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To Dual-Space Theory of Set-Valued Optimization

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. The primary goal of this paper is to review and further develop the dual-space approach to multiobjective optimization, focusing mainly on problems with set-valued objectives. This approach is based on employing advanced tools of variational analysis and generalized differentiation defined in duals to Banach spaces. Developing this approach, we present new and updated results on existence of Pareto-type optimal solutions, necessary optimality and suboptimality conditions, and also sufficient conditions for global optimality that have never been considered in the literature in such a generality.

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1. Introduction

This paper is devoted to developing a dual-space approach to multiobjective optimization. Considering general problems with set-valued objectives, we essentially improve some recent results on the existence of Pareto-type minimizers, necessary and sufficient conditions for optimality by using advanced tools of generalized differentiation. The basic unconstrained and constrained problems of our study in this paper are written, respectively, as follows:

$$\text{(P)} \qquad \text{minimize} \quad F(x) \quad \text{subject to} \quad x \in X,$$

$$\text{(CP)} \qquad \text{minimize} \quad F(x) \quad \text{subject to} \quad x \in \Omega,$$

where $F : X \rightrightarrows Z$ is a set-valued mapping between Banach spaces, Ω is a closed and generally nonconvex subset of X, and "minimization" is defined via some ordering cone in the sense of *Pareto efficiency*. To be precise, let Z be a linear space "partially" ordered by a proper, closed, and convex cone Θ with $\Theta \setminus (-\Theta) \neq \emptyset$. Denote this order by " $\leq_\Theta$ ". Its ordering (generalized Pareto) relation is described by

$$z_1 \leq_\Theta z_2 \ \text{ if and only if } \ z_2 - z_1 \in \Theta \ \text{ for all } \ z_1, z_2 \in Z. \tag{1}$$

We consider the following set of *Pareto minimal points* of Ξ with respect to Θ:

$$\min(\Xi;\Theta) := \big\{\bar z \in \Xi \mid \Xi \cap (\bar z - \Theta) = \{\bar z\}\big\}. \tag{2}$$

This Pareto minimal point set is different from the usual set given by

$$\text{ParetoMin}(\Xi;\Theta) := \{\bar z \in \Xi \mid \Xi \cap (\bar z - \Theta) \subset \bar z + \Theta\},$$

but they coincide if Θ is pointed, i.e., $\Theta \cap (-\Theta) = \{\mathbf{0}\}$.

Note that the majority of publications on vector/multiobjective optimization concerns the *weak* ordering relation "$<_\Theta$" defined by replacing the cone Θ in (1) by its interior, and the *weak* Pareto minimal points defined by replacing the optimization condition in (2) by its weak counterpart $\Xi \cap (\bar z - \text{int}\,\Theta) = \emptyset$ under the additional nonempty interiority assumption:

$$\text{int}\,\Theta \neq \emptyset. \tag{3}$$

Denote the set of all the weak Pareto points of Ξ with respect to Θ by

$$\text{WMin}\,(\Xi,\Theta) := \{\bar z \in \Xi \mid \Xi \cap (\bar z - \text{int}\,\Theta) = \emptyset\}.$$

The most significant characteristic of the weak notion is the possibility to study its optimality conditions via numerous powerful *scalarization techniques* mainly relied on condition (3). Since they convert a vector optimization problem to an equivalent scalar one in the sense of having the same solution sets, known results in scalar optimization can be applied to the scalar problem. However, the nonempty interiority condition is a very serious restriction in infinite-dimensional spaces, since the natural ordering cones in, e.g., the classical Lebesgue spaces l^p and L^p for $1 \le p < \infty$ have empty interiors.

To challenge the absence of the interiority condition, a *variational dual-space approach* has been developed in [24, Chapter 5]; see also the references therein. In contrast to any scalarization technique, this approach deals with multiobjective optimization problems directly, and it relies on advanced tools of variational analysis and generalized differentiation. This approach is mainly based on the *extremal principle*, which has been well-recognized as a variational counterpart of the separation theorem in convex analysis for nonconvex sets. It has been further developed to set-valued optimization in [4]–[10], where it was used to study not only necessary conditions for Pareto-type minimizers but also conditions ensuring the existence of minimizers and also sufficient conditions for global optimality.

In this paper, we revisit some of the major results established in [4]–[10] and developed their new, much simplified proofs based on the *fuzzy intersection rule* from [23, Lemma 3.1]. Furthermore, we obtain new results in these directions based on the aforementioned fuzzy sum rule, which is in turn proved by using the extremal principle.

The rest of this paper is organized as follow. Section 2 contains some basic definitions and preliminaries from variational analysis and generalized differentiation, as well as new definitions of subdifferentials for set-valued mappings and relationships between coderivatives and subdifferentials of mappings. In Section 3 we review basic concepts in vector(-valued) and set-valued optimization and present a version of the Ekeland variational principle for set-valued mappings used in what follows. Section 4 is devoted to deriving three independent (but closely related) extensions of the Fermat (stationary) rule to set-valued mappings with no explicit constraints; namely, the fuzzy Fermat rule, the (exact) Fermat rule for Pareto minimizers, and the fuzzy suboptimality Fermat rule for Pareto ε-optimal solutions. In Section 5 we obtain, by employing variational principles and the fuzzy Fermat rule from Section 4, general results on the existence of Pareto-type optimal solutions for unconstrained problems and describe, in particular, rather broad settings ensuring the validity of the subdifferential Palais-Smale condition, which plays a major role in the existence results for optimal solutions.

The final two sections deal with constrained problems of set-valued optimization. Section 6 is devoted to deducing Lagrange-type multiplier rules from the necessary optimality conditions obtained in Section 3 by using appropriate results of generalized differential calculus for the constructions involved. In Section 7 we establish new sufficient conditions for global optimality in constrained multiobjective optimization.

2. Tools of variational analysis

Throughout the paper we use the conventional notation of variational analysis and generalized differentiation; see the books [23, 25, 26]. For a Banach space X, denote its norm by $\|\cdot\|$, its dual space equipped with the weak* topology w^* by X^*, and the canonical pairing between X and X^* by $\langle\cdot,\cdot\rangle$. The unit ball of a space under consideration is denoted by $\mathbb{B}$ and the vector zero is $\mathbf{0}$. Given a set Ω in X, the expressions $\mathrm{cl}\,\Omega$, $\mathrm{bd}\,\Omega$, and $\mathrm{int}\,\Omega$ stand for the standard notions of closure, boundary, and interior of Ω. We use the notation $x \xrightarrow{\Omega} \bar{x}$ to indicate that $x \to \bar{x}$ with $x \in \Omega$. Given a set-valued mapping $F : X \rightrightarrows Z$, the domain, the range, and the graph of F are given, respectively, by

$$\mathrm{dom}\,F := \big\{x \in X \mid F(x) \neq \emptyset\big\},$$

$$F(X) := \big\{z \in Z \mid \exists\, x \in X,\ z \in F(x)\big\} = \bigcup \big\{F(x) \mid x \in \mathrm{dom}\,F\big\},$$

$$\mathrm{gph}\,F := \big\{(x,z) \in X \times Z \mid z \in F(x)\big\}.$$

The inverse of F denoted by $F^{-1} : Z \rightrightarrows X$ is defined by

$$F^{-1}(z) := \{x \in X \mid z \in F(x)\}.$$

The product space $X \times Z$ is Banach with respect to the sum norm $\|(x,z)\| := \|x\| + \|z\|$ unless otherwise stated. The dual product space $X^* \times Z^*$ is equipped in this case with the norm $\|(x^*,z^*)\| := \max\{\|x^*\|,\|z^*\|\}$. Given a set-valued mapping $F\colon X \rightrightarrows X^*$ between the space X and its topological dual X^*, the symbol

$$\mathop{\mathrm{Lim\,sup}}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big|\ \exists\,\text{sequences}\ \ x_k \to \bar{x}\ \ \text{and}\ \ x_k^* \xrightarrow{w^*} x^*\ \ \text{with}\ \ x_k^* \in F(x_k)\ \ \text{for all}\ \ k \in \mathbb{N}\Big\}$$

with $\mathbb{N} := \{1,2,\ldots\}$ signifies the *sequential Painlevé-Kuratowski outer limit* of F at $\bar{x}$ in the norm topology of X and weak* topology of X^*.

Since the main results of this paper require the Asplund property of the spaces in question, all the primal spaces under consideration are *assumed to be Asplund* unless otherwise stated.

Definition 2.1 (Asplund spaces). A Banach space X is *Asplund* if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U.

There are a variety of characterizations of Asplund spaces; see, e.g., [23] and the references therein. One of the most striking and useful ones is: X is Asplund if and only if any of its separable subspaces has a separable dual. Note that the class of Asplund spaces contains all reflexive Banach spaces and all Banach spaces with separable duals; in particular, c_0 and ℓ^p, $L^p[0,1]$ for $1 < p < \infty$ are Asplund spaces while ℓ_1 and ℓ_∞ are not.

Now we define the basic generalized differential constructions in Asplund spaces that enjoy comprehensive calculus rules and numerous applications; see [23, 24]. Their useful modifications in general Banach spaces can be found in [23, Chapter 1].

Definition 2.2 (Normal cones to sets). Let Ω be a nonempty subset of X.

(i) The *regular/Fréchet normal cone* to Ω at $x \in \Omega$ is defined by

$$\widehat{N}(x;\Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le 0 \right\}; \tag{4}$$

(ii) The *limiting/Mordukhovich normal cone* to Ω at $\bar{x} \in \Omega$ is defined by

$$\begin{aligned} N(\bar{x};\Omega) &:= \operatorname*{Lim\,sup}_{x \to \bar{x}} \widehat{N}(x;\Omega) \\ &= \left\{ x^* \in X^* \;\middle|\; \exists\, x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \ \text{ with } \ x_k^* \in \widehat{N}(x_k;\Omega) \right\}. \end{aligned} \tag{5}$$

Note that both normal cones (4) and (5) reduce to the normal cone of convex analysis when the set Ω is convex. The constructions (4) and (5) are also known as the *prenormal* and *basic normal cones*, respectively; see, e.g., [23, 24].

Definition 2.3 (Coderivatives of mappings). Let $F\colon X \rightrightarrows Z$ be a set-valued mapping, and let $(\bar{x},\bar{z}) \in \operatorname{gph} F$.

(i) The *regular coderivative* of F at $(\bar{x},\bar{z})$ denoted by $\widehat{D}^*F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ is defined by

$$\widehat{D}^*F(\bar{x},\bar{z})(z^*) := \left\{ x^* \in X^* \middle| \ (x^*, -z^*) \in \widehat{N}\big((\bar{x},\bar{z}); \operatorname{gph} F\big)\right\}.$$

(ii) The *normal coderivative* of F at $(\bar{x},\bar{z})$ denoted by $D_N^*F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ is defined by

$$\begin{aligned} D_N^*F(\bar{x},\bar{z})(z^*) :=& \left\{ x^* \in X^* \middle| \ (x^*, -z^*) \in N\big((\bar{x},\bar{z}); \operatorname{gph} F\big)\right\} \\ =& \Big\{ x^* \in X^* \ \Big|\ \exists\, (x_k, z_k) \xrightarrow{\operatorname{gph} F} (\bar{x},\bar{z}),\ (x_k^*, z_k^*) \xrightarrow{w^*} (x^*, z^*) \\ & \quad \text{with } \ (x_k^*, -z_k^*) \in \widehat{N}\big((x_k, z_k); \operatorname{gph} F\big) \Big\}. \end{aligned} \tag{6}$$

(iii) The *mixed coderivative* of F at $(\bar{x},\bar{z})$ denoted by $D_M^*F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ is defined by replacing the weak* convergence $z_k^* \xrightarrow{w^*} z^*$ in (6) with the norm convergence $z_k^* \xrightarrow{\|\cdot\|} z^*$, i.e.,

$$D^*_M F(\bar{x},\bar{z})(z^*) := \Big\{x^* \in X^* \Big| \exists\,(x_k,z_k) \stackrel{\mathrm{gph}\,F}{\longrightarrow} (\bar{x},\bar{z}),\ x^*_k \stackrel{w^*}{\longrightarrow} x^*,\ z^*_k \stackrel{\|\cdot\|}{\longrightarrow} z^* \\ \text{with}\ \ (x^*_k,-z^*_k) \in \widehat{N}\big((x_k,z_k);\mathrm{gph}\,F\big)\Big\}. \tag{7}$$

We always omit $\bar{z} = f(\bar{x})$ in the coderivative notation if $F = f\colon X \to Z$ is single-valued. Note that the very concept of coderivatives was introduced in [22] independently of the normal cone to the graph of the mapping under consideration.

We obviously have from (6) and (7) that

$$D^*_M F(\bar{x},\bar{z})(z^*) \subset D^*_N F(\bar{x},\bar{z})(z^*)\ \ \text{for all}\ \ z^* \in Z^*. \tag{8}$$

The equality in (8) surely holds when the space Z is finite-dimensional. It also holds for broad classes of *strong coderivative normality* mappings with infinite-dimensional image spaces listed in [23, Proposition 4.9].

In infinite-dimensional spaces, the validity of calculus and characterizations for these generalized differential objects requires certain additional "sequential normal compactness" properties of sets and mappings, which are automatic in finite dimensions while being a crucial ingredient of variational analysis in infinite dimensions; see the books [23, 24] for a comprehensive theory and numerous applications of various properties of this type. The formulations used in the paper are as follows.

