Vietnam Journal of Mathematics **40**:2&3(2012) 165-180

Vietnam Journal of MATHEMATICS

Error Bounds for Vector-Valued Functions on Metric Spaces

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

Received November 6, 2011
Revised December 19, 2011

Abstract. In this paper, we attempt to extend the definition and existing local error bound criteria to vector-valued functions, or more generally, to functions taking values in a normed linear space. Some new primal space derivative-like objects – slopes – are introduced and a classification scheme of error bound criteria is presented.

2000 Mathematics Subject Classification. 49J52, 49J53, 58C06.

Key words. Variational analysis, error bounds, slope, vector variational principle.

1. Introduction

Error bounds play a key role in variational analysis. They are of great importance for optimality conditions, subdifferential calculus, stability and sensitivity issues, convergence of numerical methods, etc. For the summary of the theory of error bounds of real-valued functions and its various applications to sensitivity analysis, convergence analysis of algorithms, and penalty function methods in mathematical programming the reader is referred to the survey papers by Azé

The research of the second author was partially supported by the Australian Research Council, grant DP110102011.

[2], Lewis & Pang [25], Pang [33], as well as the book by Auslender & Teboule [1].

Numerous characterizations of the error bound property have been established in terms of various derivative-like objects either in the primal space (directional derivatives, slopes, etc.) or in the dual space (subdifferentials, normal cones) [3, 4, 7–12, 14–17, 19, 21–23, 28–34, 38–40]

In the present paper which continues [5], we attempt to extend the concept of error bounds as well as local error bound criteria to vector-valued functions defined on a metric space and taking values in a normed linear space. The presentation, terminology and notation follow that of [11, 12, 5]. Some new types of primal space derivative-like objects – slopes – which can be of independent interest, are introduced and a classification scheme of error bound criteria is presented.

The plan of the paper is as follows. In Section 2, we introduce an abstract ordering operation in the image space and define the error bound property for vector-valued functions. In Section 3, various kinds of slopes for a vector-valued function are defined. In Section 4, we establish primal space error bound criteria in terms of slopes, the main tool being the *vector variational principle* due to Bednarczuk and Zagrodny [6].

Our basic notation is standard, see [26, 35]. Depending on the context, X is either a metric or a normed space. Y is always a normed space. The closed unit ball in a normed space is denoted by $\mathbb{B}$. If A is a set in a metric or normed space, then $\operatorname{cl} A$, $\operatorname{int} A$, and $\operatorname{bd} A$ denote its closure, interior, and boundary respectively; $d(x, A) = \inf_{a \in A} \|x - a\|$ is the point-to-set distance. We also use the notation $\alpha^+ = \max(\alpha, 0)$, where $\alpha \in \mathbb{R}$.

2. Definitions

In this section, we define the error bound property for vector-valued functions and introduce other definitions which will be used in the rest of the paper.

Let $f : X \to Y$ where X is a metric space and Y is a real normed linear space.

We are going to consider an order operation in Y defined by a proper (i.e., $C \neq \{0\}$ and $C \neq Y$) closed convex cone $C \subset Y$ with nonempty interior: we say that v is dominated by y in Y if $v \in V_y := \{v \in Y |\ y - v \in C\} = y - C$.

If $Y = \mathbb{R}$, we will always assume that $C = \mathbb{R}_+$ and consider the usual distance: $d(v_1, v_2) = |v_1 - v_2|$.

We say that f is *C-lower semicontinuous* at $x \in X$ if for each neighborhood V of $f(x)$ there exists a neighborhood U of x such that $f(U) \subset V + C$. f is C-lower semicontinuous if it is C-lower semicontinuous at each $x \in X$. We say that f is *C-bounded* if there exists a bounded set $M \subset Y$ such that $f(X) \subset M + C$. Given subsets $A \subset X$ and $D \subset C$ such that $0 \notin D$ (D can be a singleton), and $\varepsilon > 0$, we say that an $x \in A$ is an *ε-minimal point* [27] of f with respect to D over A if $(f(x) - \varepsilon D - C) \cap f(A) = \emptyset$. If x is an ε-minimal point of f with respect to D

over A for any $\varepsilon > 0$, then it is called a *minimal point* of f with respect to D over A.

Let $y \in Y$ be given. We consider the y-sublevel set of f (with respect to the order defined by C):

$$S_y(f) := \{u \in X \mid y - f(u) \in C\}. \tag{1}$$

Condition $x \in S_y(f)$ means that x solves the inclusion $f(u) \in y - C$. If $x \notin S_y(f)$, it can be important to have estimates of the distance $d(x, S_y(f))$ in terms of the value $f(x)$.

Given f and y, we construct two new functions $f_y : X \to Y$ and $f_y^+ : X \to \mathbb{R}_+$ defined as follows:

$$f_y(u) := \begin{cases} f(u) & \text{if } u \notin S_y(f), \\ y & \text{if } u \in S_y(f); \end{cases} \tag{2}$$

$$f_y^+(u) := d(y - f(u), C). \tag{3}$$

The function f_y differs from f only on the sublevel set $S_y(f)$, where it is assigned the constant value y. Inclusion $x \in S_y(f)$ is obviously equivalent to equalities $f_y(x) = y$ and $f_y^+(x) = 0$.

Some simple properties of the y-sublevel set (1) and functions (2) and (3) are summarized in the next two propositions.

