence

$$\cdots \to H^i_{\mathfrak{a}+(x)}(M, N) \to H^i_{\mathfrak{a}}(M, N) \to H^i_{\mathfrak{a}R_x}(M, N) \to H^{i+1}_{\mathfrak{a}+(x)}(M, N) \to \cdots,$$

of generalized local cohomology modules. Furthermore, it is easy to see that, if R, M, N and $\mathfrak{a}$ are graded and x is a homogeneous element of R, then all the maps in this exact sequence are homogeneous, so that for each $j \in \mathbb{Z}$, there exists the long exact sequence

$$\cdots \to [H^i_{\mathfrak{a}+(x)}(M,N)]_j \to [H^i_{\mathfrak{a}}(M,N)]_j \to [H^i_{\mathfrak{a}R_x}(M,N)]_j \\ \to [H^{i+1}_{\mathfrak{a}+(x)}(M,N)]_j \to \cdots,$$

of R_0-modules.

D) ([6, Theorem 3.1]) Assume that R is local and $\mathrm{pd}(M) < \infty$. Let $\mathfrak{a}$ be an ideal of R. Then $H^i_{\mathfrak{a}}(M,N) = 0$ for all $i > \dim(R)$.

E) If R' is another commutative Noetherian ring and $f : R \to R'$ is a flat homomorphism, then for each ideal $\mathfrak{a}$ of R

$$H^i_{\mathfrak{a}}(M,N) \otimes_R R' \cong H^i_{\mathfrak{a}R'}(M \otimes_R R', N \otimes_R R').$$

Thus for a multiplicatively closed subset S of R,

$$S^{-1}H^i_{\mathfrak{a}}(M,N) \cong H^i_{S^{-1}\mathfrak{a}}(S^{-1}M, S^{-1}N).$$

If R, M, N and $\mathfrak{a}$ are graded and $S \subseteq R_0$, then for each $j \in \mathbb{Z}$,

$$S^{-1}([H^i_{\mathfrak{a}}(M,N)]_j) \cong [H^i_{S^{-1}\mathfrak{a}}(S^{-1}M, S^{-1}N)]_j,$$

as R_0-modules. It follows that given $j \in \mathbb{Z}$ and $r_0 \in R_0$, $[H^i_{\mathfrak{a}R_{r_0}}(M_{r_0}, N_{r_0})]_j = 0$ if and only if for each $\mathfrak{p}_{0r_0} \in \mathrm{Spec}(R_{0r_0})$, $[H^i_{(\mathfrak{a}R_{r_0})_{\mathfrak{p}_{0r_0}}}(M_{\mathfrak{p}_{0r_0}}, N_{\mathfrak{p}_{0r_0}})]_j = 0$.

Now we are ready to prove the following theorem.

Theorem 2.2. *Let R be a positively graded Noetherian ring with local base ring $(R_0, \mathfrak{m}_0)$. Assume that M and N are two finitely generated graded R-modules such that pd$(M) < \infty$. Let n be a given integer. Then the following hold:*

(1) $[H^i_{\mathfrak{p}_0R_{\mathfrak{p}_0}+(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})]_j = 0$ *for all $j > n$, all $\mathfrak{p}_0 \in$ Spec(R_0) and all $0 \le i \le s(\mathfrak{p}_0, M, N)$ if and only if $[H^i_{\mathfrak{a}}(M,N)]_j = 0$ for all graded ideal $\mathfrak{a} \supseteq R_+$, all $j > n$ and all $0 \le i \le s(M,N)$.*

(2) *If $[H^{s(\mathfrak{p}_0,M,N)}_{(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})]_j = 0$ for all $j > n$ and all $\mathfrak{p}_0 \in$ Spec(R_0), then $[H^{s(M,N)}_{\mathfrak{a}}(M,N)]_j = 0$ for all graded ideal $\mathfrak{a} \supseteq R_+$ and for all $j > n$.*

