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On Second-Order Conditions for Nonsmooth Problems with Constraints*

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. To study the sufficiency of an optimization problem, one either imposes some convexity assumptions or consider second order optimality conditions. In this paper we establish second order optimality conditions for nonsmooth optimization problems by considering second order approximations of the functions involved and by introducing the concept of second order tangentiability.

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1. Introduction

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Over the years a lot of work has been done with respect to the study of first order optimality conditions. To obtain the sufficiency of these conditions, one assumes the convexity or generalized convexity assumptions. But in absence of convexity, how to go about discussing the sufficiency of the optimality conditions. To do so, one moves ahead with the second order optimality conditions which provide an alternative in absence of convexity. Work has been done for the second order optimality under twice differentiability as well as for nonsmooth scenarios [2, 10, 22]. To study the second order optimality conditions for problems involving nonsmooth functions, various concepts of first and second order directional derivatives have been introduced [6, 25, 26]. Over the years, second order optimality conditions have been obtained for problems involving $C^{1,1}$ data [9, 11, 24] with the optimality conditions expressed in terms of the Clarke Hessian. Work has been reported for problems with C^1 data [12–14] wherein the optimality conditions are given by pseudo-Jacobian and pseudo-Hessian.

In the works mentioned above, authors make use of the second order directional derivative of the functions involved in the optimization problem. In 1993, Jourani and Thibault [15] introduced the notion of *approximations* thereby replacing the concept of directional derivatives. In 1997, Allali and Amahroq [1] used a relaxed notion of approximation to obtain the second order optimality conditions. Further developments of approximation are given by Khanh and Tuan in [16–19] for vector problems. The topic we are going to deal with now is inspired by Khanh's research to whom this paper is dedicated. Namely we broaden the idea of approximations to cater to larger class of problems using which we obtain the second order optimality conditions. The paper is organized as follows. In Section 2, we present second order tangent sets and introduce the concept of second order tangentiability of a set while we define new notions of approximations for a function in Section 3. In Sections 4 and 5, we establish second order optimality conditions for problems without explicit constraint and problems with equality and inequality constraints respectively. As an application of the results obtained in Section 5, we obtain the second order optimality conditions for minimax programming problems in Section 6.

2. Tangent directions

To study local optimality conditions, one needs to approximate the objective function of the problem by its first or second order Taylor's expansions and the feasible set by tangent sets [3, 5]. In this section we recall some concepts of tangent sets and introduce the concept of tangentiable sets which will be subsequently used to establish new second order optimality conditions.

Let X be a normed space and S a nonempty set in X. Given $\bar{x} \in S$, the *feasible direction cone* of S at $\bar{x}$ is defined by

$$T_0(S;\bar{x}) = \{v \in X : \exists\ \delta > 0 \text{ such that } [\bar{x}, \bar{x} + \delta v] \subset S\}$$

and the *first order tangent cone* to S at $\bar{x} \in S$ is defined by

$$T(S;\bar{x}) = \limsup_{t\downarrow 0} \frac{1}{t}(S-\bar{x}) = \{v \in X : \exists\ t_n \downarrow 0, v_n \to v \text{ such that } x_n := \bar{x} + t_n v_n \in S\}.$$

It is clear that the feasible direction cone is contained in the first order tangent cone, and the converse is true when S is a polyhedral convex set. Given $\bar{x} \in S$ as before and $u \in X$, the *second order tangent cone* of S at $(\bar{x}, u)$ in the sense of Penot [23] is the set

$$T^{''}(S;\bar{x};u) = \limsup_{t,s\downarrow 0} \frac{1}{s}\left(\frac{1}{t}(S-\bar{x}) - u\right) = \{z \in X : \exists\ s_n, t_n \downarrow 0, z_n \to z \text{ such that } x_n := \bar{x} + t_n u + t_n s_n z_n \in S\}.$$

In particular, $T^{''}(S;\bar{x};0) = T(S;\bar{x})$ because $x_n = \bar{x} + \alpha_n z_n \in S$ with $\alpha_n = t_n s_n \to 0$ and $z_n \to z$.

The *parabolic second order tangent set* [23] is defined as

$$T^2(S;\bar{x};u) = \{z \in X : \exists\ t_n \downarrow 0, z_n \to z \text{ such that } x_n := \bar{x} + t_n u + t_n^2 z_n \in S\}.$$

The *second-order tangent set of index* $\gamma \geq 0$ given by Cambini et al. [4] is

$$T^2_\gamma(S;\bar{x};u) = \{z \in X : \exists\ s_n, t_n \downarrow 0,\ t_n/s_n \to \gamma, z_n \to z \text{ such that } x_n := \bar{x} + t_n u + t_n s_n z_n \in S\}.$$

When $\gamma = 0$, the set $T^2_0(S;\bar{x};u)$ is a cone and called the *asymptotic second order tangent cone*. Also

$$T^{''}(S;\bar{x};u) = \bigcup_{0\leq\gamma\leq\infty} T^2_\gamma(S;\bar{x};u) \quad \text{and} \quad T^2_\gamma(S;\bar{x};u) = \gamma T^2_1(S;\bar{x};u)\ \forall\ \gamma > 0.$$

The following concept of tangentiability is similar to the concept of asymptotability of [21] and plays an important role in establishing sufficient condition of optimal solutions.