Proposition 2.1. (i) *$f_y^+(x) = d(y - f_y(x), C)$;*

(ii) *$S_y(f) = S_y(f_y)$;*

(iii) *$y - f_y(x) \notin \operatorname{int} C$ for any $x \in X$;*

(iv) *If $x \in S_y(f)$ and $f(x) - f(u) \in C$, then $u \in S_y(f)$;*

(v) *If $f(x) - f(u) \in C$, then $f_y^+(x) \geq f_y^+(u)$;*

(vi) *If $x \notin S_y(f)$ and f is C-lower semicontinuous at x, then there exists a neighborhood U of x such that $u \notin S_y(f)$ and consequently $f_y(u) = f(u)$ for all $u \in U$;*

(vii) *If f_y is C-lower semicontinuous at x, then f_y^+ is lower semicontinuous at x.*

Proof. (i) If $x \notin S_y(f)$, then $f_y(x) = f(x)$ and (i) coincides with (3). If $x \in S_y(f)$, then $f_y^+(x) = 0$ and $f_y(x) = y$. It follows from the last equality that $d(y - f_y(x), C) = d(0, C) = 0$.

(ii) Let $x \notin S_y(f)$, i.e., $y - f(x) \notin C$. By (2), $f_y(x) = f(x)$. Hence, $y - f_y(x) \notin C$, i.e., $x \notin S_y(f_y)$.

Let $x \in S_y(f)$. By (2), $f_y(x) = y$, and consequently $y - f_y(x) = 0 \in C$, i.e., $x \in S_y(f_y)$.

(iii) If $y - f_y(x) \in C$, i.e., $x \in S_y(f_y)$, then by (ii), $x \in S_y(f)$, and consequently $f_y(x) = y$. Hence, $y - f_y(x) = 0 \notin \operatorname{int} C$ since C is a proper cone.

(iv) The first inclusion can be rewritten equivalently as $y - f(x) \in C$. The conclusion follows after adding this condition with the second one and taking into account the convexity of C.

(v) Let $f(x) - f(u) \in C$. Then

$$f_y^+(x) = d(y - f(x), C) \geq d(y - f(u), C) - d(f(x) - f(u), C) \\ = d(y - f(u), C) = f_y^+(u).$$

(vi) Let $x \notin S_y(f)$ and f be C-lower semicontinuous at x. Then $y - f(x) \notin C$ and for a sufficiently small neighborhood V of $f(x)$ it holds $y \notin V + C$. By the definition of C-lower semicontinuity, there exists a neighborhood U of x such that $f(U) \subset V + C$. Hence, for all $u \in U$, we have $f(u) \neq y$, and consequently $f_y(u) = f(u)$.

(vii) Let f_y be C-lower semicontinuous at x.

Consider first the case $x \notin S_y(f)$, i.e., $y - f(x) \notin C$. Take an $\varepsilon > 0$. For some $\varepsilon' \in (0, \varepsilon)$ it holds $y - f(x) \notin C + \varepsilon'\mathbb{B}$. By the definition of C-lower semicontinuity, there exists a neighborhood U of x such that $f_y(U) \subset f(x) + \varepsilon'\mathbb{B} + C$. Hence, for all $u \in U$, we have $f_y(u) \neq y$, and consequently $f_y(u) = f(u) = f(x) + v + c$ for some $v \in Y$ with $\|v\| \leq \varepsilon'$ and $c \in C$. Then

$$f_y^+(u) = d(y - f(u), C) \geq d(y - f(x), C) - d(v + c, C) \geq f_y^+(x) - \varepsilon,$$

and consequently f_y^+ is lower semicontinuous at x.

If $x \in S_y(f)$, then $f_y^+(x) = 0 \leq f_y^+(u)$ for all $u \in X$, and f_y^+ is trivially lower semicontinuous at x. ■

Note that, by (vi), the C-lower semicontinuity of f implies the closedness of the sublevel sets $S_y(f)$ of f. In general, the converse implication is not true unless f is a real-valued function (cf. [13]). It is easily seen that, if f is C-lower semicontinuous at x, then f_y is C-lower semicontinuous at x. In general, the converse implication is not true unless $x \notin S_y(f)$.

Proposition 2.2. *If $Y = \mathbb{R}$, then*

$$S_y(f) = \{u \in X | \ f(u) \leq y\}, \\ f_y(u) = \max(f(u), y), \\ f_y^+(u) = (f(u) - y)^+ = f_y(u) - y.$$

For a function $f : X \to Y$, its directional derivative at x in the direction u, if it exists, is denoted by $f'(x; u)$. If $f'(x; u)$ exists for all $u \in X$, we say that f is directionally differentiable at x.

If $A \subset X$ and $x \in \operatorname{cl} A$, then $I_A(x) := \{u \in X | \ \exists t_k \downarrow 0, \ x + t_k u \in A\}$ is the cone of internal directions to A at x.

Proposition 2.3. *Let X be a normed linear space, f be directionally differentiable at $x \in X$ and $y = f(x)$. If $u \notin I_{S_y(f)}(x)$, then $f'_y(x;u) = f'(x;u) \notin -\operatorname{int} C$.*

Proof. Let $u \notin I_{S_y(f)}(x)$. Then $x + tu \notin S_y(f)$, and consequently $f_y(x+tu) = f(x+tu)$ for all sufficiently small $t > 0$. By the definition of directional derivative, $f'_y(x;u) = f'(x;u)$. If $f'(x;u) \in -\operatorname{int} C$, then $f(x+tu) - f(x) \in -C$, and consequently $y - f(x+tu) \in C$ for all sufficiently small $t > 0$. This contradicts the assumption that $u \notin I_{S_y(f)}(x)$. ■

Given $x \in S_y(f)$, we say that f satisfies the (local) *error bound* property (cf. [5]) at x relative to y if there exist a $\gamma > 0$ and a $\delta > 0$ such that

$$d(u, S_y(f)) \leq \gamma f_y^+(u) \quad \forall u \in B_\delta(x). \tag{4}$$

It is easy to see that the inequality in (4) needs to be checked only for $u \notin S_y(f)$.

