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Viscosity Approximation Method for Lipschitzian Pseudocontraction Semigroups in Banach Spaces

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Abstract. Let E be a real Banach space which admits a weakly sequentially continuous duality mapping from E to E^*, and K be a nonempty closed convex subset of E. Let $\{T(t) : t \geq 0\}$ be a Lipschitzian pseudocontractive semigroup on K such that $F := \bigcap_{t\geq 0} \mathrm{Fix}(T(t)) \neq \emptyset$, and $f : K \to K$ be a fixed contractive mapping. When $\{\alpha_n\}, \{t_n\}$ satisfy some appropriate conditions, the iterative process given by

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n \quad \text{for } n \in \mathbb{N},$$

converges strongly to $p \in F$, which is the unique solution in F to the following variational inequality:

$$\langle (f-I)p, j(x-p)\rangle \leq 0 \quad \forall x \in F.$$

Our results presented in this paper extend and improve recent results of R. Chen and H. He [1], Y. Song and R. Chen [8], Xu [13].

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Key words. Lipschitzian pseudocontraction semigroup, demiclosed principle, common fixed point, Opial's condition, implicit iteration process.

1. Introduction

Let E be a real Banach space and let J denote the normalized duality mapping from E into 2^{E^*} given by $J(x) := \{f \in E^*, \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}$ $\forall x \in E$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In the following, we shall denote the single-valued duality mapping by j, and denote $F(T) = \{x \in E; Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x, x_n \overset{*}{\rightharpoonup} x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x.

Recall that the norm of a Banach spaces E is said to be Gâteaux differentiable (or E is said to be smooth), if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for each x, y on the unit sphere $S(E)$ of E. Moreover, if for each $y \in S(E)$ the limit defined by (1) is uniformly attained for x in $S(E)$, we say that the norm E is uniformly Gâteaux differentiable. The norm of E is said to be Fréchet differentiable, if for each $x \in S(E)$, the limit (1) is attained uniformly for $y \in S(E)$. The norm of E is said to be uniformly Fréchet differentiable (and E is said to be uniformly smooth), if the limit (1) is attained uniformly for $(x, y) \in S(E) \times S(E)$.

The following results are well known:

i) The duality mapping J in a smooth Banach space E is single valued and strong-weak* continuous [11, Lemma 4.3.3].

ii) In a uniformly smooth Banach E, the mapping $J : E \to E^*$ is single valued and norm to norm uniformly continuous on bounded sets of E [11, Theorem 4.3.6].

iii) If E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition, and E is smooth; for the details, see [4].

Definition 1.1. (Zhang [17]) A one-parameter family $\{T(t) : t \geq 0\}$ of mappings from K into itself is said to be a pseudo-contraction semigroup on K, if the following conditions are satisfied:

i) $T(0)x = x$ for each $x \in K$;

ii) $T(t + s)x = T(t)T(s)x$ for any $t, s \in \mathbb{R}_+$ and $x \in K$;

iii) for each $x \in E$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into K is continuous;

iv) for any $x, y \in C$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle T(t)x - T(t)y, j(x - y)\rangle \leq \|x - y\|^2 \quad \text{for each } t > 0.$$

A pseudocontraction semigroup $\{T(t) : t \geq 0\}$ is said to be Lipschitzian [5], [16], if the conditions (1)–(4) and the following condition (5) are satisfied:

v) There exists a bounded measurable function $L : (0, \infty) \to [0, \infty)$ such that, for any $x, y \in K$,

$$\|T(t)x - T(t)y\| \leq L(t)\|x - y\| \quad \text{for each } t > 0.$$

In the sequel, we denote

$$M := \sup_{t\geq 0} L(t) < \infty \quad \text{and} \quad F := \bigcap_{t\geq 0} \operatorname{Fix}(T(t)).$$

In [6], Shioji and Takahashi introduce in a Hilbert space the implicit iteration

$$x_n = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{t_n\}$ a sequence of positive real numbers divergent to ∞, and for each $n \geq 0$ and $x \in C$. Under certain restrictions on the sequence $\{\alpha_n\}$, Shioji and Takahashi [6] prove the strong convergence of $\{x_n\}$ to a member of F (see also [15]). In [7], Shimizu and Takahashi studied the strong convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in \mathbb{N}$$

in a Hilbert space H, where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of H, $t_n \geq 0$, and $t_n \to \infty$.