Definition 2.1. Let $\bar{x} \in S$. We say that S is *first order tangentiable* at $\bar{x}$ if for every $\varepsilon > 0$, there is a neighborhood U of the origin such that

$$(S-\bar{x}) \cap U \subset [T(S;\bar{x})]_\varepsilon,$$

where $[T(S;\bar{x})]_\varepsilon$ is the ε-conic neighborhood of $T(S;\bar{x})$ which consists of $x \in X$ such that the distance between $x/\|x\|$ and the cone $T(S;\bar{x})$ is less than ε.

S is said to be *second order tangentiable* at $\bar{x}$ if for every $u \in T(S;\bar{x})$ and for every $\varepsilon > 0$, there is a neighborhood U of the origin such that

$$\text{cone}[(S-\bar{x})\cap U]\cap(u+U)\subset u+[T^{''}(S;\bar{x};u)]_{\varepsilon}.$$

We remark that if S is second order tangentiable at $\bar{x}$, then it is also first order tangentiable at that point. To see this, take $u=0$ and apply equality $T^{''}(S,\bar{x},0)=T(S;\bar{x})$ already discussed before.

Proposition 2.2. *Every set in a finite dimensional space is first order and second order tangentiable.*

Proof. Let S be a nonempty set in $X=\mathbb{R}^k$ and let $\bar{x}\in S$. By an aforementioned remark we need only to prove the second order tangentiability of S at $\bar{x}$. If it is not the case, one may find a vector $u\in T(S;\bar{x})$ and $\epsilon>0$ such that for every $n\geq 1$ there exist some $x_n\in S,\alpha_n>0,b_n\in X$ with $\|x_n-\bar{x}\|<\frac{1}{n},\|b_n\|<\frac{1}{n}$ satisfying

$$\alpha_n(x_n-\bar{x})=u+b_n \tag{1}$$

and

$$\alpha_n(x_n-\bar{x})\notin u+[T^{''}(S;\bar{x},u)]_{\varepsilon}. \tag{2}$$

If $b_n=0$ for all $n\geq 1$, then (1) implies that $0\notin[T^{''}(S;\bar{x},u)]_{\varepsilon}$, a contradiction. We may assume $b_n\neq 0$ and $\frac{b_n}{\|b_n\|}$ converges to some unit norm vector b. We deduce from (1) that

$$x_n=\bar{x}+\frac{1}{\alpha_n}u+\frac{\|b_n\|}{\alpha_n}\frac{b_n}{\|b_n\|}\in S.$$

If $u=0$, we deduce $b\in T(S;\bar{x})=[T^{''}(S;\bar{x},0)]$ and hence $b_n\in[T^{''}(S;\bar{x},0)]_{\varepsilon}$ for n sufficiently large. This contradicts (2).
If $u\neq 0$, we deduce $b\in T^{''}(S;\bar{x},u)$ and hence $\frac{b_n}{\|b_n\|}\in[T^{''}(S;\bar{x},u)]_{\varepsilon}$ for n large. As the latter set is a cone, one obtains $b_n\in[T^{''}(S;\bar{x},u)]_{\varepsilon}$ contradicting (2) again. The proof is complete. ■

Of course, in an infinite dimensional space not every set is first order/second order tangentiable and also there exist second order tangentiable sets that are not contained in a finite dimentional subspace. For instance let H be an infinite dimensional Hilbert space with a countable base $\{e_1,e_2,\dots,e_n,\dots\}$ such that

$$\langle e_i,e_j\rangle=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

Denote $S_n:=\{te_n:\ t\geq 0\},\ n=1,2,\dots$ and

$$S:=\bigcup_{n=1}^{\infty}S_n.$$

Then one can see that S is second order tangentiable at 0 and no finite dimentional subspace of H contains it.