Definition 2.4 (SNC properties of sets). Let $\Omega \subset X \times Z$. Then

(i) Ω is *sequentially normally compact* (SNC) at $\bar{v} = (\bar{x},\bar{z}) \in \Omega$ if for any sequences

$$v_k \stackrel{\Omega}{\to} \bar{v},\ \ \text{and}\ \ (x^*_k,z^*_k) \in \widehat{N}(v_k;\Omega),\quad k \in \mathbb{N}, \tag{9}$$

we have the implication $(x^*_k,z^*_k) \stackrel{w^*}{\longrightarrow} \mathbf{0} \Longrightarrow (x^*_k,z^*_k) \stackrel{\|\cdot\|}{\longrightarrow} \mathbf{0}$ as $k \to \infty$;

(ii) Ω is *partially SNC* (PSNC) with respect to X at $\bar{v} \in \Omega$ if for any sequences (v_k,x^*_k,z^*_k) satisfying (9) we have the implication

$$\Big[x^*_k \stackrel{w^*}{\to} \mathbf{0},\quad z^*_k \stackrel{\|\cdot\|}{\longrightarrow} \mathbf{0}\Big] \Longrightarrow x^*_k \stackrel{\|\cdot\|}{\longrightarrow} \mathbf{0}\ \ \text{as}\ \ k \to \infty;$$

(iii) Ω is *strongly PSNC* with respect to X at $\bar{v} \in \Omega$ if for any sequences (v_k,x^*_k,z^*_k) satisfying (9) we have the implication

$$\Big[(x^*_k,z^*_k) \stackrel{w^*}{\to} \mathbf{0}\Big] \Longrightarrow x^*_k \stackrel{\|\cdot\|}{\longrightarrow} \mathbf{0}\ \ \text{as}\ \ k \to \infty.$$

Note that the product structure of the space in question plays no role in the SNC property (we can put $Z = \{\mathbf{0}\}$ without loss of generality) in contrast to its partial modifications. The SNC property is automatic in finite-dimensional spaces. The strongly PSNC property holds for any Cartesian product set $\Omega_1 \times \Omega_2$ provided that one component set is SNC. The PSNC property is satisfied under some kind of Lipschitzian property.

Since a set in a product space can be viewed as the graph of some set-valued mapping, we have the corresponding SNC properties for mappings.

Definition 2.5 (SNC properties to mappings). Let $F\colon X \rightrightarrows Z$ be a set-valued mapping and let $(\bar{x}, \bar{z}) \in \operatorname{gph} F$. We say that

(i) F is *SNC* at $(\bar{x}, \bar{z})$ if $\operatorname{gph} F$ is SNC at this point;

(ii) F is *PSNC* at $(\bar{x}, \bar{z})$ if $\operatorname{gph} F$ is PSNC at this point with respect to X;

(iii) F is *strongly PSNC* at $(\bar{x}, \bar{z})$ if $\operatorname{gph} F$ is strongly PSNC there with respect to X.

Definition 2.6 (Lipschitz-like property). We say that a set-valued mapping $F\colon X \rightrightarrows Z$ is *Lipschitz-like* (or enjoys the Aubin property) around a point $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{z}$ and a number $\ell \geq 0$ such that

$$F(x) \cap V \subset F(u) + \ell \|x - u\| \mathbb{B} \quad \text{for all} \quad x, u \in U.$$

It is known that the Lipschitz-like property of F is equivalent to the fundamental properties of metric regularity and linear openness of the inverse mapping F^{-1}. Furthermore, by [23, Theorem 4.10] a closed-graph set-valued mapping $F : X \rightrightarrows Z$ is Lipschitz-like around $(\bar{x}, \bar{z})$ if and only if it is PSNC at $(\bar{x}, \bar{z})$ and $D^*_M F(\bar{x}, \bar{z})(\mathbf{0}) = \{\mathbf{0}\}$.

Next we recall several definitions and properties of subdifferentials for set-valued mappings with values in ordered spaces. Let $F\colon X \rightrightarrows Z$, and let $\Theta \subset Z$ be an ordering cone that is proper, closed, and convex in Z. Since Θ is not assumed to be pointed, the ordering relation defined by (1) does not generally have the antisymmetry property, and thus it is not a partial order. The *epigraph* of F with respect to Θ is given by

$$\operatorname{epi} F := \big\{(x, z) \in X \times Z \big|\ z \in F(x) + \Theta\big\}$$

with $\operatorname{epi} F = \operatorname{gph} F$ if $\Theta = \{\mathbf{0}\}$ and the strict inclusion $\operatorname{gph} F \subset \operatorname{epi} F$ holding otherwise; we omit Θ in the epigraph notation $\operatorname{epi}_\Theta F$ for simplicity. Let $\mathcal{E}_F\colon X \rightrightarrows Z$ be the *epigraphical multifunction* of F with respect to Θ defined by

$$\mathcal{E}_F(x) := F(x) + \Theta.$$

We obviously have $\operatorname{gph} \mathcal{E}_F = \operatorname{epi} F$. The following subdifferential constructions for set-valued mappings were introduced in [4] as extensions of the corresponding subdifferentials for extended-real-valued functions [23].

Definition 2.7 (Subdifferentials of mappings). Let $F\colon X \rightrightarrows Z$ with an ordered space Z, and let $(\bar{x},\bar{z}) \in \operatorname{epi} F$.

(i) The *regular subdifferential* and the *basic subdifferential* of F at $(\bar{x},\bar{z})$ in the direction z^* with $\|z^*\|=1$ are defined, respectively, by

$$\widehat{\partial} F(\bar{x},\bar{z})(z^*) := \widehat{D}^* \mathcal{E}_F(\bar{x},\bar{z})(z^*) \text{ and}$$
$$\partial F(\bar{x},\bar{z})(z^*) := D^*_N \mathcal{E}_F(\bar{x},\bar{z})(z^*).$$

(ii) The *regular subdifferential* and the *basic subdifferential* of F at $(\bar{x},\bar{z})$ are defined, respectively, by

$$\widehat{\partial} F(\bar{x},\bar{z}) := \bigcup_{\|z^*\|=1} \widehat{\partial} F(\bar{x},\bar{z})(z^*) = \bigcup \Big\{ \widehat{\partial} F(\bar{x},\bar{z})(z^*) \big| -z^* \in N(\mathbf{0};\Theta), \|z^*\|=1 \Big\},$$
$$\partial F(\bar{x},\bar{z}) := \bigcup_{\|z^*\|=1} \partial F(\bar{x},\bar{z})(z^*) = \bigcup \Big\{ \partial F(\bar{x},\bar{z})(z^*) \big| -z^* \in N(\mathbf{0};\Theta), \|z^*\|=1 \Big\}.$$

(iii) The *singular subdifferential* of F at $(\bar{x},\bar{z})$ is defined by

$$\partial^\infty F(\bar{x},\bar{z}) := D^*_M \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0}).$$

As usual, we drop $\bar{z}=f(\bar{x})$ in the subdifferential notation if $F=f\colon X\to Z$ is single-valued, and we do not mention the cone Θ therein for simplicity. When $F=\varphi\colon X\to\mathbb{R}\cup\{\infty\}$ is an extended-real-valued function with the standard order $\Theta=\mathbb{R}_+$ on $\mathbb{R}$, the subdifferential constructions of Definition 2.7 reduce, respectively, to the standard regular/Fréchet subdifferential, and the Mordukhovich basic/limiting and singular subdifferentials.

It was proved in [7, Proposition 4.3] that

$$\partial F(\bar{x},\bar{z})(z^*) \neq \emptyset \quad \Longrightarrow \quad -z^* \in N(\mathbf{0};\Theta) \text{ and}$$
$$\widehat{\partial} F(\bar{x},\bar{z})(z^*) \subset \widehat{\partial} F(\bar{x},\bar{v})(z^*) \subset \widehat{D}^* F(\bar{x},\bar{v})(z^*)$$

for all $(\bar{x},\bar{z}) \in \operatorname{epi} F$, $\bar{v} \in F(\bar{x})$ with $\bar{v} \le_\Theta \bar{z}$. This verifies the equivalence of two alternative subdifferential expressions in (ii).

Based on properties of epigraphical multifunctions, we arrive at the following notions.

Definition 2.8 (SNEC and ELL properties). Let $\mathcal{E}_F$ be the epigraphical multifunction of the mapping $F\colon X \rightrightarrows Z$, and let $(\bar{x},\bar{z}) \in \operatorname{epi} F$. Then we say that

(i) F is *sequentially normally epi-compact* (SNEC) at $(\bar{x},\bar{z})$ if $\mathcal{E}_F$ is SNC at this point;

(ii) F is *partially SNEC* (PSNEC) at $(\bar{x},\bar{z})$ if $\mathcal{E}_F$ is PSNC at this point;

(iii) F is *strongly PSNEC* at $(\bar{x},\bar{z})$ if $\mathcal{E}_F$ is strongly PSNC at this point;

(iv) F is *epigraphically Lipschitz-like* (ELL) at $(\bar{x}, \bar{z})$ if $\mathcal{E}_F$ is Lipschitz-like at this point.

It is important to emphasize that the most significant advantage of the coderivative approach in defining subdifferentials for set-valued mappings is the inheritance of the calculus for coderivative systematically developed in [23, Section 3]. Other advantages include coderivative characterizations for metric regularity, linear openness, and robust Lipschitzian stability of set-valued mappings; see [23, 25].

Next we state important relationships between coderivatives and subdifferentials for the class of *order semicontinuous* set-valued mappings (see Definition 3.2 (vii) below) including continuous single-valued functions and lower semicontinuous extended-real-valued functions.

Theorem 2.9 (Relationships between subdifferentials and coderivatives of order semicontinuous mappings). *Let $F\colon X \rightrightarrows Z$, where Z is ordered by an ordering cone $\Theta \subset Z$. Assume that* $\operatorname{gph} F$ *and Θ are locally closed around $(\bar{x}, \bar{z})$ and* $\mathbf{0}$*, respectively. Suppose also that F is order semicontinuous at $(\bar{x}, \bar{z}) \in \operatorname{gph} F$. Then we have*

(i) $\partial F(\bar{x}, \bar{z})(z^*) \subset D^*_N F(\bar{x}, \bar{z})(z^*)$ *for all* $z^* \in Z^*$*;*

(ii) $\partial^\infty F(\bar{x}, \bar{z}) \subset D^*_M F(\bar{x}, \bar{z})(\mathbf{0})$*;*

(iii) *F is PSNEC at $(\bar{x}, \bar{z})$ provided that F is PSNC at this point;*

(iv) *F is ELL at $(\bar{x}, \bar{z})$ provided that F is Lipschitz-like at $(\bar{x}, \bar{z})$ and* $\operatorname{epi} F$ *is locally closed around $(\bar{x}, \bar{z})$.*

Proof. See [1, Section 3]. ■

These relationships clearly imply that the subdifferential necessary conditions supersede the corresponding coderivative ones in [4, 7, 8] and the references therein.

In the subsequent sections we need three calculus rules.

- The fuzzy intersection rule [23, Lemma 3.1].
- The basic normals to set intersections in product spaces [23, Theorem 3.4].
- The sum rule for coderivatives [23, Theorem 3.10].

The reader can find all the details in the book [23]. We do not recall them for the sake of simplicity. It is important to mention that the driving force in establishing these (in fact, all the) calculus rules is the *extremal principle*. In this paper, we need the following basic version of the extremal principle for two sets.

Definition 2.10 (Extremality property). Given Ω_1 and Ω_2 two subsets of the space X, we say that a point $\bar{x} \in \Omega_1 \cap \Omega_2$ is *local extremal* for the set system $\{\Omega_1, \Omega_2\}$ if there exist a neighborhood U of $\bar{x}$ and a sequence $\{a_k\} \subset X$ with $\|a_k\| \to 0$ as $k \to \infty$ satisfying

$$\Omega_1 \cap (\Omega_2 - a_k) \cap U = \emptyset \text{ for all } k \in \mathbb{N}.$$

In this case $\{\Omega_1, \Omega_2, \bar{x}\}$ is said to be an *extremal system* in X.

If we can choose $U = X$, we talk about the *global extremality.* Note that if $\bar{x}$ is a boundary point of a closed set Ω, then $\{\Omega, \{\bar{x}\}, \bar{x}\}$ is an extremal system. Another important example of extremal systems is given in Theorem 4.1.

Theorem 2.11 (Approximate extremal principle). *Let $\bar{x}$ be a local extremal point of the set system $\{\Omega_1, \Omega_2\}$, where Ω_1 and Ω_2 are locally closed around $\bar{x}$ in an Asplund space. Then the following approximate extremal principle holds: for every $\varepsilon > 0$ there are elements*

$$x_1 \in \Omega_1 \cap \big(\bar{x} + \varepsilon\mathbb{B}\big) \quad \textit{and} \quad x_2 \in \Omega_2 \cap \big(\bar{x} + \varepsilon\mathbb{B}\big)$$

satisfying the relationships

$$x^* \in \big(\widehat{N}(x_1; \Omega_1) + \varepsilon\mathbb{B}^*\big) \cap \big(-\widehat{N}(x_2; \Omega_2) + \varepsilon\mathbb{B}^*\big) \quad \textit{with} \quad \|x^*\| = 1.$$

Proof. See [23, Theorem 2.20]. ■

The extremal principle can be viewed as a variational counterpart of convex separation theorems in nonconvex settings. It plays a fundamental role in variational analysis similar to that of convex separation theorems in convex analysis.