In [2] Chen and Yunyan Song studied the strong convergence of the following sequences (2) and (3) for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ with $F \neq \emptyset$ in a Banach space:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in \mathbb{N}, \tag{2}$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \quad n \in \mathbb{N}. \tag{3}$$

In 2002, Suzuki [10] was the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1 \tag{4}$$

for the nonexpansive semigroup case. In 2005, Xu [14] estabilshed a Banach space version of sequence (4) of Suzuki [10].

Recently, in [1] R. Chen and H. He studied the vicosity approximation process for a nonexpansive semigroup and proved another strong convergence theorem for a nonexpansive semigroup in a Banach space, which is defined by

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n \quad \forall n \in \mathbb{N}, \tag{5}$$

$$y_{n+1} = \beta_n f(x_n) + (1-\beta_n)T(t_n)y_n \quad \forall n \in \mathbb{N}.$$

For a continuous pseudocontractive mapping $T: K \to K$ and a fixed contractive mapping $f: K \to K$, the mapping $T_f = tf + (1-t)T$ obviously is a continuous strongly pseudocontractive mapping from K to K. Therefore, T_f has a unique fixed point in K [3, Corollary 2], i.e., for any given $t \in (0,1)$ there exists $x_t \in K$ such that $x_t = f(x_t) + (1-t)T(x_t)$. This implies that the viscosity iterative process also applies to continuous pesudocontractive mappings.

In this paper, motivated by the above results, we study the vicosity approximation process for a Lipschitzian pseudocontractive semigroup and prove another strong convergence theorem for a continuous pseudocontractive mapping in a Banach space which is defined by (5).

Recall that $S: K \to K$ is called accretive if $I - S$ is pseudocontractive. We denote by J_r the resolvent of S, i.e., $J_r = (I + rS)^{-1}$. It is well known that J_r is nonexpansive, single valued and $F(J_r) = S^{-1}(0) := \{z \in D(S); 0 = S(z)\}$ for all $r > 0$. For more details, see [11].

Let K be a nonempty closed convex subset of a real Banach space E and $T: K \to K$ be a pseudocontractive map, then $I - T$ is accretive. We denote $A = J_1 = (2I - T)^{-1}$, then $F(A) = F(T)$, $A: R(2I - T) \to K$ is nonexpansive and single valued.

In the sequel, we will need the following definition and results.

Definition 1.2. A Banach space E is said to satisfy Opial's condition if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, as $n \to \infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| \quad \forall y \in E, y \neq x.$$

It is well known that every Hilbert space and $l^p(1 < p < \infty)$ space satisfy Opial's condition.

Lemma 1.3. [12, Lemma 1] *Let $\{t_n\}$ be a real sequence and τ be a real number such that $\liminf_{n\to\infty} t_n \leq \tau \leq \limsup_{n\to\infty} t_n$. Suppose that either of the following holds*

i) $\limsup_{n\to\infty}(t_{n+1} - t_n) \leq 0$, *or*
ii) $\liminf_{n\to\infty}(t_{n+1} - t_n) \geq 0$.

Then τ is a cluster point of $\{t_n\}$. Moreover, for $\epsilon > 0, k, m \in \mathbb{N}$, there exists $m_0 \geq m$ such that $|t_j - \tau| < \epsilon$ for every integer j with $m_0 \leq j \leq m_0 + k$.