3. Approximations

Let $\bar{x}, u, v \in X$. A sequence $\{x_n\} \subset X$ is said to *converge to* $\bar{x}$ *in the direction* u, denoted $x_n \to_u \bar{x}$, if $\exists\, t_n \downarrow 0,\ u_n \to u$ such that $x_n = \bar{x} + t_n u_n\ \forall\, n$, and it is said to *converge to* $\bar{x}$ *in the direction* (u,v), denoted by $x_n \to_{(u,v)} \bar{x}$, if

$$\exists\, t_n, s_n \downarrow 0,\ u_n \to u,\ v_n \to v \text{ such that } x_n = \bar{x} + t_n u_n + t_n s_n v_n \quad \forall\, n.$$

Let Y be another normed space. A function $r : X \to Y$ is said to have the limit 0 as x converges to 0 in the direction u, denoted by $\lim\limits_{x \to_u 0} r(x) = 0$, if

$$\forall\, \{x_n\} \subset X,\ x_n \to_u 0 \ \Rightarrow\ r(x_n) \to 0.$$

The function r is said to have the limit 0 as x converges to 0 in the direction (u,v), denoted by $\lim\limits_{x \to_{(u,v)} 0} r(x) = 0$, if

$$\forall\, \{x_n\} \subset X,\ x_n \to_{(u,v)} 0 \ \Rightarrow\ r(x_n) \to 0.$$

Some notations are in order. Let $L(X,Y)$ $(B(X,Y))$ denote the space of continuous linear mappings from X to Y (continuous bilinear mappings from $X \times X$ to Y). We denote $L(X,\mathbb{R})$ by X^* and $B(X,\mathbb{R})$ by $B(X)$. For $A \subset L(X,Y)$ and $x \in X$, $A(x) = \{a(x) : a \in A\} \subset Y$. For $f : X \to Y$, $\lambda \in Y^*$ and $x, y \in X$, $(\lambda f)(x) = \langle \lambda, f(x) \rangle$ while for $F : X \to L(X,Y)$, $\langle \lambda, F(x)(y) \rangle = \langle F(x)^* \lambda, y \rangle$, where $F(x)^*$ denotes the adjoint of $F(x)$. For $B \subset B(X,Y)$ and $x, x' \in X$, $B(x,x') = \{b(x,x') : b \in B\}$.

Definition 3.1. A nonempty subset $A_f \subset L(X,Y)$ is called a *first order approximation* of f at $\bar{x}$ if for every direction $u \neq 0$, there exists a function $r_u : X \to Y$ with $\lim\limits_{x \to_u 0} r_u(x) = 0$, such that for every sequence $\{x_n\}$ converging to $\bar{x}$ in the direction u,

$$f(x_n) \in f(\bar{x}) + A_f(x_n - \bar{x}) + \|x_n - \bar{x}\| r_u(x_n - \bar{x}).$$

The concept of approximation was first given by Jourani and Thibault [15] in a stronger form for a real-valued function, that is, $Y = \mathbb{R}$. Namely, it requires that

$$f(x) \in f(x') + A_f(x - x') + \|x - x'\| r(x, x'),$$

where $r(x,x') \to 0$ as $x, x' \to \bar{x}$. Allali and Amahroq [1] give a weaker definition by taking $x' = \bar{x}$ in the above relation, which corresponds to the concept of Fréchet pseudo-Jacobian of Jeyakumar and Luc [13, 20].

It is clear from the above definitions that an approximation in the sense of Jourani and Thibault is an approximation in the sense of Allali and Amahroq, which, in its turn, is an approximation in the sense of Definition 3.1. However, the converse is not true in general as is obvious from Example 3.2 below. The definition by Jourani and Thibault evokes the idea of strict derivatives and is very useful in the study of metric regularity and stability properties, while Definition 3.1 is more sensitive to behavior of the function in directions and so it allows us to treat certain questions such as existence conditions for a larger class of problems. This is a reason why we choose a modified version of approximations to develop second-order analysis in this work.

Example 3.2. Let H be an infinite dimensional Hilbert space as in the example in Section 2.

Denote

$$S_n := \{te_n : \ t \geq 0\}, \ n = 1, 2, \dots \quad \text{and} \quad S := \bigcup_{n=1}^{\infty} S_n.$$

Define a function $f : H \to \mathbb{R}$ as

$$f(x) = \begin{cases} 0, & x \in H \setminus S, \\ \|x\|^{1+\frac{1}{n}}, & x \in S_n, \ n = 1, 2, \dots \end{cases}$$

Set $A_f := \{0\}$. We claim that A_f is a first order approximation of f at 0. Indeed, let $u \neq 0$ in H be arbitrarily given. Consider the following cases.

i) $u \notin S$. Set $r_u(x) = 0$ for every x. Obviously $\lim\limits_{x \to_u 0} r_u(x) = 0$. Let $\{x_n\}$ be a sequence converging to 0 in direction u. We may assume that $x_n \neq 0$ for every n. Then for n sufficiently large one has $x_n \notin S$ and

$$f(x_n) = 0 \in f(0) + A_f(x_n) + \|x_n\| r_u(x_n).$$

ii) $u \in S_n$, for some $n = 1, 2, \dots$. Set

$$r_u(x) = \begin{cases} 0, & x \notin S_n \setminus \{0\}, \\ \|x\|^{\frac{1}{n}}, & x \in S_n \setminus \{0\}. \end{cases}$$