The error bound property can be characterized by certain derivative-like objects defined in terms of functions f_y and f_y^+.

3. Slopes of vector valued functions

In this section, several kinds of slopes are defined with the help of the function (2). A different approach to defining slopes in terms of the scalar function (3) was considered in [5].

3.1. Slopes. Recall that in the case $f : X \to \mathbb{R}_\infty$, the *slope* of f at $x \in X$ (with $f(x) < \infty$) is defined as (see, for instance, [18])

$$|\nabla f|(x) := \limsup_{u \to x} \frac{(f(x) - f(u))^+}{d(u,x)}. \tag{5}$$

There are several ways of extending this definition to the vector setting $f : X \to Y$. In [5], it was suggested to apply definition (5) to the scalar function (3), namely the *slope of f at $x \in X$ relative to $y \in Y$* was defined as

$$|\nabla f|_y(x) := |\nabla f_y^+|(x) = \limsup_{u \to x} \frac{(f_y^+(x) - f_y^+(u))^+}{d(u,x)}. \tag{6}$$

Below we introduce two more definitions, formulated in terms of the original vector-valued function f.

Denote $\overset{\circ}{C} := \{c \in \operatorname{int} C |\ d(c, \operatorname{bd} C) = 1\}$. It is easy to check that $\overset{\circ}{C}$ generates $\operatorname{int} C$, i.e., $\operatorname{int} C = \cup_{t>0}(t\overset{\circ}{C})$. If $Y = \mathbb{R}$, then $\overset{\circ}{C} = \{1\}$.

We define respectively the *lower* and *upper slopes* of f at $x \in X$ as

$$^{-}|\nabla f|(x) := \limsup_{u\to x} \sup\left\{ r\geq 0 \;\middle|\; \frac{f(x)-f(u)}{d(u,x)} \in C + r\overset{\circ}{C} \right\}, \tag{7}$$

$$^{+}|\nabla f|(x) := \limsup_{u\to x} \inf\left\{ r\geq 0 \;\middle|\; \frac{f(u)-f(x)}{d(u,x)} \in C + r\mathbb{B} \right\}. \tag{8}$$

The conventions $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$ are in force in definitions (7) and (8).

It always holds $^{-}|\nabla f|(x) \geq 0$ and $^{+}|\nabla f|(x) \geq 0$ while each of the equalities $^{-}|\nabla f|(x) = 0$ and $^{+}|\nabla f|(x) = 0$ defines a kind of *stationarity* property of f. If either of the slopes is strictly positive, then it characterizes quantitatively the *descent rate* of f at x.

Proposition 3.1. *Let $u \neq x$. If $r_1 \geq 0$, $r_2 \geq 0$,*

$$\frac{f(x)-f(u)}{d(u,x)} \in C + r_1\overset{\circ}{C}, \quad \text{and} \quad \frac{f(u)-f(x)}{d(u,x)} \in C + r_2\mathbb{B}, \tag{9}$$

then $r_1 \leq r_2$.

Proof. When $r_1 = 0$, the inequality holds automatically. Let $r_1 > 0$. Inclusions (9) imply

$$0 \in C + r_1\overset{\circ}{C} + r_2\mathbb{B} = C + r_1(\overset{\circ}{C} + (r_2/r_1)\mathbb{B}).$$

If $r_2 < r_1$, then $\overset{\circ}{C} + (r_2/r_1)\mathbb{B} \subset \operatorname{int} C$. This implies $0 \in \operatorname{int} C$, a contradiction. ■

The last proposition together with definitions (7) and (8) immediately implies the inequality

$$^{-}|\nabla f|(x) \leq {}^{+}|\nabla f|(x) \tag{10}$$

which justifies the terms "lower" and "upper" in the names of the two slopes.

Inequality (10) can be strict rather often. It is easy to see from definition (7) that only those points u can contribute to the value of $^{-}|\nabla f|(x)$ for which $f(x) - f(u) \in \operatorname{int} C$ because otherwise $(f(x) - f(u))/d(u,x) \notin C + r\overset{\circ}{C}$ for any $r > 0$. Hence, $^{-}|\nabla f|(x) = 0$ if x is minimal.

Example 3.2. Let $X = \mathbb{R}$, $Y = (\mathbb{R}^2, \|\cdot\|)$, where $\|(y_1, y_2)\| = \max(|y_1|, |y_2|)$, $C = \mathbb{R}^2_+$, and $f(u) = (-2u, u)$, $u \in \mathbb{R}$. Then $(f(x) - f(u))/d(u,x) = (-2, 1)\operatorname{sgn}(x-u)$ for any $x \in \mathbb{R}$ and $u \in \mathbb{R}$, $u \neq x$. Hence, $(f(x)-f(u))/d(u,x) \notin C + rc$ for any $c \in \operatorname{int} C$ and any $r > 0$, while $(f(u) - f(x))/d(u,x) \in C + r\mathbb{B}$ if and only if either $u < x$, $r \geq 1$ or $r \geq 2$. This results in $^{-}|\nabla f|(x) = 0$ and $^{+}|\nabla f|(x) = 2$ for all $x \in \mathbb{R}$.