Lemma 1.4. [9, Lemma 1.1] *Let K be a nonempty closed convex subset of a real Banach space E and $T: K \to K$ be a continuous pseudocontractive map. We denote $A = (2I - T)^{-1}$. Then*

i) *The map A is a nonexpansive self-mapping on K, i.e., for all $x, y \in K$ we have $\|Ax - Ay\| \leq \|x - y\|$ and $Ax \in K$.*

ii) *If* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, *then* $\lim_{n\to\infty} \|x_n - Ax_n\| = 0$.

Lemma 1.5. (Zhou [18]) *Let E be a real reflexive Banach space with the Opial condition. Let C be a nonempty closed convex subset of E and $T : C \to C$ be a continuous pseudocontractive mapping. Then T is demiclosed at zero, i.e., for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup y$ and $\|(I-T)x_n\| \to 0$, then $(I-T)y = 0$.*

2. Main results

Theorem 2.1. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*, suppose K is a nonempty closed convex subset of E. Let $\{T(t) : t \geq 0\}$ be a Lipschitzian pseudocontractive semigroup on K such that $F := \bigcap_{t\geq 0} \mathrm{Fix}(T(t)) \neq \emptyset$, and $f : K \to K$ be a fixed contractive mapping with contractive coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1, t_n > 0$, $\lim_{n\to\infty} \alpha_n = 0, \liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Define a sequence $\{x_n\}$ in K by*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n. \tag{6}$$

Suppose that for any bounded subset $C \subset K$,

$$\lim_{s\to 0} \sup_{x\in C} \|T(s)x - x\| = 0. \tag{7}$$

Then $\{x_n\}$ converges strongly to p, as $n \to \infty$, where p is an element of F which is the unique solution in F to the following variational inequality

$$\langle (f-I)p, J(x-p)\rangle \leq 0 \quad \text{for all } x \in F. \tag{8}$$

Proof. First, we show the uniqueness of solutions of the variational inequality (8). In fact, supposing $p, q \in F$ satisfy (8), we get

$$\langle (f-I)p, J(q-p)\rangle \leq 0,$$
$$\langle (f-I)q, J(p-q)\rangle \leq 0.$$

From the above inequalities we have

$$(1-\alpha)\|p-q\|^2 \leq \langle (I-f)p - (I-f)q, J(p-q)\rangle \leq 0.$$

We must have $p = q$ and the uniqueness is proved. Below we use $p \in F$ to denote the unique solution of (8). Now we show that $\{x_n\}$ is bounded. In fact, for any fixed $x \in F$, it follows from Eq. (6) that

$$\begin{aligned}\|x_n - x\|^2 &= \alpha_n\langle f(x_n) - x, J(x_n - x)\rangle + (1-\alpha_n)\langle T(t_n)x_n - x, J(x_n - x)\rangle \\ &\leq \alpha_n\langle f(x_n) - f(x), J(x_n - x)\rangle + \alpha_n\langle f(x) - x, J(x_n - x)\rangle\end{aligned}$$

$$+ (1-\alpha_n)\|x_n - x\|^2$$
$$\leq [1-(1-\alpha)\alpha_n]\|x_n - x\|^2 + \alpha_n\langle f(x) - x, J(x_n - x)\rangle.$$

Therefore

$$\|x_n - x\|^2 \leq \frac{1}{1-\alpha}\langle f(x) - x, J(x_n - x)\rangle \leq \frac{1}{1-\alpha}\|f(x) - x\|\|x_n - x\|. \quad (9)$$

Thus

$$\|x_n - x\| \leq \frac{1}{1-\alpha}\|f(x) - x\|.$$

We claim that $\{x_n\}$ is relatively sequentially compact. Indeed, we have

$$\|x_n - T(t_n)x_n\| = \alpha_n\|T(t_n)x_n - f(x_n)\| \to 0 \quad \text{as} \quad n \to \infty. \quad (10)$$

We choose a sequence $\{t_{n_j}\}$ of positive real numbers such that

$$t_{n_j} \to 0, \quad \frac{1}{t_{n_j}}\|x_{n_j} - T(t_{n_j})x_{n_j}\| \to 0. \quad (11)$$

