On the Regularity of Generalized Local Cohomology

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Received October 13, 2011
Revised April 3, 2012

Abstract. Let \( R = \bigoplus_{j \geq 0} R_j \) be a positively graded Noetherian ring with local base ring \((R_0, m_0)\) and the irrelevant ideal \( R_+ = \bigoplus_{j \geq 1} R_j \). Let \( M, N \) be two finitely generated graded \( R \)-modules with \( \text{pd}(M) < \infty \). The vanishing of homogeneous components of generalized local cohomology modules with support in different graded ideals in a certain graded ring will be compared and using the result a bound for the regularity of the pair \((M, N)\) relative to the graded ideal \( a \supseteq R_+ \) will be obtained.

2000 Mathematics Subject Classification. 13D45, 13E10.

Key words. Generalized local cohomology, regularity.

1. Introduction

Let \( R = \bigoplus_{j \in \mathbb{Z}} R_j \) be a graded Noetherian ring, \( a \) a graded ideal of \( R \) and \( M, N \) be two finitely generated graded \( R \)-modules. Let \( H^i_a(M, N) \) be the \( i \)-th generalized local cohomology of \( M \) and \( N \) with support in \( a \) [7]. Then, as it has been shown in [9], the \( R \)-module \( H^i_a(M, N) = \bigoplus_{j \in \mathbb{Z}} [H^i_a(M, N)]_j \) has a natural graded structure in such a way that the connected sequence of functors \((H^i_a(M, -))[i \geq 0]\) from the category of finitely generated \( R \)-modules to itself has the \( \ast \)restriction property in the sense of [2, Chapter 12]. Furthermore, whenever \( R = \bigoplus_{j=0}^{\infty} R_j \) is positively graded, the \( R_0 \)-module \([H^i_a(M, N)]_j \) is finitely generated for all \( j \) and is zero for all \( j \gg 0 \).

Now, assume that \( \text{pd}(M) \) (the projective dimension of \( M \)) is finite. Then, by [8, 2.5], \( H^i_a(M, N) = 0 \) for all \( i > \text{pd}(M) + \text{ara}(a) \), where \( \text{ara}(a) \) is the arithmetic
rank of the ideal \(a\), i.e., the least number of elements of \(R\) required to generate an ideal of \(R\) with the same radical as \(a\). In the light of the above properties of the generalized local cohomology, in [3], the authors defined the generalized regularity of the pair \((M, N)\) of finitely generated graded \(R = \oplus_{j \geq 0} R_j\)-modules as

\[
\text{reg}(M, N) = \max\{\text{end}(H^i_{R+}(M, N)) + i \mid 0 \leq i \leq \text{pd}(M) + \text{ara}(R_+)\},
\]

and obtained interesting concerning results, whereas for a graded \(R\)-module \(X\), we denote by \(\text{end}(X)\) the supremum of \(j\)-th such that \(X_j \neq 0\), with the convention that \(\text{end}(0) = -\infty\). In the following, by abuse of terminology, for each graded ideal \(a\) containing \(R_+\), we shall define the regularity of the pair \((M, N)\) relative to \(a\) as

\[
\text{reg}_a(M, N) = \max\{\text{end}(H^i_a(M, N)) + i \mid 0 \leq i \leq \text{pd}(M) + \text{ara}(a)\}.
\]

Assume that \(R\) is a positively graded (Noetherian) ring with local base ring \((R_0, m_0)\) and \(M, N\) be two finitely generated graded \(R\)-modules with \(\text{pd}(M) < \infty\). We put \(s(M, N) = \text{pd}_R(M) + \dim_R(N)\), and for each prime ideal \(p_0\) of \(R_0\), \(s(p_0, M, N) = \text{pd}_{R_{p_0}}(M_{p_0}) + \dim_{R_{p_0}}(N_{p_0})\). The aim of this paper is to prove the following theorem.