Proof. (1) ($\Rightarrow$) Let $j > n$ and $\mathfrak{a} \supseteq R_+$ be a graded ideal of R. We use the induction argument on $\dim(R/\mathfrak{a})$ to show that $[H^i_{\mathfrak{a}}(M,N)]_j = 0$ for all $0 \le i \le s(M,N)$. If $\dim(R/\mathfrak{a}) = 0$, then we have $\sqrt{\mathfrak{a}} = \mathfrak{m}_0 + R_+$ the graded maximal ideal of R. Thus by Remark 2.1 B) we have $H^i_{\mathfrak{a}}(M,N) \cong H^i_{\mathfrak{m}_0+R_+}(M,N)$ and hence

$$[H^i_{\mathfrak{a}}(M,N)]_j \cong [H^i_{\mathfrak{m}_0+R_+}(M,N)]_j,$$

for all $j \in \mathbb{Z}$. Now by Remark 2.1 E) we have

$$([H^i_{\mathfrak{m}_0+R_+}(M,N)]_j)_{\mathfrak{m}_0} \cong [H^i_{\mathfrak{m}_0 R_{\mathfrak{m}_0}+R_+R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0},N_{\mathfrak{m}_0})]_j,$$

so that using our assumption together with Remark 2.1 A) we deduce the result (note that the right hand side module is zero for $0 \leq i \leq s(\mathfrak{m}_0, M_{\mathfrak{m}_0}, N_{\mathfrak{m}_0})$ by our assumption, and is zero for $i > s(\mathfrak{m}_0, M_{\mathfrak{m}_0}, N_{\mathfrak{m}_0})$ by Remark 2.1 A)).

Now assume that $\dim(R/\mathfrak{a}) \geq 1$. Then $\exists r_0 \in R_0$ such that $\dim(R/(\mathfrak{a}, r_0)) < \dim(R/\mathfrak{a})$. By Remark 2.1 C) for each $j \in \mathbb{Z}$ there exists the long exact sequence

$$\begin{aligned}\cdots \to [H^{i-1}_{\mathfrak{a}R_{r_0}}(M_{r_0},N_{r_0})]_j \to [H^i_{(\mathfrak{a},r_0)}(M,N)]_j \to [H^i_{\mathfrak{a}}(M,N)]_j \\ \to [H^i_{\mathfrak{a}R_{r_0}}(M_{r_0},N_{r_0})]_j \to \cdots \qquad (*)\end{aligned}$$

of R_0-modules.

Let $\mathfrak{p}_{0_{r_0}} \in \operatorname{Spec}(R_{0_{r_0}})$. Then $r_0 \notin \mathfrak{p}_0$, $\dim(R_{\mathfrak{p}_0}/\mathfrak{a}R_{\mathfrak{p}_0}) < \dim(R/\mathfrak{a})$ and we have

$$\begin{aligned}([H^i_{\mathfrak{a}R_{r_0}}(M_{r_0},N_{r_0})]_j)_{\mathfrak{p}_{0_{r_0}}} &\cong [H^i_{(\mathfrak{a}R_{r_0})_{\mathfrak{p}_{0_{r_0}}}}((M_{r_0})_{\mathfrak{p}_{0_{r_0}}},(N_{r_0})_{\mathfrak{p}_{0_{r_0}}})]_j \\ &\cong [H^i_{\mathfrak{a}R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0},N_{\mathfrak{p}_0})]_j. \qquad (**)\end{aligned}$$

But replacing the ring R by $R' := R_{\mathfrak{p}_0}$, the ideal $\mathfrak{a}$ by $\mathfrak{a}' = \mathfrak{a}R'$ and using our assumption, one sees that for all $j > n$, all $\mathfrak{q}'_0 \in \operatorname{Spec}(R'_0)$ and all $0 \leq i \leq s(\mathfrak{q}'_0, M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})$

$$[H^i_{\mathfrak{q}'_0 R'_{\mathfrak{q}'_0}+(R'_{\mathfrak{q}'_0})_+}((M_{\mathfrak{p}_0})_{\mathfrak{q}'_0},(N_{\mathfrak{p}_0})_{\mathfrak{q}'_0})]_j = 0.$$

Therefore by our induction on the dimension of $R/\mathfrak{a}$,

$$[H^i_{\mathfrak{a}'}(M_{\mathfrak{p}_0},N_{\mathfrak{p}_0})]_j = 0$$

for all $j > n$ and all $0 \leq i \leq s(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})$. Hence using Remark 2.1 A) we get

$$[H^i_{\mathfrak{a}'}(M_{\mathfrak{p}_0},N_{\mathfrak{p}_0})]_j = 0$$

for all $j > n$ and all $0 \leq i \leq s(M,N)$. Thus by $(**)$, we have

$$[H^i_{\mathfrak{a}'}(M_{r_0},N_{r_0})]_j = 0$$

for all $0 \leq i \leq s(M,N)$ as $R_{0_{r_0}}$-module.
On the other hand, by our induction hypothesis we also have

$$[H^i_{(\mathfrak{a},r_0)}(M,N)]_j = 0$$

for all $0 \leq i \leq s(M,N)$. Now the result follows by the exact sequence $(*)$.