We now show that how such a special subsequence can be constructed. Fix $\delta > 0$ such that

$$\liminf_{n\to\infty} t_n = 0 < \delta < \limsup_{n\to\infty} t_n.$$

From (10), there exists $m_1 \in \mathbb{N}$ such that $\|T(t_n)x_n - x_n\| < \frac{1}{3^2}$ for all $n \geq m_1$. By Lemma 1.3, $\frac{\delta}{2}$ is a cluster point of $\{t_n\}$. In particular, there exists $n_1 > m_1$ such that $\frac{\delta}{3} < t_{n_1} < \delta$. Next, we choose $m_2 > n_1$ such that $\|T(t_n)x_n - x_n\| < \frac{1}{4^2}$ for all $n \geq m_2$. Again, by Lemma 1.3, $\frac{\delta}{3}$ is a cluster point of $\{t_n\}$ and this implies that there exists $n_2 > m_2$ such that $\frac{\delta}{4} < t_{n_2} < \frac{\delta}{2}$. Continuing in this way, we obtain a subsequence $\{n_j\}$ of n satisfying

$$\|T(t_{n_j})x_{n_j} - x_{n_j}\| < \frac{1}{(j+2)^2}, \quad \frac{\delta}{j+2} < t_{n_j} < \frac{\delta}{j} \quad \text{for all } j \in \mathbb{N}.$$

Consequently, (11) is satisfied.

Since $\{x_n\}$ is bounded, without loss of generality we may assume that the subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converges weakly to some $q \in K$. Now, we prove that $q = T(t)q$ for any fixed $t > 0$. Indeed,

$$
\begin{aligned}
\|x_{n_j} - T(t)x_{n_j}\| &\le \sum_{k=0}^{\left[\frac{t}{t_{n_j}}\right]-1} \|T((k+1)t_{n_j})x_{n_j} - T(kt_{n_j})x_{n_j}\| \\
&\quad + \left\| T\left(\left[\frac{t}{t_{n_j}} \right] t_{n_j} \right) x_{n_j} - T(t)x_{n_j} \right\| \\
&\le \left[\frac{t}{t_{n_j}} \right] M \|T(t_{n_j})x_{n_j} - x_{n_j}\| + M \left\| T\left(t - \left[\frac{t}{t_{n_j}} \right] t_{n_j} \right) x_{n_j} - x_{n_j} \right\| \\
&\le Mt \frac{\|T(t_{n_j})x_{n_j} - x_{n_j}\|}{t_{n_j}} + M \max_{0 \le s \le t_{n_j}} \{\|T(s)x_{n_j} - x_{n_j}\|\}
\end{aligned}
$$

for all $j \in \mathbb{N}$, $[t]$ denotes the integral part of t. From (11) and the continuity of the mapping $t \mapsto T(t)x$, $x \in K$, we get

$$\lim_{j\to\infty} \|x_{n_j} - T(t)x_{n_j}\| = 0.$$

By Lemma 1.5, then $T(t)q = q$, therefore $q \in F$. In inequality (9), we have

$$\|x_{n_j} - q\|^2 \le \frac{1}{1-\alpha} \langle f(q) - q, J(x_{n_j} - q) \rangle.$$

Since the duality map J is single-valued and weakly sequentially continuous from E to E^*, we get

$$\lim_{j\to\infty} \|x_{n_j} - q\|^2 \le \frac{1}{1-\alpha} \lim_{j\to\infty} \langle f(q) - q, J(x_{n_j} - q) \rangle = 0,$$

i.e., $x_{n_j} \to q$ as $j \to \infty$. Hence, $\{x_n\}$ is relatively sequentially compact. Next we show that q is a solution in F to the variational inequality (8). In fact, for any $x \in F$, from Eq. (6) we obtain

$$
\begin{aligned}
\|x_n - x\|^2 &= \alpha_n \langle f(x_n) - x, J(x_n - x) \rangle + (1 - \alpha_n) \langle T(t_n)x_n - x, J(x_n - x) \rangle \\
&\le \alpha_n \langle f(x_n) - x_n, J(x_n - x) \rangle + \alpha_n \|x_n - x\|^2 + (1 - \alpha_n) \|x_n - x\|^2 \\
&= \alpha_n \langle f(x_n) - x_n, J(x_n - x) \rangle + \|x_n - x\|^2.
\end{aligned}
$$