3. Basic concepts of set-valued optimization

Multiobjective optimization problems typically have at least two conflicting objectives. A gain of one objective is an expense of the other in most of the cases. The concept of Pareto optimality has been widely recognized to be very useful in studying such problems. A number of variants and extensions of this fundamental concept have been introduced and studied in the literature; see, e.g., the books [15, 19, 21, 24] and the references therein.

Let Z be an ordered space, where the order "$\leq_\Theta$" is generated by a closed and convex cone Θ via (1). We do not require in general that the cone Θ is pointed, and the order $\leq_\Theta$ might not be a partial order due to the absence of the antisymmetry property.

Given a mapping $F\colon X \rightrightarrows Z$ taking values in an ordered space with the ordering cone $\Theta \subset Z$ and a set $\Omega \subset X$, we consider the unconstrained problem (P) and the constrained one (CP). The latter problem can be viewed as a unconstrained problem (P), where the cost mapping is the restriction of F over Ω denoted by $F_\Omega : X \rightrightarrows Z$ with images

$$F_\Omega(x) := \begin{cases} F(x) & \text{if } x \in \Omega, \\ \emptyset & \text{if } x \notin \Omega. \end{cases}$$

Obviously, standard vector optimization problems can be seen as a special case of set-valued optimization when all the values of the cost are singletons.

In this paper we study the following notions of exact and approximate minimizers to set-valued and vector-valued mappings.

Definition 3.1 (Pareto-type minimizers and approximate minimizers in set-valued optimization). We say that

(i) $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is a *minimizer* of (P)–or just to the mapping F–if $\bar{z} \in F(\bar{x})$ is a minimal point of the range of F, denoted by $F(X)$, i.e., $\bar{z} \in \min\big(F(X);\Theta\big)$, which is equivalent to

$$(\bar{z}-\Theta)\cap F(X)=\{\bar{z}\}; \tag{10}$$

(ii) $(\bar{x},\bar{z})$ is a *weak minimizer* of (P) if $\bar{z} \in \operatorname{WMin}\big(F(X);\Theta\big)$, i.e., (10) holds with the replacement of Θ by $\operatorname{int}\Theta \neq \emptyset$ and $\{\bar{z}\}$ by $\emptyset$;

(iii) Given $\varepsilon > 0$ and $\mathbf{e} \in \Theta \setminus (-\Theta)$ with $\|\mathbf{e}\| = 1$, the pair $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is an *approximate ε-minimizer of problem* (P) *in the direction* $\mathbf{e}$–or just an *approximate $\varepsilon\mathbf{e}$-minimizer* of (P)–if

$$z+\varepsilon\mathbf{e} \not\leq_\Theta \bar{z} \ \text{ for all } \ z \in F(x) \ \text{ with } \ x \neq \bar{x};$$

(iv) $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is a *strict approximate $\varepsilon\mathbf{e}$-minimizer* of (P) if there is $\widetilde{\varepsilon} < \varepsilon$ such that it is an approximate $\widetilde{\varepsilon}\mathbf{e}$-minimizer of (P);

(v) $(\bar{x},\bar{z})$ is a *local* minimizer, weak minimizer, approximate $\varepsilon\mathbf{e}$-minimizer, or strict approximate $\varepsilon\mathbf{e}$-minimizer of F if there is a neighborhood U of $\bar{x}$ such that it is a minimizer of F_U of the same kind, where F_U is the restriction of F over the neighborhood U;

(vi) $(\bar{x},\bar{z})$ is a (local) minimizer, weak minimizer, approximate $\varepsilon\mathbf{e}$-minimizer, or strict approximate $\varepsilon\mathbf{e}$-minimizer of F over Ω–or that of the *constrained* optimization problem (CP)–if it is a corresponding minimizer of F_Ω, the restriction of F over Ω.

When $F = f : X \to Z$ is single-valued, we omit $\bar{x} = f(\bar{x})$ in the notion of minimizers, i.e., we simply say $\bar{x}$ is a minimizer of f instead of $(\bar{x}, f(\bar{x}))$ is a minimizer of f.

Next let us recall several properties of set-valued mappings $F\colon X \rightrightarrows Z$ with respect to the ordering cone Θ of Z. As always, we do not explicitly mention Θ in the names of properties for the sake of simplicity.

Definition 3.2 (Properties of set-valued mappings). Let $F : X \rightrightarrows Z$ be a mapping and $\Theta \subset Z$ be an ordering cone.

(i) F is *epiclosed* if its epigraph with respect to Θ is closed in $X \times Z$.

(ii) F is *level-closed* if for all $z \in Z$ its z-level set

$$\text{lev}(F;z) := \{x \in X \mid \exists\, v \in F(x) \ \text{ with } \ v \leq_\Theta z\}$$

is closed in X.

(iii) F is *quasibounded from below* if there exists a bounded and closed subset $M \subset Z$ such that $F(X) \subset M + \Theta$.

(iv) F is *bounded from below* if the set M in (iii) can be chosen as a singleton.

(v) F has the *domination property* at $\bar{x} \in \text{dom}\, F$ if $F(\bar{x}) \subset \text{Min}\,(F(\bar{x});\Theta) + \Theta$, i.e.,

$$\text{for every } \ z \in F(\bar{x}) \ \text{ there is } \ v \in \text{Min}\,\big(F(\bar{x});\Theta\big) \ \text{ with } \ v \leq_\Theta z.$$

(vi) F satisfies the *limiting monotonicity condition* at $\bar{x}$ if for any sequence of pairs $\{(x_k, z_k)\} \subset \text{gph}\, F$ with $x_k \to \bar{x}$ as $k \to \infty$ one has the implication

$$[z_{k+1} \leq z_k \ \text{ for all } \ k \in \mathbb{N}] \Longrightarrow \big[\exists\, \bar{z} \in \text{Min}\,\big(F(\bar{x});\Theta\big) \ \text{ with } \ \bar{z} \leq_\Theta z_k \ \ \forall\, k \in \mathbb{N}\big].$$

(vii) F is *order semicontinuous* at $(\bar{x}, \bar{z}) \in \text{gph}\, F$ if for any sequence $\{(x_k, z_k)\} \subset \text{epi}\, F$ converging to $(\bar{x}, \bar{z})$ there is a sequence $\{(x_k, v_k)\} \subset \text{gph}\, F$ with $v_k \leq_\Theta z_k$ for all $k \in \mathbb{N}$ such that $\{v_k\}$ contains a subsequence converging to $\bar{z}$.

(viii) F satisfies the *subdifferential Palais-Smale condition* if every sequence $\{x_k\} \subset \text{dom}\, F$ satisfying the relationships

$$\text{there are } \ z_k \in F(x_k) \ \text{ and } \ x_k^* \in \widehat{\partial} F(x_k, z_k) \ \text{ with } \ \|x_k^*\| \to 0 \ \text{ as } \ k \to \infty$$

contains a convergent subsequence when $\{z_k\} \subset Z$ is quasibounded from below.

It is clear that every epiclosed mapping is level-closed but not vice versa, and that the limiting monotonicity condition implies the domination property.

We conclude this section with a recent version of the celebrated Ekeland variational principle for set-valued mappings in [8, Theorem 3.4]; cf. [4, 15–17].

Theorem 3.3 (Ekeland variational principle for set-valued mappings). *Let (X, d) be a complete metric space, and let Z be ordered via the order relation (1) by a proper, closed, and convex cone $\Theta \subset Z$ with $\Theta \setminus (-\Theta) \neq \emptyset$, i.e., Θ is not a linear subspace of Z. Consider a set-valued mapping $F\colon X \rightrightarrows Z$ and assume that F is quasibounded from below, level-closed, and has the domination property. Then for any $\varepsilon > 0$, $\lambda > 0$, $\mathbf{e} \in \Theta \setminus (-\Theta)$ with $\|\mathbf{e}\| = 1$, and $(x_0, z_0) \in \text{gph}\, F$ there is $(\bar{x}, \bar{z}) \in \text{gph}\, F$ satisfying the relationships*

$$\bar{z} - z_0 + \frac{\varepsilon}{\lambda} d(\bar{x}, x_0)\,\mathbf{e} \ \leq_\Theta \ \mathbf{0}, \quad \bar{z} \in \text{Min}\,\big(F(\bar{x});\Theta\big), \tag{11}$$

$$z - \bar{z} + \frac{\varepsilon}{\lambda} d(\bar{x}, x)\mathbf{e} \ \not\leq_\Theta \ \mathbf{0} \ \ \textit{for all} \ \ (x, z) \in \operatorname{gph} F \ \ \textit{with} \ \ (x, z) \neq (\bar{x}, \bar{z}). \quad (12)$$

If (x_0, z_0) is an approximate $\varepsilon\mathbf{e}$-minimizer of F, then $\bar{x}$ can be chosen such that in addition to (11) *and* (12) *we have*

$$d(\bar{x}, x_0) \leq \lambda.$$

Proof. See [8, Theorem 3.4]. ■

Note that if $F = \varphi : X \to \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function, then it reduces to the origin version established by Ekeland [14]. There are several statements that are equivalent to Ekeland's variational principle; see, e.g., [3, 15]. One of the earliest versions of this principle for vector-valued functions was formulated by Khanh [20] while several latest extensions to set-valued mappings can be found in [4, 8, 15–17] and the references therein.

4. Generalized Fermat rules

In this section we formulate several versions of the Fermat rule, known also as the stationary principle, for set-valued mappings. Among them the fuzzy results are new and more efficient than the existing ones in the literature, while the exact (pointbased) versions are formulated for the sake of being self-contained.

Theorem 4.1 (Generalized Fermat rules for set-valued mappings). *Let $F\colon X \rightrightarrows Z$ be a set-valued mapping, where the image space Z is ordered by a proper, closed, and convex cone $\Theta \subset Z$ with $\Theta \setminus (-\Theta) \neq \emptyset$, and let $(\bar{x}, \bar{z})$ be a local minimizer of F. Assume that* $\operatorname{epi} F$ *is locally closed around $(\bar{x}, \bar{z})$. The following hold:*

- **(Fuzzy Fermat rule)** *For every $\varepsilon > 0$ there are $(x_\varepsilon, z_\varepsilon) \in \operatorname{epi} F$ with $\|(x_\varepsilon, z_\varepsilon) - (\bar{x}, \bar{z})\| \leq \varepsilon$ and $x^*_\varepsilon \in \widehat{\partial} F(x_\varepsilon, z_\varepsilon)$ with $\|x^*_\varepsilon\| \leq \varepsilon$. Moreover, we can find $(x_\varepsilon, z_\varepsilon) \in \operatorname{gph} F$ with $\|x_\varepsilon - \bar{x}\| \leq \varepsilon$ such that*

$$\|x^*_\varepsilon\| \leq \varepsilon \ \ \textit{for some} \ x^*_\varepsilon \in \widehat{\partial} F(x_\varepsilon, z_\varepsilon);$$

- **(Exact Fermat rule)** *Assume further that either Θ is SNC at the origin or F^{-1} is PSNC at $(\bar{z}, \bar{x})$. Then we have the inclusion*

$$\mathbf{0} \in \partial F(\bar{x}, \bar{z}),$$

  which is equivalent to

$$\mathbf{0} \in \partial F(\bar{x}, \bar{z})(z^*) \ \ \textit{for some} \ \ z^* \in -N(\mathbf{0}; \Theta) \ \ \textit{with} \ \ \|z^*\| = 1.$$

Proof. The proof of the exact Fermat rule can be found in [5, Theorem 4.1]; cf. also [8, Theorem 5.1]. Therefore, it remains to justify the fuzzy one. We proceed

in two steps.
Step 1. Construct an extremal set system at $(\bar{x}, \bar{z})$ from the optimality of this point.
Step 2. Apply the approximate extremal principle with appropriate parameters to the afore-constructed extremal system. Details follow.

Define two sets in the product space $X \times Z$ by

$$\Xi_1 := \operatorname{epi} F \quad \text{and} \quad \Xi_2 := X \times \{\bar{z}\}.$$

It is easy to check that both sets are locally closed around $(\bar{x}, \bar{z})$, and that $\{\Xi_1, \Xi_2, (\bar{x}, \bar{z})\}$ is an extremal system. To verify the extremality property of the system, take any element $\bar{c} \in \Theta \setminus (-\Theta)$ and construct a sequence $\{c_k\}$ by $c_k := k^{-1}\bar{c}$ for all $k \in \mathbb{N}$. We have $\|c_k\| \to 0$ as $k \to \infty$ and

$$\Xi_1 \cap \big(\Xi_2 - (\mathbf{0}, c_k)\big) \cap (U \times Z) = \emptyset \quad \text{for all } k \in \mathbb{N},$$

where U is the neighborhood of $\bar{x}$ taken from the extremality condition of $(\bar{x}, \bar{z})$ as defined in Definition 3.1 (v). To justify the validity of the latter, we proceed by contradiction and assume that it does not hold. Then there is a pair from the left-hand side set above, say

$$(x, z) \in \Xi_1 \cap \big(\Xi_2 - (\mathbf{0}, c_k)\big) \cap (U \times Z).$$

Taking into account the structures of Ξ_1 and Ξ_2, we find $\theta \in \Theta$ such that

$$(x, z) \in \operatorname{epi} F, \quad (x, z - \theta) \in \operatorname{gph} F, \quad (x, z) \in U \times \{\bar{z} - c_k\}, \quad z = \bar{z} - c_k, \quad \text{and} \quad x \in U.$$

Combining this together gives us

$$\bar{z} - \theta - c_k \in F(\bar{x}) \cap (\bar{z} - \Theta),$$

which contradicts the minimality of $(\bar{x}, \bar{z})$ and thus justifies the claim.