($\Leftarrow$) Since by Remark 2.1 A), $H^i_{R_+}(M,N) = 0$ for all $i > s(M,N)$, we may assume that $[H^i_{R_+}(M,N)]_j = 0$ for all $i \geq 0$ and all $j > n$. Let $\mathfrak{p}_0 \in \operatorname{Spec}(R_0)$. We prove that $[H^i_{\mathfrak{p}_0 R_{\mathfrak{p}_0}+(R_{\mathfrak{p}_0})_+}(M_{\mathfrak{p}_0},N_{\mathfrak{p}_0})]_j = 0$ for all $i \geq 0$ and all $j > n$.

We do this by induction argument on $\dim(R_0)$. If $\dim(R_0) = 0$, then evidently $\mathfrak{m}_0$ is the only prime ideal of R_0 and $\dim(R_{\mathfrak{m}_0}) = 0$. Thus for $i > 0$, we have $H^i_{\mathfrak{m}_0 R_{\mathfrak{m}_0} + R_+ R_{\mathfrak{m}_0}}(M_{\mathfrak{m}_0}, N_{\mathfrak{m}_0}) = 0$ by Remark 2.1 D), while for $i = 0$, $[H^i_{\mathfrak{m}_0 + R_+}(M,N)]_j \subseteq [H^i_{R_+}(M,N)]_j = 0$.

So, assume that $\dim(R_0) > 0$. Localizing all at $\mathfrak{p}_0$ and using our induction hypothesis, we may assume that $\mathfrak{p}_0 = \mathfrak{m}_0$. Put

$$\begin{aligned} \mathcal{A} = \{\mathfrak{u} \mid \mathfrak{u} \ &\text{is a graded ideal of } R \text{ such that } R_+ \subseteq \mathfrak{u} \text{ and that} \\ &[H^i_{\mathfrak{u}}(M,N)]_j = 0 \quad \forall i \geq 0, j > n\}. \end{aligned}$$

We claim that $\mathfrak{m}_0 + R_+$ is the only maximal element of $\mathcal{A}$. To see this, let $\mathfrak{a}$ be a maximal element of $\mathcal{A}$ such that $\mathfrak{a} \subset \mathfrak{m}_0 + R_+$. Then, $\exists r_0 \in \mathfrak{m}_0 \setminus \mathfrak{a}, i \geq 0$ and $j > n$ such that $[H^i_{(\mathfrak{a}, r_0)}(M,N)]_j \neq 0$. On the other hand let $\mathfrak{q}_{0_{r_0}} \in \operatorname{Spec}(R_{0_{r_0}})$. Then $r_0 \notin \mathfrak{q}_0$ and we have

$$\begin{aligned} ([H^{i-1}_{\mathfrak{a}R_{r_0}}(M_{r_0}, N_{r_0})]_j)_{\mathfrak{q}_{0_{r_0}}} &\cong [H^{i-1}_{\mathfrak{a}R_{\mathfrak{q}_0}}(M_{\mathfrak{q}_0}, N_{\mathfrak{q}_0})]_j \\ &\cong ([H^{i-1}_{\mathfrak{a}}(M,N)]_j)_{\mathfrak{q}_0} = 0 \end{aligned}$$

for all $j > n$ by the choice of $\mathfrak{a}$. This gives that

$$[H^{i-1}_{\mathfrak{a}R_{r_0}}(M_{r_0}, N_{r_0})]_j = 0$$

for all $j > n$ and so from the long exact sequence $(*)$ we have

$$[H^i_{(\mathfrak{a}, r_0)}(M,N)]_j = 0$$

for all $i \geq 0$ and all $j > n$, which is a contradiction. The claim now follows.

