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Some Inequalities for Differentiable Functions and Applications

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Abstract. Two general inequalities for functions whose modulus of derivatives at certain powers are convex or concave are established. Special cases with applications to special means and numerical integration are discussed.

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1. Introduction

In [1], by using the Hölder inequality, Dragomir and Agarwal first proved the following trapezoid type inequality.

Theorem 1.1. *Suppose $a, b \in I \subseteq \mathbf{R}$ with $a < b$ and $f : I^0 \to \mathbf{R}$ is differentiable and let $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then*

$$\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{b-a}{2(p+1)^{\frac{1}{p}}}\Big[\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2}\Big]^{\frac{p-1}{p}}. \quad (1)$$

In [3], Pearce and Pečarić not only provide an improvement of the inequality (1) in Theorem 1.1 and consider midpoint type inequality, but also develop analogous results based on concavity. The following results were established.

Theorem 1.2. *Suppose $a, b \in I \subseteq \mathbf{R}$ with $a < b$ and $f : I^0 \to \mathbf{R}$ is differentiable. If the function $|f'|^q$ $(q \geq 1)$ is convex on $[a, b]$, then*

$$\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq \frac{b-a}{4}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{\frac{1}{q}} \tag{2}$$

and

$$\Big|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq \frac{b-a}{4}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{\frac{1}{q}}. \tag{3}$$

Theorem 1.3. *Suppose $a, b \in I \subseteq \mathbf{R}$ with $a < b$ and $f : I^0 \to \mathbf{R}$ is differentiable. If the function $|f'|^q$ $(q \geq 1)$ is concave on $[a, b]$, then*

$$\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq \frac{b-a}{4}\Big|f'(\frac{a+b}{2})\Big| \tag{4}$$

and

$$\Big|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq \frac{b-a}{4}\Big|f'(\frac{a+b}{2})\Big|. \tag{5}$$

Remark 1.4. It is interesting to point out that bounds in (2) and (3) as well as (4) and (5) are the same for both trapezoid rule and midpoint rule.

In this work, we will provide two general inequalities for functions whose modulus of derivatives at certain powers are convex or concave. Special cases with applications of the averaged midpoint-trapezoid inequalities to special means and numerical integration are discussed.

2. Main results

In order to prove our main theorems, we need the following lemma (see [2]):

Lemma 2.1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L([a, b])$, then for any $\lambda \in [0, 1]$ and $x \in [a, b]$ the following equality holds:*

$$\frac{1}{b-a}\int_a^b K(x,t)f'(t)dt = (1-\lambda)f(x) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt, \tag{6}$$

where

$$K(x,t) := \begin{cases} t - (a + \lambda\frac{b-a}{2}),\ t \in [a, x], \\ t - (b - \lambda\frac{b-a}{2}),\ t \in (x, b]. \end{cases}$$

Theorem 2.2. *Let the assumptions of Theorem 1.2 hold. Then for all $\lambda \in [0, 1]$ and $x \in [a + \lambda\frac{b-a}{2}, b - \lambda\frac{b-a}{2}]$, the following inequality holds:*

$$\Big|(1-\lambda)f(x)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\Big|$$
$$\le \frac{1}{b-a}\Big[\frac{2\lambda^2-2\lambda+1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\Big]^{\frac{q-1}{q}}$$
$$\times\Big\{\frac{(3a+b-4x)(b-x)^2}{6(b-a)}|f'(a)|^q+\frac{(4x-a-3b)(x-a)^2}{6(b-a)}|f'(b)|^q$$
$$+\frac{[2(\frac{\lambda}{2})^3+2(1-\frac{\lambda}{2})^3+\frac{\lambda}{2}-(1-\lambda)](b-a)^2}{6}[|f'(a)|^q+|f'(b)|^q]\Big\}^{\frac{1}{q}}. \quad (7)$$