Given further $\varepsilon > 0$ and applying the approximate extremal principle to this extremal system $\{\Xi_1, \Xi_2\}$ with the parameter $\eta := \varepsilon/(1 + \varepsilon) < \varepsilon$, we have

$$\begin{cases} (u, v) \in \operatorname{epi} F \quad \text{with} \quad \|(u, v) - (\bar{x}, \bar{z})\| \le \eta \le \varepsilon, \\ (y, \bar{z}) \in \Omega \times \{\bar{z}\} \quad \text{with} \quad \|y - \bar{x}\| \le \eta \le \varepsilon, \\ (u^*, -v^*) \in \widehat{N}\big((u, v); \operatorname{epi} F\big) + \eta \mathbb{B}^*, \\ (y^*, -w^*) \in \widehat{N}\big((y, \bar{z}); X \times \{\bar{z}\}\big) + \eta \mathbb{B}^* = \{\mathbf{0}\} \times Z^* + \eta \mathbb{B}^*, \\ (u^*, -v^*) + (\mathbf{0}, -w^*) = \mathbf{0} \quad \text{and} \quad \|(\mathbf{0}, -w^*)\| = \|w^*\| = 1. \end{cases} \tag{13}$$

The last line in (13) gives $u^* = \mathbf{0}$, $v^* = w^*$ and $\|v^*\| = 1$. The third line of (13) ensures the existence of $(\tilde{u}^*, \tilde{v}^*) \in X^* \times Z^*$ satisfying

$$(\tilde{u}^*, \tilde{v}^*) \in \widehat{N}\big((u, v); \operatorname{epi} F\big) \quad \text{and} \quad \|(\tilde{u}^*, \tilde{v}^*) - (u^*, v^*)\| \le \eta,$$

and thus we have the relationships

$$\begin{cases} \left(\dfrac{\tilde{u}^*}{\|\tilde{v}^*\|}, \dfrac{-\tilde{v}^*}{\|\tilde{v}^*\|}\right) \in \widehat{N}\big((u,v); \operatorname{epi} F\big), \text{ i.e., } \dfrac{\tilde{u}^*}{\|\tilde{v}^*\|} \in \widehat{\partial} F(u,v), \\ \|\tilde{u}^*\| \le \eta, \quad \text{and} \quad \|\tilde{v}^*\| \ge \|v^*\| - \eta = 1 - \eta. \end{cases}$$

To complete the proof, let us set $(x_\varepsilon, z_\varepsilon) := (u, v)$ and $x_\varepsilon^* := \frac{\tilde{u}^*}{\|\tilde{v}^*\|}$ and get all the required estimates satisfied. Indeed, we have

$$\begin{cases} \|(x_\varepsilon, z_\varepsilon) - (\bar{x}, \bar{z})\| \le \varepsilon \ \text{ is valid due to (13) and} \\ \|x_\varepsilon^*\| = \dfrac{\|\tilde{u}^*\|}{\|\tilde{v}^*\|} \le \dfrac{\eta}{1-\eta} = \varepsilon, \end{cases}$$

which completes the proof of the theorem. ■

Remark 4.2 (Comparisons with other Fermat rules).

(1) The fuzzy Fermat rule obtained is more efficient than that in [27, Theorem 4.1], which is given in the form

$$\mathbf{0} \in \widehat{D}^* F(x_\varepsilon, z_\varepsilon)(z^* + \varepsilon \mathbb{B}_X^*) + \varepsilon \mathbb{B}_Z^*, \quad z^* \in -\widehat{N}(\mathbf{0}; \Theta), \quad \text{and} \quad \|z^*\| = 1.$$

Obviously, an element $\tilde{z}^* \in z^* + \varepsilon \mathbb{B}_X^*$ might not be in the set $-\widehat{N}(\mathbf{0}; \Theta)$, and the inclusion $\widehat{\partial} F(x_\varepsilon, z_\varepsilon)(z^*) \subset \widehat{D}^* F(x_\varepsilon, z_\varepsilon)(z^*)$ holds for any $(x_\varepsilon, z_\varepsilon) \in \operatorname{gph} F$ and $z^* \in Z^*$.

(2) The fuzzy Fermat rule and the characterization of Lipschitz-like behavior conclude that if $(\bar{x}, \bar{z})$ is a local minimizer of F, then the inverse of $\mathcal{E}_F$ denoting by $(\mathcal{E}_F)^{-1}$ is not Lipschitz-like at $(\bar{z}, \bar{x})$. Note that this implication has been proven in [11, Lemma 3.1] under the openness property of the cost mapping. It can be deduced from the fuzzy Fermat rule and the characterization of Lipschitz-like behavior by arguing by contradiction. Indeed, assume that $(\mathcal{E}_F)^{-1}$ is Lipschitz-like at $(\bar{z}, \bar{x})$ with modulus $\ell > 0$. Then we have from [23, Theorem 1.43] that there is $\eta > 0$ such that for all $(x, z) \in \operatorname{epi} F$ with $\|(x,z) - (\bar{x}, \bar{z})\| < \eta$ we have

$$(x^*, z^*) \in \widehat{N}\big((x,z); \operatorname{epi} F\big) \implies \|z^*\| \le \ell \|x^*\|,$$

which is violated the fuzzy Fermat rule of Theorem 4.1 with $\varepsilon < \min\{\eta, \ell^{-1}\}$.

(3) It is important to emphasize that the exact Fermat rule in Theorem 4.1 is established in [8] for various Pareto-type minimizers with closed and convex ordering cone, in [10] for the so-called extended Pareto minimizers determined by any (ordering) set containing the origin, and in [1] for the so-called extremal solutions determined by some extremality condition.

(4) The corresponding rules formulated in terms of coderivatives of set-valued cost mappings can be found in [27, Theorems 4.1 and 4.2], where they are established by employing an extended version of the separation theorem

for nonconvex sets. Further results in this direction, including those for constrained problems, are given in [28].

In the rest of this section we derive a new suboptimality Fermat rule for set-valued mappings, in which we do not assume the existence of minimizers while operate with almost optimal solutions that always exist.

Theorem 4.3 (Fuzzy suboptimality Fermat rule for set-valued mappings). *Let* $F\colon X \rightrightarrows Z$ *with the image space* Z *ordered by a proper, closed, and convex cone* $\Theta \subset Z$ *satisfying* $\Theta\setminus(-\Theta) \neq \emptyset$. *Given* $\varepsilon > 0$ *and* $\mathbf{e} \in \Theta\setminus(-\Theta)$ *with* $\|\mathbf{e}\| = 1$, *let* $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ *be a strict approximate* $\varepsilon\,\mathbf{e}$*-minimizer of* F. *Assume that* F *is quasibounded from below, epiclosed and satisfies the limiting monotonicity condition on* $\operatorname{dom} F$. *Then for any* $\lambda > 0$ *there are* $(x_\lambda, z_\lambda) \in \operatorname{gph} F$ *and* $x^*_\lambda \in \widehat{\partial} F(x_\lambda, z_\lambda)$ *such that*

$$\|x_\lambda - \bar{x}\| \le \lambda \;\; \textit{and} \;\; \|x^*_\lambda\| \le \frac{\varepsilon}{\lambda}; \tag{14}$$

the latter inequality is equivalent to that of $x^*_\lambda \in \widehat{\partial} F(x_\lambda, z_\lambda)(z^*_\lambda) = \widehat{D}^*\mathcal{E}_F(x_\lambda, z_\lambda)(z^*_\lambda)$ *for some* $z^*_\lambda \in -N(\mathbf{0};\Theta)$ *with* $\|z^*_\lambda\| = 1$ *satisfying* $\|x^*_\lambda\| \le \frac{\varepsilon}{\lambda}$.

Proof. Since $(\bar{x}, \bar{z})$ is a *strict* approximate $\varepsilon\,\mathbf{e}$-minimizer of F, there is a positive number $\widetilde{\varepsilon} < \varepsilon$ such that it is an approximate $\widetilde{\varepsilon}\,\mathbf{e}$-minimizer of F as well. Without loss of generality, assume that $\frac{\varepsilon}{\lambda} \le 1$; otherwise, we use the equivalent norm $|\!|\!|\cdot|\!|\!| := \frac{\varepsilon}{\lambda}\|\cdot\|$ with both parameters ε and λ being ε. Pick a positive number $\widetilde{\lambda} < \lambda$ such that $\frac{\widetilde{\varepsilon}}{\widetilde{\lambda}} < \frac{\varepsilon}{\lambda}(\le 1)$ and employ the *Ekeland variational principle* for set-valued mappings applied to the mapping F and its approximate $\widetilde{\varepsilon}\,\mathbf{e}$-minimizer $(\bar{x}, \bar{z})$ with the chosen parameters $\widetilde{\varepsilon}$ and $\widetilde{\lambda}$. In this way we find $(\bar{u}, \bar{v}) \in \operatorname{gph} F$ satisfying the relationships

$$\bar{v} \in \operatorname{Min}\big(F(\bar{u});\Theta\big), \quad \|\bar{x} - \bar{u}\| \le \widetilde{\lambda}, \quad \text{and} \tag{15}$$

$$z - \bar{v} + \tfrac{\widetilde{\varepsilon}}{\widetilde{\lambda}}\|x - \bar{u}\|\,\mathbf{e} \;\not\le_\Theta\; \mathbf{0} \;\;\forall\,(x,z) \in \operatorname{gph} F \setminus \{(\bar{u},\bar{v})\}. \tag{16}$$

Considering further the Lipschitz continuous vector-valued mapping $g : X \to Z$ given by

$$g(x) := \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}\|x - \bar{u}\|\,\mathbf{e}$$

and the sum mapping $\widetilde{F}(x) := F(x) + g(x)$ for all $x \in X$, we have from (16) that $\big(\bar{u}, \bar{v} + g(\bar{u})\big)$ is a minimizer of $\widetilde{F}$. Set

$$\bar{\delta} := \frac{\frac{\varepsilon}{\lambda} - \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}}{\frac{\varepsilon}{\lambda} + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}} + 2}$$

and employ the fuzzy Fermat rule from Theorem 4.1 to the mapping $\widetilde{F}$, the minimizer $(\bar{u}, \bar{v} + g(\bar{u}))$ with the ε-parameter $\eta = \min\{\bar{\delta},\ 0.5(\lambda - \widetilde{\lambda})\}$. Then we find elements $(\bar{y}, \overline{w}_1 + \overline{w}_2) \in \operatorname{epi}\widetilde{F}$ with $\overline{w}_2 = g(\bar{y})$ and $\overline{w}_1 \in \mathcal{E}_F(\bar{y}) = F(\bar{y}) + \Theta$, and dual elements $(y^*, w^*) \in X^* \times Z^*$ with $w^* \in -\widehat{N}(\mathbf{0}; \Theta)$ and $\|w^*\| = 1$ satisfying

$$\|(\bar{y}, \overline{w}_1 + \overline{w}_2) - (\bar{u}, \bar{v} + g(\bar{u}))\| \le \eta \ \text{ (and thus } \|\bar{y} - \bar{u}\| \le \eta); \tag{17}$$

$$(y^*, -w^*) \in \widehat{N}\big((\bar{y}, \overline{w}_1 + \overline{w}_2); \widetilde{F}\big) \ \text{ with } \ \|y^*\| \le \eta.$$

Construct now two sets in the space $H := X \times Z \times Z$ equipped with the sum norm by

$$\Lambda_1 := \{(y, w_1, w_2) \in H \mid (y, w_1) \in \operatorname{epi} F\} \ \text{ and}$$
$$\Lambda_2 := \{(y, w_1, w_2) \in H \mid w_2 = g(y)\}.$$

It is easy to check that $(\bar{y}, \overline{w}_1, \overline{w}_2) \in \Lambda_1 \cap \Lambda_2$, and that

$$(y^*, -w^*, -w^*) \in \widehat{N}\big((\bar{y}, \overline{w}_1, \overline{w}_2); \Lambda_1 \cap \Lambda_2\big).$$

Defining further the positive number

$$\nu := \min\left\{\frac{1 - \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}}{1 + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}}, \frac{\bar{\delta}}{2}, \eta\right\} \tag{18}$$

and employing [23, Lemma 3.1] (*a fuzzy intersection rule from the extremal principle*) to the sets Λ_1 and Λ_2 with the ε-parameter ν, we find:

- $(y_1, w_{11}, w_{21}) \in \Lambda_1$ with $\|(y_1, w_{11}, w_{21}) - (\bar{y}, \overline{w}_1, \overline{w}_2)\| \le \nu$, and thus

$$(y_1, w_{11}) \in \operatorname{epi} F \ \text{ and } \|y_1 - \bar{y}\| \le \nu; \tag{19}$$

- $(y_2, w_{12}, w_{22}) \in \Lambda_2$ with $\|(y_2, w_{12}, w_{22}) - (\bar{y}, \overline{w}_1, \overline{w}_2)\| \le \nu$, and thus $w_{22} = g(y_2)$;
- $(y_1^*, -w_1^*, \mathbf{0}) \in \widehat{N}\big((y_1, w_{11}, w_{21}); \Lambda_1\big) + \nu \mathbb{B}^*$, and thus

$$(y_1^*, -w_1^*) \in \widehat{N}\big((y_1, w_{11}); \operatorname{epi} F\big) + \nu \mathbb{B}^*; \tag{20}$$

- $(y_2^*, \mathbf{0}, -w_2^*) \in \widehat{N}\big((y_2, w_{12}, w_{22}); \Lambda_2\big) + \eta \mathbb{B}^*$, and thus

$$(y_2^*, -w_2^*) \in \widehat{N}\big((y_2, w_{12}); \operatorname{gph} g\big) + \nu \mathbb{B}^*; \tag{21}$$

- a nonnegative number ρ satisfying

$$\begin{aligned} &\rho(y^*, -w^*, -w^*) = (y_1^*, -w_1^*, \mathbf{0}) + (y_2^*, \mathbf{0}, -w_2^*) \ \text{ and} \\ &\max\{\rho, \|(y_2^*, \mathbf{0}, -w_2^*)\|\} = 1. \end{aligned} \tag{22}$$

It is obvious by (22) that $w_1^* = w_2^* = \rho w^*$ and $\|w_1^*\| = \|w_2^*\| = \|\rho w^*\| = \rho$.