(2) We again proceed by induction on $\dim(R_0)$. If $\dim(R_0) = 0$, then $\dim(R_{\mathfrak{m}_0}) = 0$, $\operatorname{pd}(M_{\mathfrak{m}_0}) = 0 = \dim(N_{\mathfrak{m}_0})$ and the result follows by Remark 2.1 E) and D). So, Assume that $\dim(R_0) > 0$ and the result has been proved over all positively graded Noetherian rings having local base ring of smaller dimension. Let $\mathfrak{a}$ be a graded ideal of R such that $R_+ \subseteq \mathfrak{a}$. Consider the set

$$\begin{aligned} \mathcal{B} = \{\mathfrak{b} \mid \mathfrak{b} \ &\text{is a graded ideal of } R \text{ such that } R_+ \subseteq \mathfrak{b} \subseteq \mathfrak{a} \\ &\text{and that } H^{s(M,N)}_{\mathfrak{b}}(M,N) = 0 \text{ for all } j > n\}. \end{aligned}$$

Let $\mathfrak{A}$ be a maximal element of $\mathcal{B}$. We show that $\mathfrak{A} = \mathfrak{a}$. If this is not the case, then $\exists r_0 \in \mathfrak{a} \setminus \mathfrak{A}$ such that $\mathfrak{A} \subsetneq (\mathfrak{A}, r_0)$ and that

$$[H^{s(M,N)}_{(\mathfrak{A}, r_0)}(M,N)]_j \neq 0$$

for some $j > n$. Now using the exact sequence $(*)$, we obtain the exact sequence

$$\cdots \to [H^{s(M,N)-1}_{\mathfrak{A}R_{r_0}}(M_{r_0}, N_{r_0})]_j \to [H^{s(M,N)}_{(\mathfrak{A}, r_0)}(M,N)]_j \to [H^{s(M,N)}_{\mathfrak{A}}(M,N)]_j \to \cdots$$

of R_0-modules.

Let $\mathfrak{p}_{0r_0} \in \mathrm{Spec}(R_{0r_0})$. Then evidently $r_0 \notin \mathfrak{p}_0$, $\dim(R_0)_{\mathfrak{p}_0} < \dim(R_0)$ and $s(\mathfrak{p}_0, M, N) \leq s(M, N) - 1$ (note that the assumption on r_0 gives that $r_0 \in \mathfrak{m}_0$). So, if $s(\mathfrak{p}_0, M, N) < s(M, N) - 1$, then by Remark 2.1 A), we have $H^{s(M,N)-1}_{\mathfrak{A}R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0}) = 0$.
Otherwise

$$[H^{s(M,N)-1}_{\mathfrak{A}R_{\mathfrak{p}_0}}(M_{\mathfrak{p}_0}, N_{\mathfrak{p}_0})]_j = 0$$

by induction on $\dim(R_0)$, which by Remark 2.1 E) means that for all $j > n$,

$$([H^{s(M,N)-1}_{\mathfrak{A}R_{r_0}}(M_{r_0}, N_{r_0})]_j)_{\mathfrak{p}_{0r_0}} = 0.$$

Thus $[H^{s(M,N)-1}_{\mathfrak{A}R_{r_0}}(M_{r_0}, N_{r_0})]_j = 0$. Hence, from the exact sequence $(*)$ we obtain $[H^{s(M,N)}_{(\mathfrak{A},r_0)}(M, N)]_j = 0$, which is a contradiction. This completes the proof. ■

Now the proof of Theorem 1.1 is immediate by the definition of $\mathrm{reg}_{\mathfrak{a}}(M, N)$ and Theorem 2.2.

Theorem 2.3. *Let R be a positively graded Noetherian ring with a local base ring and N be a finitely generated graded R-module. Let n be an integer such that $[H^i_{\mathfrak{p}_0 R_{\mathfrak{p}_0}+(R_{\mathfrak{p}_0})_+}(N_{\mathfrak{p}_0})]_j = 0$ for all $j > n$, all $\mathfrak{p}_0 \in Spec(R_0)$ and all $0 \leq i \leq dim_{R_{\mathfrak{p}_0}}(N_{\mathfrak{p}_0})$. Then for each homogeneous ideal $\mathfrak{a} \supseteq R_+$,*

$$reg_{\mathfrak{a}}(N) < dim_R(N) + n + 1.$$

Proof. In Theorem 1.1 consider $M = R$. ■

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