Proof. By (6) of Lemma 2.1 and using the Hölder inequality, we have

$$\Big|(1-\lambda)f(x)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\Big|$$
$$=\Big|\frac{1}{b-a}\int_a^b K(x,t)f'(t)dt\Big|$$
$$\le\frac{1}{b-a}\int_a^b |K(x,t)||f'(t)|\,dt$$
$$=\frac{1}{b-a}\int_a^b [|K(x,t)|^{1-\frac{1}{q}}][|K(x,t)|^{\frac{1}{q}}|f'(t)|]\,dt$$
$$\le\frac{1}{b-a}\Big(\int_a^b |K(x,t)|dt\Big)^{1-\frac{1}{q}}\Big(\int_a^b |K(x,t)||f'(t)|^q dt\Big)^{\frac{1}{q}}$$
$$=\frac{1}{b-a}\Big(\int_a^b |K(x,t)|dt\Big)^{1-\frac{1}{q}}\Big(\int_a^b |K(x,t)|\Big|f'\Big(\frac{b-t}{b-a}a+\frac{t-a}{b-a}b\Big)\Big|^q dt\Big)^{\frac{1}{q}}$$
$$\le\frac{1}{b-a}\Big(\int_a^b |K(x,t)|dt\Big)^{1-\frac{1}{q}}\Big[\int_a^b |K(x,t)|(\frac{b-t}{b-a}|f'(a)|^q+\frac{t-a}{b-a}|f'(b)|^q)dt\Big]^{\frac{1}{q}}$$
$$=\frac{1}{b-a}\Big(\int_a^b |K(x,t)|dt\Big)^{1-\frac{1}{q}}\Big[\frac{|f'(a)|^q}{b-a}\int_a^b |K(x,t)|(b-t)dt$$
$$+\frac{|f'(b)|^q}{b-a}\int_a^b |K(x,t)|(t-a)dt\Big]^{\frac{1}{q}}$$
$$=\frac{1}{b-a}\Big[\frac{2\lambda^2-2\lambda+1}{4}(b-a)^2+(x-\frac{a+b}{2})^2\Big]^{\frac{q-1}{q}}$$
$$\times\Big\{\frac{(3a+b-4x)(b-x)^2}{6(b-a)}|f'(a)|^q+\frac{(4x-a-3b)(x-a)^2}{6(b-a)}|f'(b)|^q$$
$$+\frac{(2-3\lambda+3\lambda^2)(b-a)^2}{12}[|f'(a)|^q+|f'(b)|^q]\Big\}^{\frac{1}{q}},$$

where

$$\int_a^b |K(x,t)|\,dt=\frac{2\lambda^2-2\lambda+1}{4}(b-a)^2+\Big(x-\frac{a+b}{2}\Big)^2,$$
$$\int_a^b |K(x,t)|(b-t)dt$$

$$= \int_a^x \Big|t - \Big(a + \lambda\frac{b-a}{2}\Big)\Big|(b-t)dt + \int_x^b \Big|t - \Big(b - \lambda\frac{b-a}{2}\Big)\Big|(b-t)dt$$

$$= \int_a^{a+\lambda\frac{b-a}{2}} \Big(a + \lambda\frac{b-a}{2} - t\Big)(b-t)dt + \int_{a+\lambda\frac{b-a}{2}}^x \Big(t - a - \lambda\frac{b-a}{2}\Big)(b-t)dt$$

$$+ \int_x^{b-\lambda\frac{b-a}{2}} \Big(b - \lambda\frac{b-a}{2} - t\Big)(b-t)dt + \int_{b-\lambda\frac{b-a}{2}}^b \Big(t - b + \lambda\frac{b-a}{2}\Big)(b-t)dt$$

$$= \frac{(\frac{3\lambda}{2} - 1)(b-a) - 2(x-a)}{6}(b-x)^2 + \frac{2(1-\frac{\lambda}{2})^3 + \frac{3\lambda}{2} - 1}{6}(b-a)^3$$

$$+ \frac{4(b-x) - 3\lambda(b-a)}{12}(b-x)^2 + \frac{\lambda^3}{24}(b-a)^3$$

$$= \frac{3a + b - 4x}{6}(b-x)^2 + \frac{2 - 3\lambda + 3\lambda^2}{12}(b-a)^3$$

and

$$\int_a^b |K(x,t)|(t-a)\,dt$$

$$= \int_a^x \Big|t - \Big(a + \lambda\frac{b-a}{2}\Big)\Big|(t-a)dt + \int_x^b \Big|t - \Big(b - \lambda\frac{b-a}{2}\Big)\Big|(t-a)dt$$