It follows from (21) that there exists $(\widetilde{y}_2^*, -\widetilde{w}_2^*) \in \widehat{N}\big((y_2, w_{12}); \mathrm{gph}\, g\big)$ satisfying

$$\|(\widetilde{y}_2^*, -\widetilde{w}_2^*) - (y_2^*, -w_2^*)\| \le \eta.$$

Since $\|w_2^*\| = \rho$, we have $\rho - \nu \ \le\ \|\widetilde{w}_2^*\| \ \le\ \rho + \nu$. The Lipschitz continuity of g with modulus $\frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}$ implies by [23, Theorem 1.43] the estimate

$$\|\widetilde{y}_2^*\| \le \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}\|\widetilde{w}_2^*\| \le \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(\rho + \nu),$$

and thus $\|y_2^*\| \le \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(\rho + \nu) + \nu < 1$ by the choice of ν in (18). Then the second equality in (22) yields $\rho = 1$. Substituting this into the first equality in (22) gives us

$$(y^*, -w^*, -w^*) = (y_1^*, -w_1^*, \mathbf{0}) + (y_2^*, \mathbf{0}, -w_2^*). \tag{23}$$

It follows from (20) that there exists $(\widetilde{y}_1^*, -\widetilde{w}_1^*) \in \widehat{N}\big((y_1, w_{11}); \mathrm{epi}\, F\big)$ satisfying

$$\|(\widetilde{y}_1^*, -\widetilde{w}_1^*) - (y_1^*, -w_1^*)\| \le \nu.$$

Since $\|w_1^*\| = \rho = 1$, we have $\|\widetilde{w}_1^*\| \ge 1 - \nu > 0$. This implies, by taking into account $y^* = y_1^* + y_2^*$ in (23) and thus $\|y_1^*\| \le \|y^*\| + \|y_2^*\|$, that

$$\begin{aligned} \|\widetilde{y}_1^*\| \le \|y_1^*\| + \nu \le \|y^*\| + \|y_2^*\| \ + \nu &\le \eta + \tfrac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1 + \nu) + 2\nu \\ &\le \bar{\delta} + \tfrac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1 + \nu) + 2\nu. \end{aligned} \tag{24}$$

Define further $x_\lambda := y_1$, $z_\lambda := w_{11}$, and

$$(x_\lambda^*, -z_\lambda^*) := \Big(\frac{\widetilde{y}_1^*}{\|\widetilde{w}_1^*\|}, \frac{-\widetilde{w}_1^*}{\|\widetilde{w}_1^*\|}\Big) \in \widehat{N}\big((y_1, w_{11}); \mathrm{epi}\, F\big) \ \text{ with } \ \|z_\lambda^*\| = 1$$

and show that (x_λ, z_λ) and x_λ^* satisfy two estimates in (14). The first holds due to (15), (17), (19), and the triangle inequality

$$\|\bar{x} - x_\lambda\| \le \|\bar{x} - \bar{u}\| + \|\bar{u} - \bar{y}\| + \|\bar{y} - y_1\| \le \widetilde{\lambda} + \eta + \nu \le \widetilde{\lambda} + 2\eta \le \lambda.$$

The second one is due to (24) and the choice of $\bar{\delta}$ and ν by

$$\|x_\lambda^*\| := \frac{\|\widetilde{y}_1^*\|}{\|\widetilde{w}_1^*\|} \le \frac{\bar{\delta} + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1 + \nu) + 2\nu}{1 - \nu} \le \frac{\frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1 + \bar{\delta}) + 2\bar{\delta}}{1 - \bar{\delta}} = \frac{\varepsilon}{\lambda},$$

where the latter inequality holds by $\nu \le \bar{\delta}$ implying that

$$\frac{\bar{\delta} + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1 + \nu) + 2\nu}{1 - \nu} \le \frac{\frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1 + \bar{\delta}) + 2\bar{\delta}}{1 - \bar{\delta}}$$

$$\Leftrightarrow \bar\delta - \bar\delta^2 + \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1+\nu)(1-\bar\delta) + 2\nu - 2\nu\bar\delta \le \frac{\widetilde{\varepsilon}}{\widetilde{\lambda}}(1+\nu)(1-\bar\delta) + 2\bar\delta - 2\nu\bar\delta$$

$$\Leftrightarrow 2\nu - \bar\delta^2 \le \bar\delta - 2\nu\bar\delta$$

$$\Leftrightarrow (1+\bar\delta)(2\nu - \bar\delta) \le 0.$$

This completes the proof of the theorem. ■

It is worth observing that the fuzzy suboptimality Fermat rule of Theorem 4.3 is *equivalent* to the set-valued version of *subdifferential variational principle* formulated in [8, Theorem 3.8]. However, the proof given in this paper is a bit different from those in [4] and [8]. A *sketch of this proof* is as follows:

Let $(\bar x, \bar z)$ be a strict approximate $\varepsilon\,\mathbf{e}$-minimizer of F. Then:

Step 1. Find $\bar u$ near $\bar x$ such that it is a minimizer of the perturbation mapping $\widetilde{F}$.

Step 2. Applying the fuzzy Fermat rule to $\widetilde{F}$, find $\bar y$ with $\|\bar y - \bar u\| \le \eta$ and $y^* \in \widehat{\partial}\widetilde{F}(\bar y, \cdot)$ with $\|y^*\| \le \eta$.

Step 3. Employing the fuzzy rule for Fréchet normals to intersections, which plays the role of the fuzzy sum rule for semi-Lipschitz functions, find x_λ with $\|\bar y - x_\lambda\| \le \nu$ and $x^*_\lambda \in \widehat{\partial}F(x_\lambda, \cdot)$ satisfying $\|x^*_\lambda\| \le \frac{\varepsilon}{\lambda}$.

Finally in this section, we present a consequence of Theorem 4.3 in the case of continuous vector-valued functions that unconditionally enjoy the limiting monotonicity property.

Corollary 4.4 (Fuzzy suboptimality Fermat rule for vector-valued mappings). *Let $f\colon X \rightrightarrows Z$ be a continuous vector-valued function, where Z is ordered by Θ as in Theorem 4.3. Given $\varepsilon > 0$ and $\mathbf{e} \in \Theta \setminus (-\Theta)$ with $\|\mathbf{e}\| = 1$, take $\bar x \in \operatorname{dom} f$ as an arbitrary strict approximate $\varepsilon\,\mathbf{e}$-minimizer of f. Assume that f is quasibounded from below. Then for any $\lambda > 0$ there are x_λ satisfying $\|x_\lambda - \bar x\| \le \lambda$ and*

$$x^*_\lambda \in \widehat{\partial} f(x_\lambda) \quad \textit{with} \quad \|x^*_\lambda\| \le \frac{\varepsilon}{\lambda}.$$

Proof. It is straightforward from Theorem 4.3. ■

Observe further that if $f = \varphi : X \to \overline{\mathbb{R}} = \mathbb{R}\cup\{+\infty\}$ is an extended-real-valued functional (and thus $\mathbf{e} = 1$), there is no difference between a strict approximate ε-minimizer of φ and an approximate ε-minimizer of φ known also as an ε-solution. Therefore, both Theorem 4.3 and Corollary 4.4 reduce to the well-recognized lower subdifferential variational principle [23]: For every proper l.s.c. function $\varphi : X \to \overline{\mathbb{R}}$ bounded from below, every $\varepsilon > 0$, $\lambda > 0$, and $x_0 \in X$ with $\varphi(x_0) < \inf_X \varphi + \varepsilon$ there are $\bar x \in X$ and $x^* \in \widehat{\partial}\varphi(\bar x)$ such that $\|\bar x - x_0\| < \lambda$, $\varphi(\bar x) < \inf_X \varphi + \varepsilon$, and $\|x^*\| \le \frac{\varepsilon}{\lambda}$. Note that our approach can be applied to derive this result in the scalar setting.

5. Existence of generalized Pareto minimizers

In this section we present a new proof of the existence theorem for weak Pareto minimizers of set-valued mappings from [8] under the subdifferential Palais-Smale condition. Our new proof is based on the fuzzy intersection rule and fuzzy Fermat rule instead of the subdifferential variational principle for set-valued mappings as in [8]. Moreover, in this way we avoid the *strong* limiting monotonicity condition imposed in [8].

Theorem 5.1 (Existence of weak minimizers for set-valued mappings). *Let $F\colon X \rightrightarrows Z$, and let $\Theta \subset Z$ be a proper, closed, convex, and solid ordering cone. Assume that F is epiclosed, quasibounded from below, and satisfies the limiting monotonicity condition on* $\operatorname{dom} F$. *Suppose furthermore that the subdifferential Palais-Smale condition in Definition* 3.2 (vii) *holds for F. Then F admits a weak Pareto minimizer.*

Proof. To justify the existence of weak (Pareto) minimizers for F, we first inductively apply the Ekeland variational principle for set-valued mappings of Theorem 3.3 to generate an appropriate sequence $\{(x_k, z_k)\} \subset \operatorname{gph} F$, $k \in \mathbb{N}$. Then we prove that the chosen sequence $\{x_k\}$ contains a subsequence converging to a weak minimizer of F. Details follow.

Pick an arbitrary pair $(x_0, z_0) \in \operatorname{gph} F$ and an element $\mathbf{e} \in \operatorname{int}\Theta$ with $\|\mathbf{e}\| = 1$. We inductively generate a sequence $\{(x_k, z_k)\} \subset \operatorname{gph} F$ by using the Ekeland variational principle from Theorem 3.3 with the $(k-1)$ iteration (x_{k-1}, z_{k-1}) and parameters $\varepsilon = k^{-2}$ and $\lambda = k^{-1}$ to get the iteration $(x_k, z_k) \in \operatorname{gph} F$ satisfying the relationships

$$z_k \in \min\big(F(x_k); \Theta\big), \quad z_k \ \le_\Theta z_{k-1}, \quad \text{and} \tag{25}$$

$$z - z_k + k^{-1}\|x - x_k\|\mathbf{e} \not\le_\Theta \mathbf{0} \ \text{ for all } \ (x, z) \in \operatorname{gph} F \ \text{ with } \ (x, z) \neq (x_k, z_k). \tag{26}$$

Suppose now that the chosen sequence $\{x_k\}$ contains a subsequence converging to some point $\bar{x} \in \operatorname{dom} F$; we show that it is indeed the case a bit later. Without loss of generality, assume that the whole sequence $\{x_k\}$ converges to some $\bar{x}$ as $k \to \infty$. We get from (25) and the limiting monotonicity condition of Definition 3.2 (vi) that

$$\text{there is } \ \bar{z} \in F(\bar{x}) \ \text{ with } \ \bar{z} \le_\Theta z_k \ \text{ for all } \ k \in \mathbb{N}.$$

Let us show that the pair $(\bar{x}, \bar{z})$ is a weak minimizer of F. Taking an arbitrary $(x, z) \in \operatorname{gph} F$ with $x \in \operatorname{dom} F$ and $(x, z) \neq (\bar{x}, \bar{z})$ and employing (25) and (26), we have by elementary transformations that

$$z - \bar{z} + \frac{1}{k}\|x_{k+1} - x\|\mathbf{e} \ \in \ z_{k+1} - \bar{z} + Z \setminus (-\Theta) \subset \Theta + Z \setminus (-\Theta)$$

for all $k \in \mathbb{N}$. This obviously implies that

$$z - \bar{z} + \frac{1}{k}\|x_{k+1} - x\|\mathbf{e} \ \in \ Z \setminus (-\Theta)$$

due to the convexity of the cone Θ. Now passing to the limit in the last inclusion as $k \to \infty$, we get $z - \bar{z} \in Z \setminus (-\operatorname{int}\Theta)$, which thus ensures the weak minimality of $(\bar{x}, \bar{z})$ of F.