Given $y = (y_1, y_2) \in \mathbb{R}^2$, we can also compute slope (6). It obviously depends on the value of x and the choice of y. One can easily compute $d(y - f(u), C) = \max\{(-y_1 - 2u)^+, (u - y_2)^+\}$. There are several possibilities for the value of $\mu(u) := (d(y - f(x), C) - d(y - f(u), C))^+$:

- if $y_1 + 2x \geq 0$ and $x - y_2 \leq 0$, i.e., $-y_1/2 \leq x \leq y_2$, then $\mu(u) = 0$ for all $u \in \mathbb{R}$, and consequently $|\nabla f|_y(x) = 0$;
- if $y_1+2x \geq 0$ and $x-y_2 > 0$, i.e., $-y_1/2 \leq x$ and $y_2 < x$, then $\mu(u) = (x-u)^+$ for all $u \in \mathbb{R}$ close to x, and consequently $|\nabla f|_y(x) = 1$;
- if $y_1+2x < 0$ and $x-y_2 \leq 0$, i.e., $-y_1/2 < x$ and $y_2 \leq x$, then $\mu(u) = 2(u-x)^+$ for all $u \in \mathbb{R}$ close to x, and consequently $|\nabla f|_y(x) = 2$;
- if $y_1 + 2x < 0$ and $x - y_2 > 0$, i.e., $y_2 < x < -y_1/2$, then $\mu(u) = \max\{2(u - x)^+, (x - u)^+\}$ for all $u \in \mathbb{R}$ close to x, and consequently $|\nabla f|_y(x) = 2$.

Thus $|\nabla f|_y(x)$ can take three different values between those of ${}^-|\nabla f|(x)$ and ${}^+|\nabla f|(x)$.

In general, it holds

$$|\nabla f|_y(x) \leq {}^+|\nabla f|(x) \tag{11}$$

for any $y \in Y$. Indeed,

$$f_y^+(x) - f_y^+(u) = d(y - f(x), C) - d(y - f(u), C) \leq d(f(u) - f(x), C),$$

and consequently

$$\frac{(f_y^+(x) - f_y^+(u))^+}{d(u,x)} \leq d\left(\frac{f(u) - f(x)}{d(u,x)}, C\right) \leq r$$

for all $u \neq x$ and any $r \geq 0$ such that $(f(u) - f(x))/d(u,x) \in C + r\mathbb{B}$. Inequality (11) follows from definitions (6) and (8).

Example 3.3. Let $X = \mathbb{R}$, $Y = (\mathbb{R}^2, \|\cdot\|)$, where $\|(y_1, y_2)\| = \max(|y_1|, |y_2|)$, $C = \mathbb{R}^2_+$, and $f(u) = (2u, u)$, $u \in \mathbb{R}$. Then $(f(x)-f(u))/d(u,x) = (2,1)\,\mathrm{sgn}(x-u)$ for any $x \in \mathbb{R}$ and $u \in \mathbb{R}$, $u \neq x$. Like in Example 3.2, $(f(u) - f(x))/d(u,x) \in C + r\mathbb{B}$ if and only if either $x > u$ or $r \geq 2$ which implies ${}^+|\nabla f|(x) = 2$ for any $x \in \mathbb{R}$. At the same time, $\overset{\circ}{C} = \{(c_1, c_2) \in \mathbb{R}^2_+ |\ \min\{c_1, c_2\} = 1\}$ and $(f(x) - f(u))/d(u,x) \in C + r\overset{\circ}{C}$ for some $r \geq 0$ if $x > u$ and $r \leq \sup_{(c_1,c_2)\in\overset{\circ}{C}} \min\{2/c_1, 1/c_2\} = 1$. Hence, ${}^-|\nabla f|(x) = 1$ for all $x \in \mathbb{R}$.

Let $y = (y_1, y_2) \in \mathbb{R}^2$. Then $d(y - f(u), C) = \max\{(2u - y_1)^+, (u - y_2)^+\}$. $|\nabla f|_y(x)$ depends on the values of $\mu(u) = (d(y - f(x), C) - d(y - f(u), C))^+$ for u near x:

- if $2x - y_1 \leq 0$ and $x - y_2 \leq 0$, i.e., $x \leq \min\{y_1/2, y_2\}$, then $\mu(u) = 0$ for all $u \in \mathbb{R}$, and consequently $|\nabla f|_y(x) = 0$;
- if $2x - y_1 \leq 0$ and $x - y_2 > 0$, i.e., $y_2 < x \leq y_1/2$, then $\mu(u) = (x - u)^+$ for all $u \in \mathbb{R}$ close to x, and consequently $|\nabla f|_y(x) = 1$;
- if $2x-y_1 > 0$ and $x-y_2 \leq 0$, i.e., $y_1/2 < x$ and $y_2 \leq x$, then $\mu(u) = 2(x-u)^+$ for all $u \in \mathbb{R}$ close to x, and consequently $|\nabla f|_y(x) = 2$;

- if $2x - y_1 > 0$ and $x - y_2 > 0$, i.e., $x > \max\{y_1/2, y_2\}$, then $\mu(u) = \max\{2(x-u)^+, (x-u)^+\} = 2(x-u)^+$ for all $u \in \mathbb{R}$ close to x, and consequently $|\nabla f|_y(x) = 2$.

Again, $|\nabla f|_y(x)$ takes three different values. For certain values of y it can be smaller than ${}^{-}|\nabla f|(x)$.

Another type of slope will be used in the sequel. The *internal slope* of f at $x \in X$ is defined as

$$ {}^{0}|\nabla f|(x) := \liminf_{u \to x} \sup \left\{ r \geq 0 \;\middle|\; \frac{f(u) - f(x)}{d(u,x)} + r\mathbb{B} \subset C \right\}. \tag{12} $$

The convention $\sup \emptyset = 0$ is in force here.