$$= \int_a^{a+\lambda\frac{b-a}{2}} \Big(a + \lambda\frac{b-a}{2} - t\Big)(t-a)dt + \int_{a+\lambda\frac{b-a}{2}}^x \Big(t - a - \lambda\frac{b-a}{2}\Big)(t-a)dt$$

$$+ \int_x^{b-\lambda\frac{b-a}{2}} \Big(b - \lambda\frac{b-a}{2} - t\Big)(t-a)dt + \int_{b-\lambda\frac{b-a}{2}}^b \Big(t - b + \lambda\frac{b-a}{2}\Big)(t-a)\,dt$$

$$= \frac{4(x-a) - 3\lambda(b-a)}{12}(x-a)^2 + \frac{\lambda^3}{24}(b-a)^3$$

$$+ \frac{(\frac{3\lambda}{2} - 1)(b-a) - 2(b-x)}{6}(x-a)^2 + \frac{2(1-\frac{\lambda}{2})^3 + \frac{3\lambda}{2} - 1}{6}(b-a)^3$$

$$= \frac{4x - a - 3b}{6}(x-a)^2 + \frac{2 - 3\lambda + 3\lambda^2}{12}(b-a)^3.$$

The proof is thus complete. ■

Corollary 2.3. *Let the assumptions of Theorem 1.2 hold. Then for any $\lambda \in [0,1]$ we have*

$$\Big|(1-\lambda)f\big(\frac{a+b}{2}\big) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\Big|$$
$$\le \frac{(2\lambda^2 - 2\lambda + 1)(b-a)}{4}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{\frac{1}{q}}. \tag{8}$$

Proof. We set $x = \frac{a+b}{2}$ in (7) to get (8). ■

Remark 2.4. If we take $\lambda = 1$ and $\lambda = 0$ in (8), we get a trapezoid type inequality

$$\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{b-a}{4}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{\frac{1}{q}}$$

and a midpoint type inequality

$$\Big|f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{b-a}{4}\Big[\frac{|f'(a)|^q + |f'(b|^q}{2}\Big]^{\frac{1}{q}}$$

which recaptures inequalities (2) and (3).

If we take $\lambda = \frac{1}{3}$ in (8), we get a Simpson type inequality

$$\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big] - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{5(b-a)}{36}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{\frac{1}{q}},$$

and if we take $\lambda = \frac{1}{2}$ in (8), we get an averaged midpoint-trapezoid type inequality as

$$\Big|\frac{1}{4}\Big[f(a) + 2f(\frac{a+b}{2}) + f(b)\Big] - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{b-a}{8}\Big[\frac{|f'(a)|^q + |f'(b)|^q}{2}\Big]^{\frac{1}{q}}. \tag{9}$$

Remark 2.5. If we take $\lambda = 0$ in (7), we have

$$\Big|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{1}{b-a}\Big[\frac{(b-a)^2}{4} + \Big(x - \frac{a+b}{2}\Big)^2\Big]^{\frac{q-1}{q}}$$
$$\times\Big\{\frac{(3a+b-4x)(b-x)^2}{6(b-a)}|f'(a)|^q + \frac{(4x-a-3b)(x-a)^2}{6(b-a)}|f'(b)|^q$$
$$+\frac{(b-a)^2}{6}[|f'(a)|^q + |f'(b)|^q]\Big\}^{\frac{1}{q}}$$

for $q \ge 1$.

Remark 2.6. It is interesting to notice that the smallest bound for inequality (7) is obtained at $x = \frac{a+b}{2}$ and $\lambda = \frac{1}{2}$. Thus the averaged midpoint-trapezoid inequality (9) is optimal in the current situation.

We now derive a comparable result to Theorem 2.2 with concavity property instead of convexity.