To complete the proof, it remains to justify the announced claim: the sequence $\{x_k\}$ generated above contains a convergent subsequence. To prove this convergence, we inductively construct another sequence $\{u_k\} \subset \operatorname{dom} F$ such that $\|u_k - x_k\| \to 0$ as $k \to \infty$ and that the *Palais-Smale condition* for set-valued mappings can be applied to this new sequence. To proceed, define for each $k \in \mathbb{N}$ a set-valued mapping $F_k \colon X \rightrightarrows Z$ given by

$$F_k(x) := F(x) + g_k(x) \ \text{ with } \ g_k(x) := k^{-1}\|x - x_k\|\mathbf{e}$$

and conclude from (26) that (x_k, z_k) is a Pareto minimizer of F_k.

By the above fuzzy Fermat rule applied to the mapping F_k, the minimizer (x_k, z_k), and the ε-error k^{-1} we find $(\bar{y}, \overline{w}) \in \operatorname{epi} F$ satisfying

$$\|\bar{y} - x_k\| \le k^{-1}, \ \ y^* \in \widehat{\partial} F_k(\bar{y}, \overline{w})(w_k^*) \ \text{ with } \ \|y^*\| \le k^{-1} \ \text{ and } \ \|w_k^*\| = 1.$$

Using the same arguments as in the proof of the fuzzy suboptimality Fermat rule, we have

$$(y^*, -w^*, -w^*) \in \widehat{N}\big((\bar{y}, \overline{w}_1, \overline{w}_2); \Lambda_1 \cap \Lambda_2\big),$$

where $\overline{w}_2 := k^{-1}\|\bar{y} - x_k\|$, $\overline{w}_1 := \overline{w} - \overline{w}_2$, and Λ_i as $i = 1, 2$ are two sets in the product space $H := X \times Z \times Z$ defined by

$$\begin{aligned} &\Lambda_1 := \{(x, z_1, z_2) \in H \mid (x, z_1) \in \operatorname{epi} F\} \\ \text{and} \qquad &\Lambda_2 := \{(x, z_1, z_2) \in H \mid (x, z_2) \in \operatorname{gph} g_k\}. \end{aligned}$$

Employing the fuzzy intersection rule from [23, Lemma 3.1] to the sets Λ_1 and Λ_2, we find:

- $(y_1, w_{11}, w_{21}) \in \Lambda_1$ with $\|(y_1, w_{11}, w_{21}) - (\bar{y}, \overline{w}_1, \overline{w}_2)\| \le k^{-1}$, and thus

$$(y_1, w_{11}) \in \operatorname{epi} F \ \text{ and } \|y_1 - \bar{y}\| \le k^{-1};$$

- $(y_2, w_{12}, w_{22}) \in \Lambda_2$ with $\|(y_2, w_{12}, w_{22}) - (\bar{y}, \overline{w}_1, \overline{w}_2)\| \le k^{-1}$, and thus

$$w_{22} = g(y_2);$$

- $(y_1^*, -w_1^*, \mathbf{0}) \in \widehat{N}\big((y_1, w_{11}, w_{21}); \Lambda_1\big) + k^{-1}\mathbb{B}^*$, and thus

$$(y_1^*, -w_1^*) \in \widehat{N}\big((y_1, w_{11}); \operatorname{epi} F\big) + k^{-1}\mathbb{B}^*; \tag{27}$$

- $(y_2^*, \mathbf{0}, -w_2^*) \in \widehat{N}\big((y_2, w_{12}, w_{22}); \Lambda_1\big) + k^{-1}\mathbb{B}^*$, and thus

$$(y_2^*, -w_2^*) \in \widehat{N}\big((y_2, w_{12}); \operatorname{gph} g\big) + k^{-1}\mathbb{B}^*;$$

- (we can show that $\rho = 1$ due to the Lipschitzian continuity of g_k; see details in the proof of Theorem 4.3)

$$(y^*, -w^*, -w^*) = (y_1^*, -w_1^*, \mathbf{0}) + (y_2^*, \mathbf{0}, -w_2^*). \tag{28}$$

It is obvious from (28) that $w_1^* = w_2^* = w^*$ and $\|w_1^*\| = \|w_2^*\| = \|w^*\| = 1$. By (28) there are $(\widetilde{y}_2^*, -\widetilde{w}_2^*) \in \widehat{N}\big((y_2, w_{12}); \operatorname{gph} g\big)$, i.e., $\widetilde{y}_2^* \in \widehat{D}^* g(y_2)(w_2^*)$ such that $\|(\widetilde{y}_2^*, -\widetilde{w}_2^*) - (y_2^*, -w_2^*)\| \le k^{-1}$. Since $\|w_2^*\| = 1$, we have $1 - k^{-1} \le \|\widetilde{w}_2^*\| \le 1 + k^{-1}$. Since g is a Lipschitz function with modulus k^{-1}, we have

$$\|\widetilde{y}_2^*\| \le k^{-1}\|\widetilde{w}_2^*\| \le k^{-1}(1 + k^{-1})$$

and thus $\|y_2^*\| \le \|\widetilde{y}_2^*\| + k^{-1} \le k^{-2} + 2k^{-1}$. Now we get from (28) an estimate for y_1^* as follows

$$\|y_1^*\| = \|y^* - y_2^*\| \le \|y^*\| + \|y_2^*\| \le k^{-2} + 3k^{-1}.$$

We also deduce from (27) the existence of some $(\widetilde{y}_1^*, -\widetilde{w}_1^*) \in \widehat{N}\big((y_1, w_{11}); \operatorname{epi} F\big)$ such that $\|(\widetilde{y}_1^*, -\widetilde{w}_1^*) - (y_1^*, -w_1^*)\| \le k^{-1}$. Furthermore, $1 - k^{-1} \le \|\widetilde{w}_1^*\| \le 1 + k^{-1}$ by $\|w_1^*\| = 1$ and

$$\|\widetilde{y}_1^*\| \le \|y_1^*\| + k^{-1} \le k^{-2} + 4k^{-1}$$

due to the above estimate of $\|y_1^*\|$. Setting now

$$u_k := y_1,\ v_k := w_{11},\ v_k^* := \frac{\widetilde{w}_1^*}{\|\widetilde{w}_1^*\|} \ \text{ and } \ u_k^* := \frac{\widetilde{y}_1^*}{\|\widetilde{w}_1^*\|},$$

we arrive at the relationships

$$u_k^* \in \widehat{\partial} F(u_k, v_k)(v_k^*),\ \|v_k^*\| \le \frac{k^{-2} + 4k^{-1}}{1 - k^{-1}}, \ \text{ and } \ \|u_k - x_k\| \le k^{-1}. \tag{29}$$

Since k was chosen arbitrarily in $\mathbb{N}$, this creates the sequences (u_k, v_k) and (u_k^*, v_k^*) as $k \in \mathbb{N}$ satisfying all the conditions in (29). Employing finally the Palais-Smale condition for set-valued mappings, we get that the sequence $\{u_k\}$ contains a convergent subsequence, and so does the sequence $\{x_k\}$. This completes the proof of the theorem. ■

Let us emphasize that in the proof above we do not need the strong limiting monotonicity property of F: for any sequence of pairs $\{(x_k, z_k)\} \subset \operatorname{gph} F$ with $x_k \to \bar{x}$ as $k \to \infty$ one has

$$\big[z_k \le v_k,\ v_{k+1} \le v_k \ \text{ for all } \ k \in \mathbb{N}\big]$$

$$\Longrightarrow \ \big[\text{there is } \ \bar{z} \in \operatorname{Min} F(\bar{x}) \ \text{ with } \ \bar{z} \le v_k, \quad k \in \mathbb{N}\big]$$

imposed in both results of [8, Theorem 4.4] and [4, Theorem 4.3]. Note also that our proof is given for the case of weak Pareto minimizers just for simplicity. It equally works for the other types of generalized Pareto minimizers considered in [8, Theorem 4.4].

Remark 5.2 (On Palais-Smale compactness conditions). The Palais-Smale compactness condition, named after Richard Palais and Stephen Smale, is a hypothesis for some theorems of the calculus of variations. It is useful for ensuring the existence of certain kinds of critical points; in particular, minimizers for set-valued mappings in Theorem 5.1.

It is easy to see that our subdifferential Palais-Smale condition in Definition 3.2(vii) reduces to the classical one for continuously Fréchet differentiable functions: a function $\varphi : X \to \mathbb{R}$ satisfies the (classical) Palais-Smale condition if every sequence $\{x_k\} \subset X$ such that $\{\varphi(x_k)\}$ is bounded and that $\lim_{k\to\infty} \nabla\varphi(x_k) = 0$ has a convergent subsequence in X. We require the boundedness from below of the sequence $\{\varphi(x_k)\}$ since our focus on minimizers φ instead of its critical points. Furthermore, it follows from the classical Bolzano-Weierstrass theorem that a set-valued mapping $F : X \rightrightarrows Z$ from a finite-dimensional space X to an ordered space Z with the ordering cone $\Theta \subset Z$ enjoys the subdifferential Palais-Smale condition of Definition 3.2(vii) if it does not map unbounded subsets of X into subset of Z quasibounded from below. It means that, given an unbounded set $\Omega \subset X$, there is no bounded set $M \subset Z$ such that $F(\Omega) \subset M + \Theta$. This requirement is essential for the validity of the Palais-Smale condition even for real-valued functions. Indeed, consider the Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(x) := e^{-|x|}$ with $\varphi(X) = (0, 1]$. Obviously, it maps any subsets of the real line into bounded ones. Our subdifferential Palais-Smale condition is not satisfied here since for the divergent sequence $\{x_n = \ln(n)\}$ the corresponding gradient sequence $\{\nabla\varphi(x_n) = e^{-ln(n)} = -\frac{1}{n}\}$ tends to zero as $n \to \infty$. Observe finally that in the case of infinite-dimensional spaces X the above arguments hold true with replacing boundedness by compactness.

Remark 5.3 (Comparisons with other sufficient results). In contrast to the existence results derived in [21, Section 5.3] based on an axiom duality scheme in which both dual and primal problems are considered simultaneously, we study here solely primal problems. Note that considering optimization problems for set-valued mapping $F : X \rightrightarrows Z$ with no explicit constraints on x while with possible empty values of F encompasses constrained problems as well due to the hidden constraints given by $x \in \operatorname{dom} F$. Furthermore, a constrained optimization problem of the above type can be equivalently converted into a unconstrained one by imposing the restriction of the cost over the feasible solution set. Having this in mind, the results of [21] can be summarized as follows. Given a *primal* mapping $F : X \rightrightarrows Z$ and a convex ordering cone Θ of Z, we say that a mapping $D : X \rightrightarrows Z$ is a dual mapping of F if the weak duality axiom

$$D(u) \cap (F(x) + \Theta \setminus \{0\}) = \emptyset \ \text{ for } \ x, u \in X$$

holds. Recall also that a dual mapping of F is *exact* if the intersection of the ranges of F and D is nonempty, i.e., $F(X) \cap D(X) \neq \emptyset$. It follows from [21, Proposition 3.3] that F has a minimizer and D has a maximizer provided that D is the exact dual mapping of F. This interesting existence result seems to be rather conceptual in our understanding by taking into that it requires the existence of the exact dual mapping for F, which might not be easy to find. Our existence results are more direct, since we impose a new kind of Palais-Smale compactness condition on the original mapping in question, which ensures the existence at least one Pareto minimizer for the primal problem under consideration.

6. Multiplier rules for constrained problems

This section is devoted to the constrained set-valued optimization problem (CP). We derive new fuzzy multiplier rules and give a refined proof of the exact multiplier rules for (CP) with the help of the fuzzy intersection rule. Note that a fuzzy suboptimality multiplier rule for (CP) can be derived by using similar arguments in the proof of Theorem 4.3.

Theorem 6.1 (Necessary conditions in constrained set-valued optimization). *Let $(\bar{x}, \bar{z})$ be a local minimizer for the constrained set-valued optimization problem (CP), where the image space Z is ordered by a proper, closed, and convex cone $\Theta \subset Z$ with $\Theta \setminus (-\Theta) \neq \emptyset$. Assume that* $\operatorname{epi} F$ *and Ω are locally closed around $(\bar{x}, \bar{z})$ and $\bar{x}$, respectively. Assume also that the following two SNC conditions are satisfied:*

(I) *either Θ is SNC at the origin, or $\big(\mathcal{E}_{F_\Omega}\big)^{-1}$ is PSNC at $(\bar{z}, \bar{x})$; the latter holds provided that $\big(\mathcal{E}_F\big)^{-1}$ is PSNC at $(\bar{z}, \bar{x})$ and Ω is SNC at $\bar{x}$;*

(II) *either F is PSNEC at $(\bar{x}, \bar{z})$, or Ω is SNC at $\bar{x}$.*

Impose also the qualification condition

$$\partial^\infty F(\bar{x}, \bar{z}) \cap \big(-N(\bar{x}; \Omega)\big) = \{\mathbf{0}\}, \tag{30}$$

which holds, in particular, when F is ELL at $(\bar{x}, \bar{z})$. Then we have the multiplier rule

$$\mathbf{0} \in \partial F(\bar{x}, \bar{z}) + N(\bar{x}; \Omega),$$

which is equivalent to

$$\mathbf{0} \in \partial F(\bar{x}, \bar{z})(z^*) + N(\bar{x}; \Omega) = D^*_N \mathcal{E}_F(\bar{x}, \bar{z})(z^*) + N(\bar{x}; \Omega)$$

for some $z^* \in -N(\mathbf{0}; \Theta)$ with $\|z^*\| = 1$.

Proof. We present only the sketch of the proof referring the reader to [8, Theorem 5.3] and [5, Theorem 4.1] for the corresponding details.