Condition ${}^{0}|\nabla f|(x) > 0$ obviously implies that $\frac{f(u)-f(x)}{d(u,x)} \in \operatorname{int} C$ for all $u \neq x$ in a neighborhood of x. Unlike slopes (7) and (8), slope (12) characterizes quantitatively the (minimal) *ascent rate* of f at x. The next proposition provides a further comparison of the slopes.

Proposition 3.4. (i) *Slopes* (8) *and* (12) *cannot be positive simultaneously;*

(ii) *Slopes* (7) *and* (12) *cannot be positive simultaneously.*

Proof. If ${}^{+}|\nabla f|(x) > 0$, then by definition (8), there exists an $r > 0$ such that $\frac{f(u)-f(x)}{d(u,x)} \notin C + r\mathbb{B}$ for all $u \neq x$ in a neighborhood of x. Particularly, $\frac{f(u)-f(x)}{d(u,x)} \notin C$ and it follows from (12) that ${}^{0}|\nabla f|(x) = 0$. This proves part (i). Part (ii) is a consequence of part (i) due to (10). ∎

The role of internal slope (12) is similar to that of the constant

$$ |\nabla f|^{0}(x) := \liminf_{u \to x} \frac{f^{+}_{f(x)}(u)}{d(u,x)} \tag{13} $$

introduced in [5]. In (13), $f^{+}_{f(x)}(u)$ denotes $f^{+}_{y}(u)$ (defined by (3)) with $y = f(x)$. Constant (13) also characterizes quantitatively the ascent rate of f at x.

Proposition 3.5. *If C is a pointed cone (i.e., $C \cap (-C) = \{0\}$), then ${}^{0}|\nabla f|(x) \leq |\nabla f|^{0}(x)$.*

Proof. If ${}^{0}|\nabla f|(x) = 0$, the inequality is trivial. If ${}^{0}|\nabla f|(x) > r > 0$, then by definition (12), for all u in a neighborhood of x it holds $f(u) - f(x) + rd(u,x)\mathbb{B} \subset C$, and consequently $d(f(u) - f(x), Y \setminus C) \geq rd(u,x)$. Since C is a pointed cone, $-C \setminus \{0\} \subset Y \setminus C$, and consequently $f^{+}_{f(x)}(u) = d(f(x) - f(u), C) \geq rd(u,x)$. It follows from definition (13) that $|\nabla f|^{0}(x) \geq r$. As $r < {}^{0}|\nabla f|(x)$ is arbitrary, the assertion is proved. ∎

If $Y = \mathbb{R}$, then (7) and (8) reduce to (5) while constants (12) and (13) coincide.

Proposition 3.6. *If* $Y = \mathbb{R}$, *then*

$$ {}^{-}|\nabla f|(x) = {}^{+}|\nabla f|(x) = \limsup_{u \to x} \frac{(f(x) - f(u))^{+}}{d(u,x)}, $$
$$ {}^{0}|\nabla f|(x) = |\nabla f|^{0}(x) = \liminf_{u \to x} \frac{(f(u) - f(x))^{+}}{d(u,x)}. $$

Proof. Let $Y = \mathbb{R}$. Then $\overset{\circ}{C} = \{1\}$ and $C + r\overset{\circ}{C} = \mathbb{R}_{+} + r$ while $C + r\mathbb{B} = \mathbb{R}_{+} - r$. Hence, $\mu \notin C + r\overset{\circ}{C} \Leftrightarrow \mu < r$ and $\mu \notin C + r\mathbb{B} \Leftrightarrow \mu < -r$, and it holds

$$ \sup \left\{ r \geq 0 \,\middle|\, \frac{f(x) - f(u)}{d(u,x)} \in C + r\overset{\circ}{C} \right\} = \inf \left\{ r \geq 0 \,\middle|\, \frac{f(u) - f(x)}{d(u,x)} \in C + r\mathbb{B} \right\} $$
$$ = \frac{(f(x) - f(u))^{+}}{d(u,x)}. $$

At the same time,

$$ \sup \left\{ r \geq 0 \,\middle|\, \frac{f(u) - f(x)}{d(u,x)} + r\mathbb{B} \subset C \right\} = \sup \left\{ r \geq 0 \,\middle|\, \frac{f(u) - f(x)}{d(u,x)} \geq r \right\} $$
$$ = \frac{(f(u) - f(x))^{+}}{d(u,x)} = \frac{d(f(x) - f(u), C)}{d(u,x)} = \frac{f^{+}_{f(x)}(u)}{d(u,x)}. $$

The conclusions follow from the definitions. ■

Proposition 3.7. *Let* X *be a normed linear space. If* f *is directionally differentiable at* x *in the direction* $u \neq 0$, *then*

$$ f'(x;u) \notin -\left(\operatorname{int} C + \|u\| \; {}^{-}|\nabla f|(x)\overset{\circ}{C} \right), $$
$$ f'(x;u) \in C + \|u\| \; {}^{+}|\nabla f|(x)\mathbb{B}. $$

Proof. If $f'(x;u) \in -(\operatorname{int} C + \|u\| \; {}^{-}|\nabla f|(x)c)$ for some $c \in \overset{\circ}{C}$, then with $x_t = x + tu$, it holds

$$ \frac{f(x) - f(x_t)}{d(x_t, x)} \in C + ({}^{-}|\nabla f|(x) + \varepsilon)c $$

for some $\varepsilon > 0$ and all sufficiently small $t > 0$. This contradicts definition (7) of the slope.