Theorem 2.7. *Let the assumptions of Theorem 1.3 hold. Then for all* $\lambda \in [0,1]$ *and* $x \in [a + \lambda\frac{b-a}{2}, b - \lambda\frac{b-a}{2}]$*, the following inequality holds:*

$$
\begin{aligned}
&|(1-\lambda)f(x)+\lambda\tfrac{f(a)+f(b)}{2}-\tfrac{1}{b-a}\int_a^b f(t)\,dt|\\
&\le [\tfrac{2\lambda^2-2\lambda+1}{4}(b-a)+\tfrac{1}{b-a}(x-\tfrac{a+b}{2})^2]\times\\
&\quad\Big|f'\Big(\tfrac{3\lambda^2(a+b)(b-a)^2+(x-a-\lambda\frac{b-a}{2})^2(8x+4a+2\lambda(b-a))+(b-\lambda\frac{b-a}{2}-x)^2(8x+4b-2\lambda(b-a))}{6(2\lambda^2-2\lambda+1)(b-a)^2+24(x-\frac{a+b}{2})^2}\Big)\Big|. \qquad (10)
\end{aligned}
$$

Proof. By Lemma 2.1

$$\Big|(1-\lambda)f(x)+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\Big| \le \frac{1}{b-a}\int_a^b |K(x,t)||f'(t)|\,dt$$

and notice that

$$
\begin{aligned}
|f'(\lambda x+(1-\lambda)y)|^q &\ge \lambda|f'(x)|^q+(1-\lambda)|f'(y)|^q\\
&\ge (\lambda|f'(x)|+(1-\lambda)|f'(y)|)^q,
\end{aligned}
$$

by the concavity of $|f'|^q$ and the power-mean inequality. Hence

$$|f'(\lambda x+(1-\lambda)y)| \ge \lambda|f'(x)|+(1-\lambda)|f'(y)|,$$

so $|f'|$ is also concave.

Accordingly to the Jensen integral inequality we have

$$\int_a^b |K(x,t)||f'(t)|\,dt \le \int_a^b |K(x,t)|\,dt\Big|f'\Big(\frac{\int_a^b |K(x,t)|t\,dt}{\int_a^b |K(x,t)|\,dt}\Big)\Big|,$$

where

$$\int_a^b |K(x,t)|\,dt = \frac{2\lambda^2-2\lambda+1}{4}(b-a)^2+\Big(x-\frac{a+b}{2}\Big)^2$$

and

$$
\begin{aligned}
&\int_a^b |K(x,t)|t\,dt\\
&= \int_a^{a+\lambda\frac{b-a}{2}}\Big(a+\lambda\frac{b-a}{2}-t\Big)tdt+\int_{a+\lambda\frac{b-a}{2}}^{x}\Big(t-a-\lambda\frac{b-a}{2}\Big)tdt\\
&\quad+\int_x^{b-\lambda\frac{b-a}{2}}\Big(b-\lambda\frac{b-a}{2}-t\Big)tdt+\int_{b-\lambda\frac{b-a}{2}}^{b}\Big(t-b+\lambda\frac{b-a}{2}\Big)tdt\\
&= \frac{\lambda^2}{8}(b-a)^2a+\frac{\lambda^3}{48}(b-a)^3+\Big(x-a-\lambda\frac{b-a}{2}\Big)^2\frac{4x+2a+\lambda(b-a)}{12}\\
&\quad+\frac{\lambda^2}{8}(b-a)^2b-\frac{\lambda^3}{48}(b-a)^3+\Big(b-x-\lambda\frac{b-a}{2}\Big)^2\frac{4x+2b-\lambda(b-a)}{12}\\
&= \frac{\lambda^2}{8}(b-a)^2(a+b)+\frac{1}{12}\Big[\Big(x-a-\lambda\frac{b-a}{2}\Big)^2(4x+2a+\lambda(b-a))
\end{aligned}
$$

$$+\Big(b-x-\lambda\frac{b-a}{2}\Big)^2(4x+2b-\lambda(b-a))\Big].$$

The proof is complete. ■

Corollary 2.8. *Let the assumptions of Theorem 1.3 hold. Then for any* $\lambda \in [0,1]$ *we have*

$$\Big|(1-\lambda)f(\frac{a+b}{2})+\lambda\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\Big| \\ \leq \frac{(2\lambda^2-2\lambda+1)(b-a)}{4}\Big|f'(\frac{a+b}{2})\Big|. \tag{11}$$