We can see that $\lim\limits_{x \to_u 0} r_u(x) = 0$. Let $\{x_k\}$ be a sequence converging to 0 in direction u. We have $\|e_n - e_m\| = \sqrt{2}, \ m \neq n$. Let $\varepsilon > 0$ be sufficiently small. For k sufficiently large, $x_k \in \bigcup\limits_{0 \leq t \leq 1} tB(e_n, \varepsilon)$ and $x_k \notin S \setminus S_n$. Thus,

$$f(x_k) = \begin{cases} 0, & x_k \notin S_n \setminus \{0\}, \\ \|x_k\| \cdot \|x_k\|^{\frac{1}{n}}, & x_k \in S_n \setminus \{0\} \end{cases}$$
$$\in f(0) + A_f(x_k) + \|x_k\| r_u(x_k).$$

Hence A_f is a first order approximation of f at 0. To show that A_f is not a first order approximation in the sense of Allali and Amahroq [1], consider a sequence defined by

$$x_n = \frac{1}{n} e_n, \ n = 1, 2, \ldots$$

We have

$$\lim_{n \to \infty} \frac{f(x_n) - f(0) - A_f(x_n)}{\|x_n\|} = \lim_{n \to \infty} \left(\frac{1}{n}\right)^{\frac{1}{n}} = 1$$

while the limit should be equal to 0 if A_f were a first order approximation in the sense of Allali and Amahroq since in this case A_f is the Frechet derivative of f at 0.

Definition 3.3. Let $A_f \subset L(X, Y)$, $B_f \subset B(X, Y)$ be nonempty sets. We say that f admits (A_f, B_f) as a *second order approximation* at $\bar{x}$ if

(i) A_f is a first order approximation of f at $\bar{x}$.

(ii) For every direction $(u, z) \neq (0, 0)$, there exists a function $r_{(u,z)} : X \to Y$ with $\lim_{x \to_{(u,z)} 0} r_{(u,z)}(x) = 0$ such that for every sequence $\{x_n\}$ converging to $\bar{x}$ in the direction (u, z),

$$f(x_n) \in f(\bar{x}) + A_f(x_n - \bar{x}) + \frac{1}{2} B_f(x_n - \bar{x}, x_n - \bar{x}) + \|x_n - \bar{x}\|^2 r_{(u,z)}(x_n - x).$$

In Allali and Amahroq [1] for $Y = \mathbb{R}$, the corresponding formula is

$$f(x) \in f(\bar{x}) + A_f(x - \bar{x}) + \frac{1}{2} B_f(x - \bar{x}, x - \bar{x}) + o(\|x - \bar{x}\|)^2 \quad \forall \ x \text{ around } \bar{x}.$$

It is clear that the first order and second order approximations need not be singleton. But under certain conditions, they may reduce to a singleton. One such a condition is the Hadamard differentiability.

Definition 3.4. A function $f : X \to Y$ is said to be *Hadamard directional differentiable* at $x \in X$ in a direction $u \in X$ if

$$df(x; u) := \lim_{t \downarrow 0; v \to u} \frac{f(x + tv) - f(x)}{t}$$

exists, and f is said to be *Hadamard differentiable* at x if there exists $Df(x) \in L(X, Y)$ such that

$$df(x; u) = Df(x)(u) \quad \forall \ u \in X.$$

Below we present a result to relate first order approximations and Hadamard derivatives.

Proposition 3.5. *Let a function $f : X \to Y$ admit a first order approximation A_f at a point $\bar{x} \in X$. If A_f is a singleton and $df(\bar{x}; 0) = 0$, then f is Hadamard differentiable at $\bar{x}$ and*

$$A_f = \{Df(\bar{x})\}.$$

Conversely, if f is Hadamard differentiable at $\bar{x}$, then $\{Df(\bar{x})\}$ is a first order approximation of f at $\bar{x}$.

Proof. For the first assertion, let $u \in X,\ u \neq 0$ be arbitrary and consider any sequences $t_n \downarrow 0,\ u_n \to u$. Then $x_n := \bar{x} + t_n u_n \to_u \bar{x}$. By Definition 3.1, there exists a function $r_u : X \to Y$ with $\lim_{x \to_u 0} r_u(x) = 0$ such that

$$f(x_n) = f(\bar{x}) + A_f(x_n - \bar{x}) + \|x_n - \bar{x}\| r_u(x_n - \bar{x}).$$

This implies

$$\lim_{n\to\infty} \frac{f(\bar{x} + t_n u_n) - f(\bar{x})}{t_n} = \lim_{n\to\infty} A_f(u_n) = A_f(u).$$

Since the preceding relation holds for every sequences $t_n \downarrow 0,\ u_n \to u$, then

$$df(\bar{x}; u) = A_f(u),$$

which along with the assumption that $df(\bar{x}; 0) = 0$ implies the Hadamard differentiability of f at $\bar{x}$.