If $f'(x;u) \notin C + \|u\| \; {}^{+}|\nabla f|(x)\mathbb{B}$, then with $x_t = x + tu$, it holds

$$ \frac{f(x_t) - f(x)}{d(x_t, x)} \notin C + ({}^{+}|\nabla f|(x) + \varepsilon)\mathbb{B} $$

for some $\varepsilon > 0$ and all sufficiently small $t > 0$. This contradicts definition (8) of the upper slope. ■

One of the consequences of Proposition 3.7 is the fact that, in the directionally differentiable case, condition ${}^{-}|\nabla f|(x) = 0$ implies that $f'(x;u) \notin -\mathrm{int}\, C$ for all $u \in X$ which means that x is a stationary point of f in the sense of Smale [36, 37].

3.2. Strict slopes. Given a fixed point $\bar{x} \in X$, denote $\bar{y} = f(\bar{x})$. One can use the collection of slopes (6), (7) and (8) computed at nearby points to define more robust objects – the *strict outer slope* [5]

$$\overline{|\nabla f|}^{>}(\bar{x}) = \liminf_{x\to\bar{x},\ f^+_{\bar{y}}(x)\downarrow 0} |\nabla f|_{\bar{y}}(x) \tag{14}$$

as well as the *strict outer lower* and *strict outer upper slopes* of f at $\bar{x}$:

$$^{-}\overline{|\nabla f|}^{>}(\bar{x}) = \liminf_{x\to\bar{x},\ f^+_{\bar{y}}(x)\downarrow 0} {}^{-}|\nabla f|(x), \tag{15}$$

$$^{+}\overline{|\nabla f|}^{>}(\bar{x}) = \liminf_{x\to\bar{x},\ f^+_{\bar{y}}(x)\downarrow 0} {}^{+}|\nabla f|(x). \tag{16}$$

Note that condition $f^+_{\bar{y}}(x) \downarrow 0$ in (14), (15) and (16) does not imply in general that $f(x) \to \bar{y} = f(\bar{x})$. It only means that $d(\bar{y} - f(x), C) \downarrow 0$ or, equivalently, $d(f(x), \bar{y} - C) \downarrow 0$.

Constants (14) – (16) provide lower estimates for the "uniform" descent rate of f near $\bar{x}$. The word "strict" reflects the fact that slopes at nearby points contribute to the definitions (14) – (16) making them analogues of the strict derivative. The word "outer" is used to emphasize that only points outside the set $S_{\bar{y}}(f)$ are taken into account.

Condition (10) implies the inequality

$$^{-}\overline{|\nabla f|}^{>}(\bar{x}) \le {}^{+}\overline{|\nabla f|}^{>}(\bar{x}).$$

If $Y = \mathbb{R}$, then, due to Proposition 3.6, definitions (15) and (16) take the form

$$^{-}\overline{|\nabla f|}^{>}(\bar{x}) = {}^{+}\overline{|\nabla f|}^{>}(\bar{x}) = \liminf_{x\to\bar{x},\ f(x)\downarrow f(\bar{x})} |\nabla f|(x)$$

and coincide with the strict outer slope defined in [11].

The strict outer slopes (14), (15) and (16) are limits of the usual slopes $|\nabla f|_{\bar{y}}(x)$, ${}^{-}|\nabla f|(x)$ and ${}^{+}|\nabla f|(x)$, respectively, which themselves are limits and do not take into account how close to $\bar{x}$ the point x is. This can be important when characterizing error bounds. In view of this observation, the next definitions can be of interest.

The *uniform strict slope* [5] of f at $\bar{x}$:

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x\to\bar{x},\ f^+_{\bar{y}}(x)\downarrow 0}\ \sup_{u\ne x} \frac{(f^+_{\bar{y}}(x) - f^+_{\bar{y}}(u))^+}{d(u,x)} \tag{17}$$

and the *uniform strict lower* and *upper slopes* of f at $\bar{x}$:

$$^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x\to\bar{x},\ f_{\bar{y}}^{+}(x)\downarrow 0}\ \sup_{u\neq x}\sup\left\{r\geq 0 \;\middle|\; \frac{f(x)-f_{\bar{y}}(u)}{d(u,x)}\in C+r\overset{\circ}{C}\right\}, \tag{18}$$

$$^{+}\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x\to\bar{x},\ f_{\bar{y}}^{+}(x)\downarrow 0}\ \sup_{u\neq x}\inf\left\{r\geq 0 \;\middle|\; \frac{f_{\bar{y}}(u)-f(x)}{d(u,x)}\in C+r\mathbb{B}\right\}. \tag{19}$$

If f is C-lower semicontinuous near $\bar{x}$, then

$$^{-}\overline{|\nabla f|}^{>}(\bar{x}) \leq {}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}), \quad {}^{+}\overline{|\nabla f|}^{>}(\bar{x}) \leq {}^{+}\overline{|\nabla f|}^{\diamond}(\bar{x}).$$

The inequalities follow from definitions (7), (8), (15), (16), (18), and (19) and Proposition 2.1 (vi), while the inequality

$$^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) \leq {}^{+}\overline{|\nabla f|}^{\diamond}(\bar{x})$$

follows from Proposition 3.1.

Using the same arguments as in the proof of Proposition 3.6, we can easily show that in the scalar case the defined above uniform strict slopes (17)–(19) reduce to the scalar uniform strict slope from [12].