Proof. We set $x=\frac{a+b}{2}$ in (10) to get (11). ■

Remark 2.9. If we take $\lambda = 1$ and $\lambda = 0$ in (11), we get a trapezoid type inequality

$$\Big|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\Big| \leq \frac{b-a}{4}\Big|f'(\frac{a+b}{2})\Big|$$

and a midpoint type inequality

$$\Big|f(\frac{a+b}{2})-\frac{1}{b-a}\int_a^b f(t)\,dt\Big| \leq \frac{b-a}{4}\Big|f'(\frac{a+b}{2})\Big|$$

which recaptures inequalities (4) and (5).

If we take $\lambda=\frac{1}{3}$ in (11), we get a Simpson type inequality

$$\Big|\frac{1}{6}\Big[f(a)+4f(\frac{a+b}{2})+f(b)\Big]-\frac{1}{b-a}\int_a^b f(t)\,dt\Big| \leq \frac{5(b-a)}{36}\Big|f'(\frac{a+b}{2})\Big|,$$

and if we take $\lambda=\frac{1}{2}$ in (11), we get an averaged midpoint-trapezoid type inequality as

$$\Big|\frac{1}{4}\Big[f(a)+2f(\frac{a+b}{2})+f(b)\Big]-\frac{1}{b-a}\int_a^b f(t)\,dt\Big| \leq \frac{b-a}{8}\Big|f'(\frac{a+b}{2})\Big|. \tag{12}$$

Remark 2.10. It should be noticed that the smallest bound for (11) is obtained at $\lambda=\frac{1}{2}$. Thus the averaged midpoint-trapezoid inequality (12) is optimal in the current situation.

Remark 2.11. If we take $\lambda=0$ in (10), we have

$$\Big|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\Big| \\ \leq \Big[\frac{b-a}{4}+\frac{1}{b-a}\Big(x-\frac{a+b}{2}\Big)^2\Big]\Big|f'\Big(\frac{(4x+2a)(x-a)^2+(4x+2b)(b-x)^2}{3(b-a)^2+12(x-\frac{a+b}{2})^2}\Big)\Big|$$

for $q \geq 1$.

3. Applications to special means

We now consider the applications of the above mentioned inequalities (9) and (12) to the following special means for positive real numbers $\alpha, \beta, \alpha \neq \beta$:

(i) The arithmetic mean:

$$A(\alpha, \beta) := \frac{\alpha + \beta}{2}.$$

(ii) The harmonic mean:

$$H(\alpha, \beta) := \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

(iii) The logarithmic mean:

$$L(\alpha, \beta) := \frac{\beta - \alpha}{\ln \beta - \ln \alpha}.$$

(iv) The identric mean:

$$I(\alpha, \beta) := \frac{1}{e}\Big(\frac{\beta^\beta}{\alpha^\alpha}\Big)^{\frac{1}{\beta - \alpha}}.$$

(v) The generalized logarithmic mean:

$$L_p(a, b) := \Big[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\Big]^{\frac{1}{p}}, \quad p \neq -1, 0.$$

Using the inequalities (9) and (12), three new inequalities are derived for the above means.

Proposition 3.1. *Let* $0 < a < b$ *and* $0 < r < 1, r \geq 2$, $r \in \mathbf{R}$. *Then for all* $q \geq 1$, *we have*

$$\Big|\frac{1}{2}A(a^r, b^r) + \frac{1}{2}A^r(a, b) - L_r^r(a, b)\Big| \leq \frac{r(b-a)}{8}[A(a^{(r-1)q}, b^{(r-1)q})]^{\frac{1}{q}}.$$

Proof. The assertion follows from applying the inequality (9) to the function $f(t) = t^r, t \in [a, b]$, which implies that $|f'(t)|^q = r^q t^{(r-1)q}, t \in [a, b]$ is a convex function. ■

Proposition 3.2. *Let* $0 < a < b$. *Then for all* $q \geq 1$, *we have*

$$\Big|\frac{1}{2}A(\ln a, \ln b) + \frac{1}{2}\ln A(a, b) - \ln I(a, b)\Big| \leq \frac{(b-a)}{8}H^{-\frac{1}{q}}(a^q, b^q).$$