**Theorem 1.1.** Let \(R\) be a positively graded Noetherian ring with a local base ring and let \(M, N\) be two finitely generated graded \(R\)-modules with pd\((M) < \infty\). Let \(n\) be a given integer such that \([H^i_{p_0R_0+(R_0)_+}(M_{p_0}, N_{p_0})]_j = 0\) for all \(j > n\), all \(p_0 \in \text{Spec}(R_0)\) and all \(0 \leq i \leq s(p_0, M, N)\). Then for each graded ideal \(a \supseteq R_+\),

\[
\text{reg}_a(M, N) < s(M, N) + n + 1.
\]

**2. Proof of Theorem 1.1**

For ease in access, we first quote some known observations about generalized local cohomology modules.

In the rest \(R\) is a commutative Noetherian ring and \(M, N\) are \(R\)-modules.

**Remark 2.1.** A) ([1, Lemma 5.1]) Assume that \(a\) is an ideal of \(R\), \(M\) a finitely generated \(R\)-module of finite projective dimension \(\text{pd}(M)\), and \(N\) an \(R\)-module of finite Krull dimension \(\dim(N)\). Then \(H^i_a(M, N) = 0\) for all \(i > \text{pd}(M) + \dim(N)\).

B) ([5, Lemma 2.1 (i)]) For ideal \(a\) of \(R\), \(H^i_a(M, N) \cong H^i_{R^a}(M, N)\).

C) ([4, Lemma 3.1]) Let \(a\) be an ideal of \(R\) and \(x \in R\). Let \(M, N\) be finitely generated \(R\)-modules. Then there is a natural long exact sequence

\[
\cdots \to H^i_{a+(x)}(M, N) \to H^i_a(M, N) \to H^i_{aR_x}(M, N) \to H^{i+1}_{a+(x)}(M, N) \to \cdots,
\]
of generalized local cohomology modules. Furthermore, it is easy to see that, if \( R, M, N \) and \( a \) are graded and \( x \) is a homogeneous element of \( R \), then all the maps in this exact sequence are homogeneous, so that for each \( j \in \mathbb{Z} \), there exists the long exact sequence

\[
\cdots \rightarrow [H^i_{a+(x)}(M, N)]_j \rightarrow [H^i_a(M, N)]_j \rightarrow [H^i_{aR_+}(M, N)]_j \\
\rightarrow [H^{i+1}_{a+(x)}(M, N)]_j \rightarrow \cdots ,
\]

of \( R_0 \)-modules.

D) ([6, Theorem 3.1]) Assume that \( R \) is local and \( \text{pd}(M) < \infty \). Let \( a \) be an ideal of \( R \). Then \( H^i_a(M, N) = 0 \) for all \( i > \dim(R) \).

E) If \( R' \) is another commutative Noetherian ring and \( f : R \rightarrow R' \) is a flat homomorphism, then for each ideal \( a \) of \( R \)

\[
H^i_a(M, N) \otimes_R R' \cong H^i_{aR'}(M \otimes_R R', N \otimes_R R').
\]

Thus for a multiplicatively closed subset \( S \) of \( R \),

\[
S^{-1}H^i_a(M, N) \cong H^i_{S^{-1}a}(S^{-1}M, S^{-1}N).
\]

If \( R, M, N \) and \( a \) are graded and \( S \subseteq R_0 \), then for each \( j \in \mathbb{Z} \),

\[
S^{-1}([H^i_a(M, N)]_j) \cong [H^i_{S^{-1}a}(S^{-1}M, S^{-1}N)]_j,
\]

as \( R_0 \)-modules. It follows that given \( j \in \mathbb{Z} \) and \( r_0 \in R_0 \), \( [H^i_{aR_+}(M_{r_0}, N_{r_0})]_j = 0 \) if and only if for each \( p_0r_0 \in \text{Spec}(R_{0r_0}) \), \( [H^i_{(aR_+)+p_0r_0}(M_{p_0r_0}, N_{p_0r_0})]_j = 0 \).

Now we are ready to prove the following theorem.