Step 1. Since $(\bar{x}, \bar{z})$ is a minimizer of F over Ω, it is a minimizer of the restriction of F over Ω, denoted as usual by F_Ω, which can be expressed as the sum of two mappings $F + \triangle(\cdot; \Omega)$ via the indicator mapping of Ω equal to $0 \in X$ if $x \in \Omega$ and $\emptyset$ otherwise. Employing the exact Fermat rule to the cost mapping F_Ω at the minimizer $(\bar{x}, \bar{z})$ under the general assumptions and the first SNC condition imposed, we have

$$\mathbf{0} \in \partial F_\Omega(\bar{x}, \bar{z})(z^*) = D_N^* \mathcal{E}_{(F+\triangle(\cdot;\Omega))}(\bar{x}, \bar{z})(z^*) = D_N^* \big(\mathcal{E}_F + \triangle(\cdot; \Omega)\big)(\bar{x}, \bar{z})(z^*).$$

Step 2. Applying the sum rule for coderivatives to $\mathcal{E}_F$ and $\triangle(\cdot; \Omega)$ under the second SNC condition imposed and the (mixed) qualification condition (30), we arrive at

$$\mathbf{0} \in D_N^* \mathcal{E}_F(\bar{x}, \bar{z})(z^*) + D_N^* \triangle(\cdot; \Omega)(\bar{x}, \bar{z})(z^*) = \partial F(\bar{x}, \bar{z})(z^*) + N(\bar{x}; \Omega).$$

■

Theorem 6.2 (Fuzzy necessary conditions in constrained set-valued optimization). *Let $(\bar{x}, \bar{z})$ be a local minimizer for the constrained set-valued optimization problem* (CP)*. Assume that the sets* epi F *and Ω are locally closed around $(\bar{x}, \bar{z})$ and $\bar{x}$, respectively, and that F is ELL* (*epigraphically Lipschitz-like*) *at $(\bar{x}, \bar{z})$. Then for any $\varepsilon > 0$ there are $(x_\varepsilon, z_\varepsilon, x_\Omega, x_\varepsilon^*, x_\Omega^*)$ satisfying*

$$\begin{cases} (x_\varepsilon, z_\varepsilon) \in \operatorname{epi} F, \quad \|(x_\varepsilon, z_\varepsilon) - (\bar{x}, \bar{z})\| \le \varepsilon, \\ x_\Omega \in \Omega, \quad \|x_\Omega - \bar{x}\| \le \varepsilon, \\ x_\varepsilon^* \in \widehat{\partial} F(x_\varepsilon, z_\varepsilon), \quad x_\Omega^* \in \widehat{N}(x_\Omega; \Omega), \quad \|x_\varepsilon^* + x_\Omega^*\| \le \varepsilon, \end{cases}$$

where the conditions in the third line can be combined as

$$\mathbf{0} \in \widehat{\partial} F(x_\varepsilon, z_\varepsilon) + \widehat{N}(x_\Omega; \Omega) + \varepsilon \mathbb{B}^*.$$

Proof. Here is the sketch of the proof. All the assumptions imposed in this theorem are needed for the fulfillment of the calculus requirements.

Step 1. Since $(\bar{x}, \bar{z})$ is a minimizer of F_Ω, we apply the fuzzy Fermat rule to the cost mapping F_Ω at the minimizer $(\bar{x}, \bar{z})$. It ensures that, given a threshold $\eta < \varepsilon$, there are $(x_\eta, z_\eta) \in \operatorname{epi} F_\Omega$ and $x_\eta^* \in \widehat{\partial} F_\Omega(x_\eta, z_\eta)$ satisfying

$$\|(x_\eta, z_\eta) - (\bar{x}, \bar{z})\| \le \eta, \quad \text{and} \quad \|x_\eta^*\| \le \eta.$$

Step 2. Observe that the inclusion $x_\eta^* \in \widehat{\partial} F_\Omega(x_\eta, z_\eta)$ implies the existence of $z_\eta^* \in -N(\mathbf{0}; \Theta)$ with $\|z_\eta^*\| = 1$ satisfying

$$x_\eta^* \in \widehat{\partial} F_\Omega(x_\eta, z_\eta)(z_\eta^*) = \widehat{D}^*(\mathcal{E}_F + \triangle(\cdot; \Omega))(x_\eta, z_\eta)(z_\eta^*).$$

Thus we arrive at the relationship

$$(x_\eta^*, -z_\eta^*, -z_\eta^*) \in \widehat{N}\big((x_\eta, z_\eta, \mathbf{0}); \Omega_1 \cap \Omega_2\big),$$

where Ω_1 and Ω_2 are two sets in the product space $H := X \times Z \times Z$ defined by

$$\Omega_1 := \{(x, z_1, z_2) \in H \mid (x, z_1) \in \operatorname{epi} F\} \quad \text{and}$$
$$\Omega_2 := \{(x, z_1, z_2) \in H \mid x \in \Omega \text{ and } z_2 = \bar{z}\}.$$

Applying the fuzzy rule for regular normals to intersections to the above triple $(x^*_\eta, -z^*_\eta, -z^*_\eta)$ with the parameter $\delta = \frac{\varepsilon-\eta}{\varepsilon+2}$ gives us $\rho \geq 0$, $(x_\delta, z_\delta, x_\Omega, x^*_\delta, z^*_\delta, x^*_\Omega) \in \operatorname{epi} F \times \Omega \times X^* \times Z^* \times X^*$ satisfying

$$\begin{cases} (x_\delta, z_\delta) \in \operatorname{epi} F, \ \ \|(x_\delta, z_\delta) - (x_\eta, z_\eta)\| \leq \delta, \\ x_\Omega \in \Omega, \ \ \|x_\Omega - x_\eta\| \leq \delta, \\ (x^*_\delta, -z^*_\delta) \in \widehat{N}\big((x_\delta, z_\delta); \operatorname{epi} F\big) + \delta \mathbb{B}^*, \ \ x^*_\Omega \in \widehat{N}(x_\Omega; \Omega) + \delta B^*, \\ \rho(x^*_\eta, -z^*_\eta, -z^*_\eta) = (x^*_\delta, -z^*_\delta, \mathbf{0}) + (x^*_\Omega, \mathbf{0}, z^*_\Omega), \ \ \max\big\{\|(x^*_\delta, -z^*_\delta, \mathbf{0})\|, \rho\big\} = 1. \end{cases} \tag{31}$$

Proceeding as in the proof of Theorem 4.1, we get $\rho = 1$, and thus $\|z^*_\delta\| = 1$ and $x^*_\eta = x^*_\delta + x^*_\Omega$.

It follows from (31) the existence of $(\tilde{x}^*_\delta, -\tilde{z}^*_\delta) \in \widehat{N}\big((x_\delta, z_\delta); \operatorname{epi} F\big)$ with $\|(\tilde{x}^*_\delta, -\tilde{z}^*_\delta) - (x^*_\delta, -z^*_\delta)\| \leq \delta$. Since $\|z^*_\delta\| = 1$, we have $\|\tilde{z}^*_\delta\| \geq 1 - \delta \neq 0$ as well as $\tilde{x}^*_\Omega \in \widehat{N}(x_\Omega; \Omega)$ with $\|\tilde{x}^*_\Omega - x^*_\Omega\| \leq \delta$. To finish the proof, let us set $(x_\varepsilon, z_\varepsilon) := (x_\delta, z_\delta)$,

$$u^*_\Omega := \frac{\tilde{x}^*_\Omega}{\|\tilde{z}^*_\delta\|} \in \widehat{N}(x_\Omega; \Omega), \quad \text{and}$$

$$(x^*_\varepsilon, -z^*_\varepsilon) := \left(\frac{\tilde{x}^*_\delta}{\|\tilde{z}^*_\delta\|}, \frac{-\tilde{z}^*_\delta}{\|\tilde{z}^*_\delta\|}\right) \in \widehat{N}\big((x_\delta, z_\delta); \operatorname{epi} F\big), \quad \text{i.e.,} \quad x^*_\varepsilon \in \widehat{\partial} F(x_\varepsilon, z_\varepsilon).$$

Then we check that all the required estimates are fulfilled; namely:

$$\begin{aligned} \|x^*_\varepsilon + u^*_\Omega\| &\leq \frac{1}{\|\tilde{z}^*_\delta\|}\Big(\|\tilde{x}^*_\delta + \tilde{x}^*_\Omega\|\Big) \\ &\leq \frac{1}{\|\tilde{z}^*_\delta\|}\Big(\|\tilde{x}^*_\delta - x^*_\delta\| + \|\tilde{x}^*_\Omega - x^*_\Omega\| + \|x^*_\delta + x^*_\Omega\|\Big) \\ &\leq \frac{1}{\|\tilde{z}^*_\delta\|}\Big(\|\tilde{x}^*_\delta - x^*_\delta\| + \|\tilde{x}^*_\Omega - x^*_\Omega\| + \|x^*_\eta\|\Big) \\ &\leq \frac{2\delta + \eta}{1 - \delta} = \varepsilon; \end{aligned}$$

$$\begin{aligned} \|(x_\varepsilon, z_\varepsilon) - (\bar{x}, \bar{z})\| &= \|(x_\delta, z_\delta) - (\bar{x}, \bar{z})\| \\ &\leq \|(x_\delta, z_\delta) - (x_\eta, z_\eta)\| + \|(x_\eta, z_\eta) - (\bar{x}, \bar{z})\| \leq \delta + \eta \leq \varepsilon; \end{aligned}$$

$$\|x^*_\Omega - \bar{x}\| \leq \|x^*_\Omega - x_\eta\| + \|x_\eta - \bar{x}\| \leq \delta + \eta \leq \varepsilon.$$

This completes the proof of the theorem. ■

Remark 6.3 (Discussions on fuzzy necessary optimality conditions). The fuzzy necessary optimality condition obtained in Theorem 6.2 is new in set-valued optimization. In scalar optimization such results are known for problems with various kinds of constraints including those given by equalities and inequalities, operator constraints, and equilibrium constraints, etc.; see [24, Section 5.1.4] and the more recent paper [2]. It is not difficult to extend the obtained fuzzy result to broader classes of set-valued optimization problems with all the types of constraints mentioned above. Note also that in the known scalar optimization results of this type the cost function may not be necessary locally Lipschitzian, while a Lipschitz-like behavior of set-valued cost mappings seems to be important for us to obtain a nontriviality fuzzy necessary optimality conditions.

Remark 6.4 (Other developments on necessary optimality conditions in constrained set-valued optimization).

(1) In [10], we introduced and studied refined notions of *extended Pareto optimality*. In particular, these notions extend the generalized Pareto order defined above via the ordering relation (1) by replacing a closed and convex ordering cone with an ordering set containing the origin and satisfying a rather mild requirement called the *local asymptotic closedness condition*. It holds under the standard assumptions imposed on ordering cones, while there are nonconvex sets enjoying this property. The results obtained in this paper can be developed to the aforementioned extended framework.

(2) An abstract notion of (f,Θ)-*optimality* was developed in [24, Chapter 5] for a single-valued cost mapping $f : X \to Z$ via a generalized order relation defined by a given subset $\Theta \subset Z$, which may be generally nonconic and nonconvex with an empty interior. Such a generalized notion of optimality is actually induced by the concept of local extremal points of set systems and extends the classical concepts of Pareto/weak Pareto optimality as well as their generalizations. The latter notion was extended to set optimization in [1]. Recall to this end that, given a set-valued mapping $F : X \rightrightarrows Z$ and an ordering set $\Theta \subset Z$ containing the origin, we say that a pair $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is a *local extremal solution* with respect to Θ of F–or (F,Θ)-optimal solution of F–if $\bar{z}$ is a global extremal point to the set system $\big\{F(U);(\bar{z}-\Theta)\big\}$ for some neighborhood U of $\bar{x}$, where

$$F(\Omega \cap U) := \bigcup \big\{F(x) \mid x \in \Omega \cap U\big\},$$

i.e., there is a sequence $\{z_k\} \subset Z$ with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$F(\Omega \cap U) \cap (\bar{z} - \Theta - z_k) = \emptyset \text{ for all } k \in \mathbb{N}.$$

7. Sufficient conditions in set-valued optimization

In this section we derive new sufficient optimality conditions for set-valued optimization problems with geometric constraints (CP). Such conditions for minimization problems are naturally formulated via *upper subdifferentials* (or superdifferentials) of cost mappings. To proceed equivalently with the (lower) subdifferentials defined above, it is more convenient to deal below with set-valued *maximization* problems given by:

$$\text{(CP-max)} \qquad \text{maximize} \quad F(x) \quad \text{subject to} \quad x \in \Omega,$$

where $F : X \rightrightarrows Z$ and $\Omega \subset X$. We say that the pair $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ is a *global weak maximizer* of F over Ω–or just a global weak maximizer of (CP-max)–if $\bar{z} \in \operatorname{WMin}\big(F(\Omega); -\Theta\big)$, i.e.,

$$F(x) \cap (\bar{z} + \operatorname{int}\Theta) = \emptyset \ \text{ for all } \ x \in \Omega.$$

Assume that the domain space X admits *Fréchet smooth renorming* (i.e., it has an equivalent norm $\|\cdot\|$ that is Fréchet differentiable at any nonzero point) and fix this norm in what follows. Furthermore, we also suppose that the geometric constraint set Ω enjoys the *normal independence* property meaning that the condition

$$-\nabla\|\cdot - x\|(u) \notin N(u;\Omega) \ \text{ for all } \ u \in \Omega \setminus \{x\} \tag{32}$$

holds for any point $x \in \Omega$. It is easy to see that the normal independence property (32) can be equivalently written as

$$x - u \notin N(u;\Omega) \ \text{ for all } \ u \in \Omega \setminus \{x\}$$

when X is Hilbert. Note that every closed and convex set in a Fréchet smooth space is normally independent; see [6, Proposition 3.1]. Observe also that there are nonconvex sets having this property such as, e.g., $\operatorname{epi} x^3$ and $\operatorname{epi}(-x^2)$ in $\mathbb{R}^2$. On the other hand, there are simple sets in finite dimensions that are not normally independent. For instance, let Ω be a circle in $\mathbb{R}^2$ centered at the origin with a radus $\sqrt{2}$, and let $x = (-1,-1)$ and $u = (1,1)$. It is easy to check that condition (32) is not fulfilled, since

$$x - u = (-2,-2) \in N(u;\Omega) = \mathbb{R} \cdot (1,1).$$

We are now ready to reestablish the sufficient optimality conditions for problem (CP-max) that first appeared in [6, Theorem 3.2]. The proof presented here is shorter than that of [6, Theorem 3.2] and is based on refined calculus rules.