Conversely, for each $u \in X,\ u \neq 0$, define the function $r_u : X \to Y$ as

$$r_u(x) = \begin{cases} 0, & x = 0, \\ \dfrac{f(\bar{x} + x) - f(\bar{x}) - Df(\bar{x})(x)}{\|x\|}, & x \neq 0. \end{cases}$$

Then $\lim_{x \to_u 0} r_u(x) = 0$. Consider any sequence $x_n \to_u \bar{x}$. By the definition of r_u,

$$f(x_n) = f(\bar{x}) + Df(\bar{x})(x_n - \bar{x}) + \|x_n - \bar{x}\| r_u(x_n - \bar{x}).$$

Hence $Df(\bar{x})$ is a first order approximation of f at $\bar{x}$. The proof is complete. ■

We also note that when f is twice Fréchet differentiable at $\bar{x}$ then $(Df(\bar{x}), D^2 f(\bar{x}))$ is a second order approximation of f at $\bar{x}$. Before moving any further, we discuss some notions of Hessian and their relation to the approximations.

Definition 3.6. (1) Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of class $C^{1,1}$. The *Clarke generalized Hessian* [11] of g at x_0 is defined as

$$\partial_C^2 g(x_0) := \text{co}\{\lim_{k\to\infty} D^2 g(x_k) :\ x_k \to x_0,\ D^2 g(x_k) \text{ exists}\}.$$

(2) Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be continuous. A closed and bounded subset $\partial_* g(x_0) \subset L(\mathbb{R}^n, \mathbb{R}^m)$ is said to be a *pseudo-Jacobian* [14] of g at $x_0 \in \mathbb{R}^n$ if for each $v \in \mathbb{R}^m,\ u \in \mathbb{R}^n$,

$$(vg)^+(x_0; u) \leq \max_{M \in \partial_* g(x_0)} \langle v, Mu \rangle,$$

where $(vg)^+(x_0;u)$ is the upper Dini directional derivative of (vg) at x_0 in the direction u defined as

$$(vg)^+(x_0;u) := \limsup_{t\downarrow 0} \frac{(vg)(x_0+tu)-(vg)(x_0)}{t}.$$

(3) Let $g : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. A closed and bounded subset $\partial_* g^2(x_0) \subset B(\mathbb{R}^n, \mathbb{R}^n, \mathbb{R})$ is said to be a *pseudo-Hessian* [13, 14] of g at $x_0 \in \mathbb{R}^n$ if $\partial_* g^2(x_0)$ is a pseudo-Jacobian of Dg at x_0.

It is known [16] that if $g : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable in a neighborhood U of x_0 and that $\partial^2_* g(\cdot)$ is a pseudo-Hessian which is upper semicontinuous at x_0, then g admits $(Dg(x_0), \operatorname{co}\partial^2_* g(x_0))$ as a second order approximation in the sense of Allali and Amahroq [1] at x_0. In particular this is true when $g : \mathbb{R}^n \to \mathbb{R}$ is $C^{1,1}$ in a neighborhood U of x_0 and $\partial^2_C g(x_0)$ is taken in the role of a pseudo-Hessian at x_0.

Next we present some properties of the second order approximation.

Proposition 3.7. *Let functions $f : X \to \mathbb{R}$ and $g : X \to Y$ admit second order approximations (A_f, B_f) and (A_g, B_g) at a point $\bar{x} \in X$, respectively. Let $\lambda \in Y^*$ be arbitrary. Then the function $f + (\lambda g)$ admits $\big(A_f + A_g^*\lambda, B_f + B_g^*\lambda\big)$ as a second order approximation at $\bar{x}$.*

Proof. Let a nonzero direction $u \in X$ be arbitrarily given. Then by Definition 3.1, there exist $r_u : X \to \mathbb{R}$ and $s_u : X \to Y$ which converge to 0 when $x \to_u 0$ such that for every sequences $\{x_n\}$ converging to $\bar{x}$ in the direction u,

$$f(x_n) \in f(\bar{x}) + A_f(x_n - \bar{x}) + \|x_n - \bar{x}\| r_u(x_n - \bar{x}),$$
$$g(x_n) \in g(\bar{x}) + A_g(x_n - \bar{x}) + \|x_n - \bar{x}\| s_u(x_n - \bar{x}).$$

Hence

$$f(x_n) + (\lambda g)(x_n) \in f(\bar{x}) + (\lambda g)(\bar{x}) + A_f(x_n - \bar{x}) + (A_g^*\lambda)(x_n - \bar{x}) + \|x_n - \bar{x}\| t_u(x_n - \bar{x}),$$

where $t_u(x) := r_u(x) + (\lambda s_u)(x)$ satisfies $\lim\limits_{x \to_u 0} t_u(x) = 0$. Thus $A_f + (A_g^*\lambda)$ is a first order approximation of $f + (\lambda g)$ at $\bar{x}$. For the second order approximation, the proof is quite similar. ∎

4. Optimality conditions

In this section, we study the second order optimality conditions in terms of the approximations presented in the preceding section. But before that we present some notions and notations (see [16]) which will be needed to establish the results.