**Theorem 2.2.** Let \( R \) be a positively graded Noetherian ring with local base ring \((R_0, \mathfrak{m}_0)\). Assume that \( M \) and \( N \) are two finitely generated graded \( R \)-modules such that \( \text{pd}(M) < \infty \). Let \( n \) be a given integer. Then the following hold:

1. \( [H^i_{aR_{0r_0}+(p_0r_0)}(M_{p_0}, N_{p_0})]_j = 0 \) for all \( j > n \), all \( p_0 \in \text{Spec}(R_0) \) and all \( 0 \leq i \leq s(p_0, M, N) \) if and only if \( [H^i_a(M, N)]_j = 0 \) for all graded ideal \( a \supseteq R_+ \), all \( j > n \) and all \( 0 \leq i \leq s(M, N) \).

2. If \( [H^i_{aR_+(p_0N)}(M_{p_0}, N_{p_0})]_j = 0 \) for all \( j > n \) and all \( p_0 \in \text{Spec}(R_0) \), then \( [H^i_{a+(p_0N)}(M, N)]_j = 0 \) for all graded ideal \( a \supseteq R_+ \) and for all \( j > n \).

**Proof.**

1. \( (\Rightarrow) \) Let \( j > n \) and \( a \supseteq R_+ \) be a graded ideal of \( R \). We use the induction argument on \( \dim(R/a) \) to show that \( [H^i_a(M, N)]_j = 0 \) for all \( 0 \leq i \leq s(M, N) \).

If \( \dim(R/a) = 0 \), then we have \( \sqrt{a} = \mathfrak{m}_0 + R_+ \) the graded maximal ideal of \( R \). Thus by Remark 2.1 B) we have \( H^i_a(M, N) \cong H^i_{\mathfrak{m}_0+R_+}(M, N) \) and hence

\[
[H^i_a(M, N)]_j \cong [H^i_{\mathfrak{m}_0+R_+}(M, N)]_j,
\]
for all \( j \in \mathbb{Z} \). Now by Remark 2.1 E) we have

\[ ([H^i_{m_0 + R_+} (M, N)])_{m_0} \approx [H^i_{m_0 + R_+} (M_{m_0}, N_{m_0})]_{j}, \]

so that using our assumption together with Remark 2.1 A) we deduce the result (note that the right hand side module is zero for \( 0 \leq i \leq s(m_0, M_{m_0}, N_{m_0}) \) by our assumption, and is zero for \( i > s(m_0, M_{m_0}, N_{m_0}) \) by Remark 2.1 A)).

Now assume that \( \dim(R/a) \geq 1 \). Then \( \exists r_0 \in R_0 \) such that \( \dim(R/(a, r_0)) < \dim(R/a) \). By Remark 2.1 C) for each \( j \in \mathbb{Z} \) there exists the long exact sequence

\[ \cdots \rightarrow [H^{i-1}_{aR_0} (M_{r_0}, N_{r_0})]_{j} \rightarrow [H^i_{(a, r_0)} (M, N)]_{j} \rightarrow [H^i_{a} (M, N)]_{j} \]

\[ \rightarrow [H^i_{aR_0} (M_{r_0}, N_{r_0})]_{j} \rightarrow \cdots \quad (\ast) \]

of \( R_0 \)-modules.

Let \( p_0 r_0 \in \text{Spec}(R_0) \). Then \( r_0 \not\in p_0 \), \( \dim(R_{p_0}/aR_{p_0}) < \dim(R/a) \) and we have

\[ ([H^i_{aR_0} (M_{r_0}, N_{r_0})])_{p_0 r_0} \approx \tilde{[H}^i_{(aR_0)p_0r_0} ((M_{r_0})_{p_0r_0}, (N_{r_0})_{p_0r_0})]_{j} \]

\[ \approx \tilde{[H}^i_{aR_0} (M_{p_0}, N_{p_0})]_{j}. \quad (\ast\ast) \]

But replacing the ring \( R \) by \( R' := R_{p_0} \), the ideal \( a \) by \( a' = aR' \) and using our assumption, one sees that for all \( j > n \), all \( q_0' \in \text{Spec}(R_0') \) and all \( 0 \leq i \leq s(q_0', M_{p_0}, N_{p_0}) \)

\[ [H^i_{q_0' R_{q_0'}} (M_{q_0'}, N_{q_0'})] = 0. \]