Proposition 3.8. *If $Y=\mathbb{R}$, then*

$$^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) = {}^{+}\overline{|\nabla f|}^{\diamond}(\bar{x}) = \overline{|\nabla f|}^{\diamond}(\bar{x}) = \limsup_{x\to\bar{x},\ f(x)\downarrow\bar{y}}\ \sup_{u\neq x}\frac{(f(x)-f_{\bar{y}}(u))^{+}}{d(u,x)}.$$

4. Criteria in terms of slopes

In this section, several primal space error bound criteria in terms of various kinds of slopes are established.

4.1. Main criterion. To establish sufficient error bound criteria we use the following *vector variational principle.*

Theorem 4.1. [6] *Let X be complete and D be a nonempty closed convex and bounded subset of C such that $0\notin \operatorname{cl}(D+C)$. Let $f: X\to Y$ be C-lower semicontinuous and C-bounded. Then for every $x\in X$, $\varepsilon>0$ and $\lambda>0$ there exists a $w\in X$ such that*

(a) $(f(x)-C)\cap(f(w)+\varepsilon d(x,w)D)\neq\emptyset$,
(b) $(f(w)-C)\cap(f(u)+\varepsilon d(u,w)D)=\emptyset$ *for every* $u\neq w$,
(c) *moreover, if x is an $\varepsilon\lambda$-minimal point of f with respect to D, then*

$$d(x,w)<\lambda.$$

Let an $\bar{x}\in X$ be fixed and $\bar{y}=f(\bar{x})$. We are examining the error bound property (4) at $\bar{x}$ relative to $\bar{y}$: there exist a $\gamma>0$ and a $\delta>0$ such that

$$d(x, S_{\bar{y}}(f)) \leq \gamma f_{\bar{y}}^{+}(x) \quad \forall x \in B_\delta(\bar{x}). \tag{20}$$

It can be equivalently defined in terms of the *error bound modulus* [5] of f at $\bar{x}$:

$$\operatorname{Er} f(\bar{x}) := \liminf_{x \to \bar{x},\, x \notin S_{\bar{y}}(f)} \frac{f_{\bar{y}}^{+}(x)}{d(x, S_{\bar{y}}(f))}, \tag{21}$$

which coincides with the supremum of γ^{-1} over all γ such that (20) holds for some $\delta > 0$. Hence, the error bound property holds for f at $\bar{x}$ if and only if $\operatorname{Er} f(\bar{x}) > 0$.

The error bound modulus (21) admits the following representation which will be used in the sequel:

$$\operatorname{Er} f(\bar{x}) = \liminf_{x \to \bar{x},\, x \notin S_{\bar{y}}(f)} \sup \left\{ r \geq 0 \;\middle|\; \frac{\bar{y} - f(x)}{d(x, S_{\bar{y}}(f))} \notin C + r\mathbb{B} \right\}. \tag{22}$$

Indeed, taking into account definition (3), we have

$$\begin{aligned} \frac{f_{\bar{y}}^{+}(x)}{d(x, S_{\bar{y}}(f))} = \frac{d(\bar{y} - f(x), C)}{d(x, S_{\bar{y}}(f))} &= d\left(\frac{\bar{y} - f(x)}{d(x, S_{\bar{y}}(f))}, C \right) \\ &= \sup \left\{ r \geq 0 \;\middle|\; \frac{\bar{y} - f(x)}{d(x, S_{\bar{y}}(f))} \notin C + r\mathbb{B} \right\}. \end{aligned}$$

The claimed representation (22) follows from definition (21).

The next theorem provides the relationship between the error bound modulus and the uniform strict outer slopes.

Theorem 4.2. (i) $\operatorname{Er} f(\bar{x}) \leq {}^{+}\overline{|\nabla f|}^{\diamond}(\bar{x})$;

(ii) *If X is complete and $f_{\bar{y}}$ is C-lower semicontinuous and C-bounded, then* $\operatorname{Er} f(\bar{x}) \geq {}^{-}\overline{|\nabla f|}^{\diamond}(\bar{x})$.

Proof. (i) If $\operatorname{Er} f(\bar{x}) = 0$, the inequality holds trivially. Let $0 < \gamma < \operatorname{Er} f(\bar{x})$. We are going to show that ${}^{+}\overline{|\nabla f|}^{\diamond}(\bar{x}) \geq \gamma$. By representation (22) of the error bound modulus, there is a $\delta > 0$ such that for any $x \in B_\delta(\bar{x})$ with $\bar{y} - f(x) \notin C$ one can find an $r > \gamma$ with the property

$$\frac{\bar{y} - f(x)}{d(x, S_{\bar{y}}(f))} \notin C + r\mathbb{B}.$$

Choose a $w \in S_{\bar{y}}(f)$ such that

$$\frac{\bar{y} - f(x)}{d(x, w)} \notin C + \gamma\mathbb{B}$$

and recall that $f_{\bar{y}}(w) = \bar{y}$. If

$$\frac{f_{\bar y}(w)-f(x)}{d(w,x)}\in C+r\mathbb{B}$$

for some $r\ge 0$, then necessarily $r>\gamma$, and consequently $\gamma\le {}^{+}\overline{|\nabla f|}^{\diamond}(\bar x)$.