Definition 4.1. (i) Let $a_n, a \in L(X,Y)$. The sequence $\{a_n\}$ *pointwise converges* to a and denoted as $a_n \overset{\text{p}}{\to} a$ or $a = \text{p-}\lim a_n$ if

$$\lim_{n\to\infty} a_n(x) = a(x) \quad \forall\, x \in X.$$

A similar definition is adopted for $b_n, b \in B(X,Y)$.

(ii) A subset $A \subset L(X,Y)$ is said to be *relatively p-compact* if each sequence $\{a_n\} \subset A$ has a subsequence which pointwise converges to some $a \in L(X,Y)$.

Definition 4.2. A subset $A \subset L(X,Y)$ is said to be *abp-compact* (*asymptotically and boundedly pointwise compact*) if every sequence $\{a_n\} \subset A$ admits either a bounded pointwise convergent subsequence or an unbounded subsequence $\{a_{n_k}\}$ with $a_{n_k}/\|a_{n_k}\|$ pointwise converging to some nonzero vector. A similar definition is adopted for $B \subset B(X,Y)$. If A and B are both abp-compact, then (A,B) is said to be abp-compact.

We will make use of the following notations. Let $A \subset L(X,Y)$ and $B \subset B(X,Y)$.

$$\begin{aligned}
&\text{p-cl}A := \{a \in L(X,Y) : \exists\, \{a_n\} \subset A,\ \text{p-}\lim a_n = a\},\\
&\text{p-cl}B := \{b \in B(X,Y) : \exists\, \{b_n\} \subset B,\ \text{p-}\lim b_n = b\},\\
&\text{p-}A_\infty := \{a \in L(X,Y) : \exists\, \{a_n\} \subset A,\ \exists\, t_n \downarrow 0,\ \text{p-}\lim t_n a_n = a\},\\
&\text{p-}B_\infty := \{b \in B(X,Y) : \exists\, \{b_n\} \subset B,\ \exists\, t_n \downarrow 0,\ \text{p-}\lim t_n b_n = b\}.
\end{aligned}$$

Consider $f : X \to \mathbb{R}$ and a nonempty set $S \subset X$. In the following result, we establish the second order optimality conditions for minimizing f over S.

Theorem 4.3 (Second order necessary condition). *Let $\bar{x}$ be a local minimizer of f on S. Assume that f admits an abp-compact second-order approximation (A_f, B_f) at $\bar{x}$. Then for every nonzero vector $u \in T(S;\bar{x})$ the following conditions hold.*

(i) *There exists $a \in \text{p-cl}A_f \cup (\text{p-}(A_f)_\infty \setminus \{0\})$ such that $a(u) \geq 0$.*

(ii) *If $\sup A_f(u) \leq 0$ and A_f is bounded, then*

1) *for every vector $z \in T^2_\gamma(S;\bar{x};u)$ and $\gamma \in [0,\infty)$ either there exists $a \in \text{p-cl}A_f$, $b \in \text{p-cl}B_f$ such that $a(z) + \frac{1}{2}\gamma b(u,u) \geq 0$ or there exists $b_* \in \text{p-}(B_f)_\infty \setminus \{0\}$ such that $b_*(u,u) \geq 0$;*

2) *if $T^2_\infty(S;\bar{x};u) \neq \emptyset$, then there exists $b \in \text{p-cl}B_f \cup (\text{p-}(B_f)_\infty \setminus \{0\})$ such that $b(u,u) \geq 0$.*

(iii) *If $\sup A_f(u) \leq 0$ and $u \in T_0(S;\bar{x})$, then*

$$\sup B_f(u,u) \geq 0.$$

Proof. (i) Let a nonzero direction $u \in T(S;\bar{x})$ be arbitrarily given. Then there exist sequences $t_n \downarrow 0,\ u_n \to u$ such that $x_n := \bar{x} + t_n u_n \in S$. Hence $x_n \to_u \bar{x}$. By Definition 3.1 of first order approximation, there exists $r_u : X \to \mathbb{R}$ which converges to 0 when $x \to_u 0$ such that for each n there exists $a_n \in A_f$ satisfying

$$f(x_n) - f(\bar{x}) = a_n(x_n - \bar{x}) + \|x_n - \bar{x}\| r_u(x_n - \bar{x}).$$