Theorem 7.1 (Sufficient conditions for global weak Pareto maximizers). *Consider a constrained set-valued optimization problem* (CP-max), *where Z is ordered by a closed, convex, and solid cone $\Theta \subset Z$. Assume that* $\operatorname{epi} F$ *and* Ω

are closed sets and that Ω *is normally independent. Given a pair* $(\bar{x}, \bar{z}) \in \operatorname{gph} F$, *define the* $\bar{z}$*-level set of* F *with respect to* Ω *by*

$$\operatorname{lev}_\Omega(F; \bar{z}) := \{x \in \Omega \mid F(x) \cap (\bar{z} - \Theta) \neq \emptyset\} = \operatorname{lev}(F; \bar{z}) \cap \Omega$$

and suppose that it is a compact set. In addition we assume the fulfillment of the implication

$$\big[F(\bar{y}) \cap \big(\bar{z} + \operatorname{int}\Theta\big) \neq \emptyset\big] \quad \Longrightarrow \quad F(\bar{y}) \cap \big(\bar{z} - \Theta\big) = \emptyset, \tag{33}$$

which holds, in particular, when F *is single-valued. Then the conditions*

$$\mathbf{0} \notin \partial F(\bar{u}, \bar{v}) + N(\bar{u}; \Omega) \quad \textit{and} \tag{34}$$

$$\partial F(\bar{u}, \bar{v}) + N(\bar{u}; \Omega) \subset N(\bar{u}; \Omega) \tag{35}$$

for all $\bar{u} \in \Omega$ *and* $\bar{v} \in F(\bar{u}) + \Theta$ *are sufficient for the global weak optimality of* $(\bar{x}, \bar{z})$ *for problem* (CP-max) *provided the validity of the following SNC and constraint qualifications:*

- *either* Ω *is SNC at* $\bar{u}$, *or* F *is PSNEC at* $(\bar{u}, \bar{v})$;
- $\partial^\infty F(\bar{u}, \bar{v}) \cap (-N(\bar{u}; \Omega)) = \{\mathbf{0}\}$ *and* $\partial^\infty F(\bar{u}, \bar{v}) + N(\bar{u}; \Omega) \subset N(\bar{u}; \Omega)$,

which all are automatic if F *is ELL at* $(\bar{u}, \bar{v})$.

Proof. Arguing by contradiction, suppose that $(\bar{x}, \bar{z})$ is not a global weak Pareto maximizer for problem (CP-max). Then we find $\bar{y} \in \Omega$ such that $F(\bar{y}) \cap \big(\bar{z} + \operatorname{int}\Theta\big) \neq \emptyset$. Since implication (33) yields that

$$F(\bar{y}) \cap \big(\bar{z} - \Theta\big) = \emptyset,$$

we conclude that $\bar{y} \notin \operatorname{lev}_\Omega(F; \bar{z}) = \operatorname{lev}(F; \bar{z}) \cap \Omega$. Consider further the auxiliary problem of *scalar constrained optimization*:

$$\begin{aligned} &\text{minimize} \quad \varphi(u) := \|u - \bar{y}\| \\ &\text{subject to} \ \ (u, v) \in \Xi := \Xi_1 \cap \Xi_2, \end{aligned} \tag{36}$$

where the closed sets Ξ_1 and Ξ_2 are given by

$$\Xi_1 := \operatorname{epi} F \quad \text{and} \quad \Xi_2 := \Omega \times \{\bar{z}\}.$$

Due to the fulfillment of the relationships

$$\begin{aligned} &(u, v) \in \Xi = \operatorname{epi} F \cap (\Omega \cap \{\bar{z}\}) \\ \Leftrightarrow\ & v - \theta \in F(u) \cap (\bar{z} - \Theta) \ \text{ for some } \ \theta \in \Theta \ \text{ and } \ u \in \Omega \\ \Leftrightarrow\ & u \in \operatorname{lev}(F; \bar{z}) \cap \Omega, \end{aligned}$$

the auxiliary problem (36) is equivalent to the problem of finding minimizers of $\varphi(u)$ over $\text{lev}_\Omega(F;\bar z)$. Since the latter set is assumed to be compact, the cost function φ attains its minimum value at some $\bar u \in \text{lev}_\Omega(F;\bar z)$. Fix an arbitrary element $\bar v \in \mathcal{E}_F(\bar u)$ with $(\bar u,\bar v)\in \Xi$.

Employing further the *necessary condition* for minimizers in (36) from [24, Proposition 5.3] and taking into account that φ is Lipschitz continuous give us

$$\mathbf{0} \in \partial\|\cdot - \bar y\|(\bar u) \times \{\mathbf{0}\} + N\big((\bar u,\bar v);\Xi_1\cap\Xi_2\big).$$

Since $\bar y \notin \text{lev}_\Omega(F;\bar z)$ and $\bar u \in \text{lev}_\Omega(F;\bar z)$, we have $\bar y \neq \bar u$ and

$$\partial\|\cdot - \bar y\|(\bar u) = \widehat{\partial}\|\cdot - \bar y\|(\bar u) = \{\nabla\|\cdot - \bar y\|(\bar u)\},$$

where the first equality is obvious by the convexity of the norm and the second one holds due to the Fréchet differentiability. Denoting $x^* := -\nabla\|\cdot - \bar y\|(\bar u)$ allows us to write

$$(x^*,\mathbf{0}) \in N\big((\bar u,\bar v);\text{epi}\,F\cap(\Omega\times\{\bar z\})\big).$$

Now we employ to the set intersection $\Xi_1\cap\Xi_2$ the result of [23, Theorem 3.4] and get

$$\begin{aligned}(x^*,\mathbf{0}) \in\ & N\big((\bar u,\bar v);\text{epi}\,F\big) + N\big((\bar u,\bar v);\Omega\times\{\bar z\}\big)\\ =\ & N\big((\bar u,\bar v);\text{epi}\,F\big) + N(\bar u;\Omega)\times Z^*\end{aligned} \tag{37}$$

provided the validity of all the assumptions imposed in this theorem. In more details:

- Since Θ is solid, Θ is SNC at the origin and $\text{epi}\,F$ is strongly PSNC with respect to Z.
- The PSNC assumption implies that either $\Xi_1 = \text{epi}\,F$ is SNC at $(\bar u,\bar v)$, or Ξ_2 is strongly PSNC at $\bar x$ with respect to X.
- Prior to checking for the sets $\{\Xi_1,\Xi_2\}$ at $(\bar u,\bar v)$ the validity of the *limiting qualification condition* required in [23, Theorem 3.4], we recall its definition: for any sequence $(u_{1k},v_{1k}) \xrightarrow{\text{epi}\ F} (\bar u,\bar v)$, $(u_{2k},v_{2k}) \xrightarrow{\Omega\times\{\bar z\}} (\bar u,\bar v)$, $(u^*_{1k},-v^*_{1k}) \in \widehat{N}\big((u_{1k},v_{1k});\text{epi}\,F\big)$ with $(u^*_{1k},-v^*_{1k}) \xrightarrow{w^*} (u^*_1,-v^*_1)$, and $(u^*_{2k},-v^*_{2k}) \in \widehat{N}(u_{2k};\Omega)\times Z^*$ with $(u^*_{2k},-v^*_{2k}) \xrightarrow{w^*} (u^*_2,-v^*_2)$ we have the following implication:

$$\|(u^*_{1k},-v^*_{1k}) + (u^*_{2k},-v^*_{2k})\| \to 0 \quad \Longrightarrow \quad (u^*_1,-v^*_1) = (u^*_2,-v^*_2) = \mathbf{0}.$$

By the assumption $\mathbf{0} \notin \partial F(\bar u,\bar v) + N(\bar u;\Omega)$, we have $v^*_1 = v^*_2 = \mathbf{0}$, since $\liminf\|v^*_{1k}\| \geq \|v^*_1\| > 0$ otherwise. Hence it gives

$$\frac{u^*_{1k}}{\|v^*_1\|} + \frac{u^*_{2k}}{\|v^*_1\|} \quad\longrightarrow\quad \mathbf{0},$$

and thus $\mathbf{0} \in \partial F(\bar{u}, \bar{v}) + N(\bar{u}; \Omega)$, which is a contradiction. Taking now into account the solidness of Θ, we get

$$u_1^* \in \partial^\infty F(\bar{u}, \bar{v}), \quad u_2^* \in N(\bar{u}; \Omega), \quad \text{and} \quad u_1^* + u_2^* = \mathbf{0}.$$

The qualification condition imposed in the theorem yields $u_1^* = u_2^* = \mathbf{0}$, which justifies the fulfilment of the limiting qualification condition of the intersection rule.

This allows us to deduce from (37) the existence of dual elements $(x_1^*, -z^*) \in N\big((\bar{u}, \bar{v}); \operatorname{epi} F\big)$ and $x_2^* \in N(\bar{u}; \Omega)$ satisfying the relationships

$$x^* = x_1^* + x_2^* \quad \text{and} \quad x_1^* \in D_N^* \mathcal{E}_F(\bar{u}, \bar{v})(z^*).$$

It is easy to see that the inclusion $x_1^* \in D_N^* \mathcal{E}_F(\bar{u}, \bar{v})(z^*)$ implies that $z^* \in -N(\mathbf{0}; \Theta)$. To finish the proof, we consider the following two cases:

Case 1. ($z^* = \mathbf{0}$). Since Θ is solid, it is SNC at the origin. Therefore, $D_M^* \mathcal{E}_F(\bar{u}, \bar{v})(\mathbf{0}) = D_N^* \mathcal{E}_F(\bar{u}, \bar{v})(\mathbf{0})$, and thus

$$x_1^* \in \partial^\infty F(\bar{x}, \bar{v}) \quad \text{with} \quad x^* = x_1^* + x_2^* \in \partial^\infty F(\bar{x}, \bar{v}) + N(\bar{u}; \Omega),$$

which contradicts the imposed qualification condition and the normal independence condition for Ω.

Case 2. ($z^* \neq \mathbf{0}$). Since $\|z^*\| \neq 0$, we have

$$\frac{x^*}{\|z^*\|} \in \partial F(\bar{u}, \bar{v}) + N(\bar{u}; \Omega) \subset N(\bar{u}, \Omega),$$

where the inclusion holds due to (35). Thus

$$x^* = -\nabla \| \cdot - \bar{y} \| (\bar{u}) \in N(\bar{x}; \Omega) \quad \text{and} \quad \bar{u} \neq \bar{y},$$

which contradicts the normal independence condition for Ω .

Since both cases lead to a contradiction, we conclude that $(\bar{x}, \bar{z})$ is a global weak maximizer for the problem (CP-max) and complete the proof of the theorem. ■

Finally in this section, we show how to derive from Theorem 7.1 the following sufficient condition in scalar Lipschitzian optimization over convex sets in finite dimensions first obtained by Dutta in [12, Theorem 3.2].

Corollary 7.2 (Sufficient condition in nonsmooth scalar optimization over convex sets). *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitzian function, and let $\Omega \subset \mathbb{R}^n$ be a closed and convex set. Take $\bar{x} \in \Omega$ such that*

$$\mathbf{0} \notin \partial \varphi(\bar{u}) + N(\bar{u}; \Omega) \quad \textit{whenever} \quad \bar{u} \in \Omega \quad \textit{with} \quad \varphi(\bar{u}) = \varphi(\bar{x}). \tag{38}$$

Then $\bar{x}$ is a global maximum of φ over Ω if

$$\partial\varphi(\bar{u}) \subset N(\bar{u};\Omega) \ \ \textit{for all} \ \ u \in \Omega \ \ \textit{with} \ \ \varphi(\bar{u}) = \varphi(\bar{x}). \tag{39}$$

Proof. It follows directly from Theorem 7.1. Indeed, implication (33) is automatic due to the single-valuedness of φ, while both SNC and qualification conditions hold by its Lipschitz continuity. The normal independence condition for the set Ω in the Euclidean space $\mathbb{R}^n$ follows from its convexity. Since the real numbers are ordered by the ordering cone $[0,\infty)$, we have $\bar{v} = f(\bar{u}) = \bar{z}$, and thus conditions (34) and (35) reduce to conditions (38) and (39), respectively. ■

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