Therefore

$$\langle f(x_n) - x_n, J(x - x_n) \rangle \le 0.$$

Since the sets $\{x_n - x\}$ and $\{x_n - f(x_n)\}$ are bounded and the duality mapping J is single-valued and weakly sequentially continuous from E to E^*, for any fixed $x \in F$, we have

$$\langle f(q) - q, J(x - q) \rangle = \lim_{j\to\infty} \langle f(x_{n_j}) - x_{n_j}, J(x - x_{n_j}) \rangle \le 0.$$

This $q \in F$ is a solution of variational inequality (8), hence $q = p$ by the uniqueness. In summary, we have proved that $\{x_n\}$ is relatively sequentially compact and each cluster point of $\{x_n\}$ (as $n \to \infty$) equals p. Therefore $x_n \to p$ as $n \to \infty$. The proof is complete. ∎

Theorem 2.2. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J from E to E^*. Suppose K is a nonempty closed convex subset of E. Let $\{T(t) : t \geq 0\}$ be a Lipschitzian pseudocontractive semigroup on K such that $F = \bigcap_{t\geq 0} \mathrm{Fix}(T(t)) \neq \emptyset$, and $f : K \to K$ be a fixed contractive mapping with the contractive coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1, t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Define a sequence $\{x_n\}$ in K by*

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n.$$

Suppose that for any bounded subset $C \subset K$,

$$\lim_{s\to 0} \sup_{x\in C} \|T(s)x - x\| = 0.$$

Then $\{x_n\}$ converges strongly to p as $n \to \infty$, where p is an element of F which is the unique solution in F to the following variational inequality

$$\langle (f-I)p, J(x-p)\rangle \leq 0 \quad \text{for all } x \in F.$$

Proof. We have proved in Theorem 2.1 that $\{x_n\}$ is bounded, so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$.

Now we show that for $t > 0$, $\|T(t)x_n - x_n\| \to 0$ as $n \to \infty$. In fact, we have

$$\begin{aligned}
\|x_n - T(t_n)x_n\| \leq & \sum_{k=0}^{[\frac{t}{t_n}]-1} \|T((k+1)t_n)x_n - T(kt_n)x_n\| \\
& + \left\| T\left(\left[\frac{t}{t_n}\right] t_n\right) x_n - T(t)x_n \right\| \\
\leq & M \left[\frac{t}{t_n}\right] \|T(t_n)x_n - x_n\| + M \left\| T\left(t - \left[\frac{t}{t_n}\right] t_n\right) x_n - x_n \right\| \\
\leq & Mt\frac{\alpha_n}{t_n} \|f(x_n) - T(t_n)x_n\| \\
& + M \max\{\|T(s)x_n - x_n\| : 0 \leq s \leq t_n\}.
\end{aligned}$$

From $\lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$ and the continuity of the mapping $t \mapsto T(t)x$, $x \in K$, it follows that the conclusion is proved. Since the remainder of the proof is the same as the proof of Theorem 2.1, we omit it. This completes the proof. ∎

Corollary 2.3. *Let E be a real reflexive Banach space which satisfies Opial's condition with a uniformly Gâteaux differentiable norm, and K be a nonempty closed convex subset of E. Let $\{T(t) : t \geq 0\}$ be a Lipschitzian pseudocontractive* *semigroup on K such that $F := \bigcap_{t\geq 0} \mathrm{Fix}(T(t)) \neq \emptyset$ and satisfying (7). Let $f : K \to K$ be a fixed contractive mapping with contractive coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1, t_n > 0$, $\lim_{n\to\infty} \alpha_n = 0, \liminf_{n\to\infty} t_n = 0$, $\limsup_{n\to\infty} t_n > 0$, $\lim_{n\to\infty}(t_{n+1} - t_n) = 0$. Define a sequence $\{x_n\}$ in K by Eq. (6). Then $\{x_n\}$ converges strongly to a point p of F which is the unique solution in F to the variational inequality (8).*