(ii) Let $\gamma>\gamma'>\mathrm{Er}_{\bar y} f(\bar x)$ and $\delta>0$. By definition (21) of the error bound modulus, there is an $x\in B_{\delta\min(1/2,\gamma^{-1})}(\bar x)$ such that $x\notin S_{\bar y}(f)$, i.e., $f^+_{\bar y}(x)>0$ and

$$f^+_{\bar y}(x)<\gamma' d(x,S_{\bar y}(f)). \tag{23}$$

In other words, $\bar y-f(x)\notin C$ and $\bar y-f(x)\in C+\gamma' d(x,S_{\bar y}(f))\mathbb{B}$. It follows that $f_{\bar y}(x)=f(x)$ and, for any $c\in\overset{\circ}{C}$ (since $c+\mathbb{B}\subset C$ or $\mathbb{B}\subset C-c$),

$$\bar y-f_{\bar y}(x)\in C-\gamma' d(x,S_{\bar y}(f))c. \tag{24}$$

Denote $\varepsilon:=\gamma d(x,S_{\bar y}(f))$ and $\lambda:=\varepsilon/\gamma$. We are going to apply Theorem 4.1.

x is an ε-minimal point of $f_{\bar y}$ with respect to c. Indeed, if there exists a point $u\in X$ such that $f_{\bar y}(u)\in f_{\bar y}(x)-\varepsilon c-C$, then comparing this inclusion with (24), we obtain

$$\bar y-f_{\bar y}(u)\in C+(\gamma-\gamma')d(x,S_{\bar y}(f))c\subset \operatorname{int} C,$$

which contradicts Proposition 2.1 (iii).

By Theorem 4.1, there exists a $w\in X$ such that

(a) $d(w,x)<\lambda$,
(b) $f_{\bar y}(w)\in f(x)-C$,
(c) $f_{\bar y}(u)+\gamma d(u,w)c\notin f_{\bar y}(w)-C$ for all $u\ne w$.

Below we discuss some implications of conditions (a)–(c).

(a) Since $\lambda=d(x,S_{\bar y}(f))$, it holds $f_{\bar y}(w)=f(w)$ and $f^+_{\bar y}(w)>0$. Furthermore, we have the following estimates:

$$d(w,\bar x)\le d(w,x)+d(x,\bar x)<d(x,S_{\bar y}(f))+d(x,\bar x)\le 2d(x,\bar x)\le\delta. \tag{25}$$

(b) By Proposition 2.1 (v) and (23),

$$0<f^+_{\bar y}(w)\le f^+_{\bar y}(x)<\gamma' d(x,\bar x)\le(\gamma'/\gamma)\delta<\delta. \tag{26}$$

(c) can be rewritten equivalently as

$$\frac{f_{\bar y}(w)-f_{\bar y}(u)}{d(u,w)}\notin C+\gamma c\quad\text{for all } u\ne w.$$

Hence,

$$\sup_{u\ne w}\sup\left\{r\ge 0\;\middle|\;\frac{f_y(w)-f_y(u)}{d(u,w)}\in C+r\overset{\circ}{C}\right\}\le\gamma. \tag{27}$$

It follows from estimates (27), (25), and (26) and definition (18) that $\overline{|\nabla f|}^{\diamond}(\bar{x}) \leq \gamma$, and consequently $\overline{|\nabla f|}^{\diamond}(\bar{x}) \leq \mathrm{Er}\, f(\bar{x})$. ■

When $Y = \mathbb{R}$, Theorem 4.2 recaptures [12, Theorem 1] and improves [11, Theorem 2]. It goes in line with a similar condition for the calmness [q] property of level set maps which first appeared in [24, Proposition 3.4]; see also [20, Corollary 4.3].

Comparing Theorem 4.2 and [5, Theorem 3.3], we can deduce the relationship between the uniform strict slopes.

Corollary 4.3. *If X is complete and $f_{\bar{y}}$ is C-lower semicontinuous and C-bounded, then* $^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) \leq \overline{|\nabla f|}^{\diamond}(\bar{x}) \leq {}^{+}\overline{|\nabla f|}^{\diamond}(\bar{x})$.

4.2. Error bound criteria. Theorem 4.2 together with [5, Theorem 3.3 and Proposition 3.4] and the relationships between the slopes and error bound modulus established above produce several error bound criteria summarized in the next theorem.

Theorem 4.4. *Let $\bar{y} = f(\bar{x})$. Consider the following conditions.*

(i) *f satisfies the error bound property at $\bar{x}$ relative to $\bar{y}$;*
(ii) $^{+}\overline{|\nabla f|}^{\diamond}(\bar{x}) > 0$;
(iii) $\overline{|\nabla f|}^{\diamond}(\bar{x}) > 0$;
(iv) $^{-}\overline{|\nabla f|}^{\diamond}(\bar{x}) > 0$;
(v) $^{-}\overline{|\nabla f|}^{>}(\bar{x}) > 0$;
(vi) $^{0}|\nabla f|^{>}(\bar{x}) > 0$.

Then (vi) $\Rightarrow$ (i) $\Rightarrow$ (ii) *and* (v) $\Rightarrow$ (iv).

If X is complete and $f_{\bar{y}}$ is C-lower semicontinuous and C-bounded, then (iv) $\Rightarrow$ (iii) $\Leftrightarrow$ (i).

Note that if X is complete and $f_{\bar{y}}$ is C-lower semicontinuous and C-bounded, then conditions (v) and (vi) are both sufficient for the error bound property of f at $\bar{x}$ relative to $\bar{y}$. At the same time, one can use functions in [5, Examples 2 and 6] to check that these conditions are independent.

5. Conclusion

To the best of our knowledge, this paper together with [5] present the first attempt to extend the definitions and theory of slopes which have proved to be very useful tools in the scalar analysis to functions with values in normed vector spaces. We consider new concepts of error bounds for vector-valued functions and demonstrate the applicability of vector slopes to developing primal space characterizations of error bounds. Subdifferential characterizations of vector error bounds are discussed in [5].

Acknowledgements. The authors wish to acknowledge the very professional refereeing of the paper and express their gratitude to the two anonymous referees for valuable and helpful comments and suggestions.

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