This equality implies

$$a_n(u_n) + \|u_n\| r_u(x_n - \bar{x}) \geq 0. \tag{3}$$

By the abp-compactness of A_f we may assume either the sequence $\{a_n\}$ is bounded and pointwise converges to some element in p-clA_f or $\{a_n\}$ is unbounded and $\{a_n/\|a_n\|\}$ pointwise converges to some nonzero vector. Then either passing (3) to the limit when $n \to \infty$ or dividing (3) by $\|a_n\|$ and letting $n \to \infty$, we find some $a \in$ p-cl$A_f \cup ($p-$(A_f)_\infty \setminus \{0\})$ such that

$$a(u) \geq 0.$$

(ii) Let $z \in T^2_\gamma(S;\bar{x};u)$ and $\gamma \in [0,\infty]$ be arbitrary. Then there exist sequences $t_n, s_n \downarrow 0$, $z_n \to z$ satisfying $t_n/s_n \to \gamma$ such that $x_n := \bar{x} + t_n u + t_n s_n z_n \in S$. Hence $x_n \to_{(u,z)} \bar{x}$. By Definition 3.3, there exists $r_{(u,z)} : X \to \mathbb{R}$ which converges to 0 when $x \to_{(u,z)} 0$ such that for each n there exist $a_n \in A_f, b_n \in B_f$ satisfying

$$f(x_n) - f(\bar{x}) = a_n(x_n - \bar{x}) + \frac{1}{2} b_n(x_n - \bar{x}, x_n - \bar{x}) + \|x_n - \bar{x}\|^2 r_{(u,z)}(x_n - \bar{x}).$$

This equality along with the hypothesis implies that

$$a_n(z_n) + \frac{1}{2}\frac{t_n}{s_n} b_n(u + s_n z_n, u + s_n z_n) + \frac{t_n}{s_n}\|u + s_n z_n\|^2 r_{(u,z)}(x_n - \bar{x}) \geq 0. \tag{4}$$

Since A_f is abp-compact and bounded, we may assume the sequence $\{a_n\}$ pointwise converges to some $a \in$ p-clA_f. Similarly, since B_f is abp-compact, either $\{b_n\}$ is bounded and pointwise converges to some $b \in$ p-clB_f, or $\{b_n\}$ is unbounded and $\{b_n/\|b_n\|\}$ pointwise converges to some $b_* \in$ p-$(B_f)_\infty \setminus \{0\}$.

1) $\gamma \in [0,\infty)$. We consider the following two cases.

Case (a). $\gamma > 0$. By either taking the limit as $n \to \infty$ in (4), $b_n \xrightarrow{\text{p}} b$ or by dividing (4) by $\|b_n\|$ and then taking the limit as $n \to \infty$,

$$a(z) + \frac{1}{2}\gamma b(u,u) \geq 0 \quad \text{or} \quad b_*(u,u) \geq 0.$$

Case (b). $\gamma = 0$. If $\{b_n\}$ is bounded and $b_n \xrightarrow{\text{p}} b$, then taking the limit as $n \to \infty$ in (4),

$$a(z) \geq 0.$$

Next consider the case when $\|b_n\|$ is unbounded and $b_n/\|b_n\| \xrightarrow{\mathrm{p}} b_*$. We have two cases. If $b_n(u+s_nz_n, u+s_nz_n) < 0$ for every n, then (4) yields

$$a_n(z_n) + \frac{t_n}{s_n}\|u+s_nz_n\|^2 r_{(u,z)}(x_n - \bar{x}) \geq 0 \quad \forall\ n.$$

As the limit $n \to \infty$,

$$a(z) \geq 0.$$

If $b_n(u+s_nz_n, u+s_nz_n) \geq 0$ for every n, dividing this inequality by $\|b_n\|$ and taking the limit as $n \to \infty$,

$$b_*(u,u) \geq 0.$$

Thus, completing the proof of (ii)1).

2) $\gamma = \infty$. Dividing (4) by $\dfrac{t_n}{s_n}$, we obtain

$$\frac{s_n}{t_n}a_n(z_n) + \frac{1}{2}b_n(u+s_nz_n, u+s_nz_n) + \|u+s_nz_n\|^2 r_{(u,z)}(x_n - \bar{x}) \geq 0. \tag{5}$$