Proof. The assertion follows from applying the inequality (9) to the function $f(t) = \ln t, t \in [a,b]$, which implies that $|f'(t)|^q = t^{-q}, t \in [a,b]$ is a convex function. ■

Proposition 3.3. *Let* $0 < a < b$ *and* $1 < r < 2$, $r \in \mathbf{R}$. *Then for all* $q \geq 1$, *we have*

$$\Big|\frac{1}{2}A(a^r, b^r) + \frac{1}{2}A^r(a,b) - L_r^r(a,b)\Big| \leq \frac{r(b-a)}{8}\Big(\frac{a+b}{2}\Big)^{r-1}.$$

Proof. The assertion follows from applying the inequality (12) to the function $f(t) = t^r, t \in [a,b]$, which implies that $|f'(t)| = rt^{r-1}, t \in [a,b]$ is a concave function. ■

4. Applications in numerical integration

Let d be a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of the interval $[a,b]$ and consider the averaged midpoint-trapezoid quadrature formula

$$\int_a^b f(x)\,dx = T(f,d) + E(f,d),$$

where

$$T(f,d) = \sum_{i=0}^{n-1} \frac{x_{i+1} - x_i}{4}\Big[f(x_i) + 2f\Big(\frac{x_i + x_{i+1}}{2}\Big) + f(x_{i+1})\Big].$$

Proposition 4.1. *Let the assumptions of Theorem 2.2 hold. Then for every partition* d *of* $[a,b]$, *we have*

$$\begin{aligned} |E(f,d)| &\leq \frac{1}{8}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^2\Big[\frac{|f(x_i)|^q + |f'(x_{i+1})|^q}{2}\Big]^{\frac{1}{q}} \\ &\leq \frac{1}{8}\max\{|f'(a)|, |f'(b)|\}\sum_{i=0}^{n-1}(x_{i+1} - x_i)^2. \end{aligned}$$

Proof. From (9) in Corollary 2.3 we obtain

$$\begin{aligned} &\Big|\frac{x_{i+1} - x_i}{4}\Big[f(x_i) + 2f\big(\frac{x_i + x_{i+1}}{2}\big) + f(x_{i+1})\Big] - \int_{x_i}^{x_{i+1}} f(t)\,dt\Big| \\ &\leq \frac{(x_{i+1} - x_i)^2}{8}\Big[\frac{|f'(x_i)|^q + |f'(x_{i+1})|^q}{2}\Big]^{\frac{1}{q}}. \end{aligned}$$

By summing over i from 0 to $n-1$ and taking into account that $|f'|^q$ is convex, using the generalized triangle inequality, we get

$$|T(f,d) - \int_a^b f(t)\,dt|$$

$$= \Big|\sum_{i=0}^{n-1}\Big\{\frac{x_{i+1}-x_i}{4}\Big[f(x_i)+2f\big(\frac{x_i+x_{i+1}}{2}\big)+f(x_{i+1})\Big]-\int_{x_i}^{x_{i+1}} f(t)\,dt\Big\}\Big|$$
$$\le \sum_{i=0}^{n-1}\Big|\frac{x_{i+1}-x_i}{4}\Big[f(x_i)+2f\big(\frac{x_i+x_{i+1}}{2}\big)+f(x_{i+1})\Big]-\int_{x_i}^{x_{i+1}} f(t)\,dt\Big|$$
$$\le \frac{1}{8}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\Big[\frac{|f'(x_i)|^q+|f'(x_{i+1})|^q}{2}\Big]^{\frac{1}{q}}$$
$$\le \frac{1}{8}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\max\{|f'(x_i)|,|f'(x_{i+1})|\}$$
$$\le \frac{1}{8}\max\{|f'(a)|,|f'(b)|\}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2.$$

■

Similarly using Theorem 2.7 we can prove the following.

Proposition 4.2. *Let the assumptions of Theorem 2.7 hold. Then for every partition d of $[a,b]$, we have*

$$|E(f,d)| \le \frac{1}{8}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\Big|f'\big(\frac{x_{i+1}+x_i}{2}\big)\Big|.$$

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