Therefore by our induction on the dimension of \( R/a \),

\[ [H^i_{a} (M_{p_0}, N_{p_0})]_{j} = 0 \]

for all \( j > n \) and all \( 0 \leq i \leq s(M_{p_0}, N_{p_0}) \). Hence using Remark 2.1 A) we get

\[ [H^i_{a} (M_{p_0}, N_{p_0})]_{j} = 0 \]

for all \( j > n \) and all \( 0 \leq i \leq s(M, N) \). Thus by (\( \ast\ast \)), we have

\[ [H^i_{a} (M_{r_0}, N_{r_0})]_{j} = 0 \]

for all \( 0 \leq i \leq s(M, N) \) as \( R_{0r_0} \)-module.

On the other hand, by our induction hypothesis we also have

\[ [H^i_{(a, r_0)} (M, N)]_{j} = 0 \]

for all \( 0 \leq i \leq s(M, N) \). Now the result follows by the exact sequence (\( \ast \)).

(\( \Rightarrow \)) Since by Remark 2.1 A), \( H^i_{R_+} (M, N) = 0 \) for all \( i > s(M, N) \), we may assume that \( [H^i_{R_+} (M, N)]_{j} = 0 \) for all \( i \geq 0 \) and all \( j > n \). Let \( p_0 \in \text{Spec}(R_0) \).

We prove that \( [H^i_{p_0 R_0} (M_{p_0}, N_{p_0})]_{j} = 0 \) for all \( i \geq 0 \) and all \( j > n \).
We do this by induction argument on \( \dim(R_0) \). If \( \dim(R_0) = 0 \), then evidently \( \mathfrak{m}_0 \) is the only prime ideal of \( R_0 \) and \( \dim(R_{m_0}) = 0 \). Thus for \( i > 0 \), we have \( H^{H_{m_0}R_{m_0}+R_+R_{m_0}}(M_{m_0}, N_{m_0}) = 0 \) by Remark 2.1 D), while for \( i = 0 \), \( H^{H_{m_0}R_{m_0}+R_+}(M, N) \) is the only maximal element of \( A \). Then, put \( \mathfrak{a} \) be a graded ideal of \( R \) such that \( R_+ \subseteq \mathfrak{a} \) and that

\[
A = \{ u \mid u \text{ is a graded ideal of } R \text{ such that } R_+ \subseteq u \text{ and that } [H^{H_{a}^{i}}(M, N)]_j = 0 \forall i \geq 0, j > n \}.
\]

We claim that \( \mathfrak{m}_0 + R_+ \) is the only maximal element of \( A \). To see this, let \( \mathfrak{a} \) be a maximal element of \( A \) such that \( \mathfrak{a} \subseteq \mathfrak{m}_0 + R_+ \). Then, \( \exists r_0 \in \mathfrak{m}_0 \setminus \mathfrak{a} \), \( i \geq 0 \) and \( j > n \) such that \( [H^{H_{a}^{i}}(M, N)]_j \neq 0 \). On the other hand let \( q_{0r_0} \in \text{Spec}(R_{0r_0}) \). Then \( r_0 \notin q_0 \) and we have

\[
([H^{H_{a}^{i-1}}_{aR_0}(M_{r_0}, N_{r_0})]_{q_{0r_0}} \cong [H^{H_{a}^{i-1}}_{aR_0}(M_{q_0}, N_{q_0})]_j \\
\cong ([H^{H_{a}^{i-1}}_{a}(M, N)]_j)_{q_0} = 0
\]

for all \( j > n \) by the choice of \( \mathfrak{a} \). This gives that

\[
[H^{H_{a}^{i-1}}_{aR_0}(M_{r_0}, N_{r_0})]_j = 0
\]

for all \( j > n \) and so from the long exact sequence (\text{*}) we have

\[
[H^{H_{a}^{i}}_{a}(M, N)]_j = 0
\]

for all \( i \geq 0 \) and all \( j > n \), which is a contradiction. The claim now follows.