Corollary 2.4. *Let E be a real reflexive Banach space which satisfies Opial's condition with a uniformly Gâteaux differentiable norm, and K be a nonempty closed convex subset of E. Let $\{T(t) : t \geq 0\}$ be a Lipschitzian pseudocontractive semigroup on K such that $F := \bigcap_{t\geq 0} \mathrm{Fix}(T(t)) \neq \emptyset$ and satisfying (7). Let $f : K \to K$ be a fixed contractive mapping with contractive coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1, t_n > 0$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Define a sequence $\{x_n\}$ in K by Eq. (6). Then $\{x_n\}$ converges strongly to a fixed point p of F which is the unique solution in F to the variational inequality (8).*

The following theorem was proved by Xu:

Theorem 2.5. [13, Theorem 4.1] *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E, and $T : K \to K$ a nonexpansive mapping with a fixed point and $f : K \to K$ a fixed contractive mapping. If there exists a bounded sequence $\{x_n\}$ such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then*

$$\limsup_{n\to\infty} \langle f(p) - p, j(x_n - p)\rangle \leq 0,$$

where p is the unique solution in $Fix(T)$ to the variational inequality

$$\langle (f - I)p, j(x - p)\rangle \quad \textit{for all } x \in \mathrm{Fix}(T). \tag{12}$$

Theorem 2.6. *Let K be a nonempty closed convex subset of a uniformly smooth Banach space E, and $T : K \to K$ be a continuous pseudocontractive mapping with a fixed point and $f : K \to K$ be a fixed contractive mapping with contractive coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ be a sequence of real numbers satisfying $0 < \alpha_n < 1$, $\lim_{n\to\infty} \alpha_n = 0$. Define a sequence $\{x_n\}$ in K by*

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n.$$

Then $\{x_n\}$ converges strongly to a fixed point p of $\mathrm{Fix}(T)$ which is the unique solution in $\mathrm{Fix}(T)$ to the variational inequality (12).

Proof. As in the proof of Theorem 2.1, we can conclude that

i) The sets $\{x_n\}, \{Tx_n\}, \{f(x_n)\}$ are bounded;

ii) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Denoting $A = (2I - T)^{-1}$, from Lemma 1.4 we get that $\mathrm{Fix}(T) = \mathrm{Fix}(A)$, and A is a nonexpansive self-mapping on K and

$$\lim_{n\to\infty} \|x_n - Ax_n\| = 0.$$

By Theorem 2.5, for the nonexpansive self-mapping A, we obtain

$$\lim_{n\to\infty} \sup\langle f(p) - p, J(x_n - p)\rangle \le 0,$$

where $p \in \mathrm{Fix}(A) = \mathrm{Fix}(T)$ is a unique solution to the variational inequality (12) in $\mathrm{Fix}(A) = \mathrm{Fix}(T)$. Finally, we show that $x_n \to p$ $(n \to \infty)$. Indeed

$$\begin{aligned}\|x_n - p\|^2 &= \alpha_n\langle f(x_n) - p, J(x_n - p)\rangle + (1 - \alpha_n)\langle Tx_n - p, J(x_n - p)\rangle \\ &\le \alpha_n\langle f(x_n) - f(p), J(x_n - p)\rangle + \alpha_n\langle f(p) - p, J(x_n - p)\rangle \\ &\quad + (1 - \alpha_n)\|x_n - p\|^2.\end{aligned}$$

Thus

$$\|x_n - p\|^2 \le \frac{1}{1-\alpha}\langle f(p) - p, J(x_n - p)\rangle.$$

Therefore $\limsup_{n\to\infty} \|x_n - p\|^2 = 0$. So, the conclusion is proved. ■

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