(2) We again proceed by induction on \( \dim(R_0) \). If \( \dim(R_0) = 0 \), then \( \dim(R_{m_0}) = 0 \), pd\( (M_{m_0}) = 0 = \dim(N_{m_0}) \) and the result follows by Remark 2.1 E) and D). So, Assume that \( \dim(R_0) > 0 \) and the result has been proved over all positively graded Noetherian rings having local base ring of smaller dimension. Let \( \mathfrak{a} \) be a graded ideal of \( R \) such that \( R_+ \subseteq \mathfrak{a} \). Consider the set

\[
\mathcal{B} = \{ \mathfrak{b} \mid \mathfrak{b} \text{ is a graded ideal of } R \text{ such that } R_+ \subseteq \mathfrak{b} \subseteq \mathfrak{a} \text{ and that } H^{H_{a}^{(M, N)}}_{\mathfrak{b}}(M, N) = 0 \forall j > n \}.
\]

Let \( \mathfrak{A} \) be a maximal element of \( \mathcal{B} \). We show that \( \mathfrak{A} = \mathfrak{a} \). If this is not the case, then \( \exists r_0 \in \mathfrak{a} \setminus \mathfrak{A} \) such that \( \mathfrak{A} \subseteq (\mathfrak{A}, r_0) \) and that

\[
[H^{H_{a}^{(M, N)}}_{(\mathfrak{A}, r_0)}(M, N)]_j \neq 0
\]

for some \( j > n \). Now using the exact sequence (\text{*}), we obtain the exact sequence

\[
\cdots \rightarrow [H^{H_{a}^{(M, N)}}_{(\mathfrak{A}, r_0)}(M_{r_0}, N_{r_0})]_j \rightarrow [H^{H_{(\mathfrak{A}, r_0)}^{(M, N)}}_{\mathfrak{A}}(M, N)]_j \rightarrow [H^{H_{a}^{(M, N)}}_{\mathfrak{A}}(M, N)]_j \rightarrow \cdots
\]
of $R_0$-modules.

Let $p_{0,r_0} \in \text{Spec}(R_{0,r_0})$. Then evidently $r_0 \notin p_0$, $\dim(R_0)_{p_0} < \dim(R_0)$ and $s(p_0, M, N) \leq s(M, N) - 1$ (note that the assumption on $r_0$ gives that $r_0 \in m_0$). So, if $s(p_0, M, N) < s(M, N) - 1$, then by Remark 2.1 A), we have $H^{s(M, N) - 1}(M_{p_0}, N_{p_0}) = 0$.

Otherwise

$$[H^{s(M, N) - 1}(M_{p_0}, N_{p_0})]_{j} = 0$$

by induction on $\dim(R_0)$, which by Remark 2.1 E) means that for all $j > n$,

$$([H^{s(M, N) - 1}(M_{r_0}, N_{r_0})]_{j})_{p_0, r_0} = 0.$$  

Thus $[H^{s(M, N) - 1}(M_{r_0}, N_{r_0})]_{j} = 0$. Hence, from the exact sequence $(\ast)$ we obtain $[H^{s(M, N)}(M, N)]_{j} = 0$, which is a contradiction. This completes the proof. \[\square\]

Now the proof of Theorem 1.1 is immediate by the definition of $\text{reg}_a(M, N)$ and Theorem 2.2.

**Theorem 2.3.** Let $R$ be a positively graded Noetherian ring with a local base ring and $N$ be a finitely generated graded $R$-module. Let $n$ be an integer such that $[H^{i}_{p_0, R_{r_0} + (R_{p_0})}, (N_{p_0})]_{j} = 0$ for all $j > n$, all $p_0 \in \text{Spec}(R_0)$ and all $0 \leq i \leq \dim_{R_{p_0}}(N_{p_0})$. Then for each homogeneous ideal $a \supseteq R_+$,

$$\text{reg}_a(N) < \dim_{R}(N) + n + 1.$$

**Proof.** In Theorem 1.1 consider $M = R$. \[\square\]

**Acknowledgments.** I would like to thank the referee for carefully reading of the paper and for useful suggestions.

**References**