By either taking the limit as $n \to \infty$ in (5) for the case $\{b_n\}$ is bounded and $b_n \xrightarrow{\mathrm{p}} b$ or dividing (5) by $\|b_n\|$ and then taking the limit as $n \to \infty$, we deduce

$$b(u,u) \geq 0 \quad \text{or} \quad b_*(u,u) \geq 0.$$

(iii) Let $u \in T_0(S;\bar{x})$ be arbitrary nonzero. Then there exists a sequence $t_n \downarrow 0$ such that $x_n := \bar{x} + t_n u \in S$. Hence $x_n \to_{(u,0)} \bar{x}$. By Definition 3.3, there exists $r_{(u,0)} : X \to \mathbb{R}$ which converges to 0 when $x \to_{(u,0)} 0$ such that for every n, there exists $a_n \in A_f$ satisfying

$$f(x_n) - f(\bar{x}) \in a_n(x_n - \bar{x}) + \frac{1}{2}B_f(x_n - \bar{x}, x_n - \bar{x}) + \|x_n - \bar{x}\|^2 r_{(u,0)}(x_n - \bar{x})$$

or equivalently,

$$\frac{f(x_n) - f(\bar{x})}{t_n^2} - \frac{a_n(u)}{t_n} - \|u\|^2 r_{(u,u)}(x_n - \bar{x}) \in \frac{1}{2}B_f(u,u).$$

Set

$$\alpha_n := \frac{f(x_n) - f(\bar{x})}{t_n^2} - \frac{a_n(u)}{t_n},\ n = 1, 2, \ldots$$

For n sufficiently large one has $\alpha_n \geq 0$. Thus,

$$0 \leq \limsup_{n\to\infty}(\alpha_n - \|u\|^2 r_{(u,u)}(x_n - \bar{x})) \leq \frac{1}{2}\sup B_f(u,u),$$

thereby completing the proof. ■

Example 4.4. Let H be a Hilbert space as in Example 3.2.

Denote

$$S_n := \{te_n : \ t \in \mathbb{R}\}, \ n = 1, 2, \dots \quad \text{and} \quad S := \bigcup_{n=1}^{\infty} S_n.$$

For $\bar{x} = 0$ we have $T(S;\bar{x}) = T_0(S;\bar{x}) = S$ and for any $u \in S \setminus \{0\}$, $T^2_\gamma(S;\bar{x};u) = \{tu \ : \ t \in \mathbb{R}\}$ $\forall \gamma \in [0, +\infty]$. Let f be a function on H which admits a second-order approximation (A_f, B_f) at $\bar{x}$ such that A_f is a singleton, B_f is abp-compact. Then a necessary optimality condition for f on S at 0 is that

$$A_f = 0 \text{ and } \forall u \in S \setminus \{0\}, \exists b \in \text{p-cl}B_f \cup (\text{p-}(B_f)_\infty \setminus \{0\}) \text{ such that } \ b(u,u) \geq 0.$$

We denote by S_X the unit sphere of X, by $\mathbb{R}_+$ the set of nonnegative real numbers and by $\mathbb{R}_{++}$ the set of positive real numbers.

Lemma 4.5. *Let $u \in S_X$ and $\{v_n\} \subset S_X$ be such that $v_n \to u$. Assume that $\left\{\frac{v_n - u}{\|v_n - u\|}\right\}$ converges to some $z \in S_X$. Then*

i) $z \notin \mathbb{R}_+ u$ (*where* $\mathbb{R}_+ u := \{tu | \ t \geq 0\}$).

ii) *Moreover, if X is a Hilbert space then $z \in u^\perp \setminus \{0\}$, where $u^\perp$ denotes the orthogonal subspace to u.*

Proof. i) By Hahn-Banach's theorem, there exists $u^* \in X^*$ such that $\langle u^*, u\rangle = \|u\| = 1, \|u^*\| = 1$. Then one has

$$\left\langle u^*, \frac{v_n - u}{\|v_n - u\|} \right\rangle = \frac{\langle u^*, v_n\rangle - \langle u^*, u\rangle}{\|v_n - u\|} \leq \frac{\|u^*\|\|v_n\| - 1}{\|v_n - u\|} = 0.$$

Passing to the limit, we have $\langle u^*, z\rangle \leq 0$. Then $z \neq u$. Hence $z \notin \mathbb{R}_+ u$ since $z \in S_X$.

ii) By choosing $u^* = u$ in i), it implies $\langle u, z\rangle \leq 0$. On the other hand, one has $\langle u, \frac{v_n - u}{\|v_n - u\|}\rangle = \langle u - v_n, \frac{v_n - u}{\|v_n - u\|}\rangle + \langle v_n, \frac{v_n - u}{\|v_n - u\|}\rangle = -\|u - v_n\| + \frac{1 - \langle v_n, u\rangle}{\|v_n - u\|} \geq -\|u - v_n\|$. Passing to the limit, we have $\langle u, z\rangle \geq 0$. Hence $\langle u, z\rangle = 0$. ■

Theorem 4.6 (Second order sufficient condition). *Assume that f admits a second order approximation (A_f, B_f) at $\bar{x} \in S$. Then the following conditions are sufficient for $\bar{x}$ to be a strict local minimizer of f on S.*

(i) *S is a second-order tangentiable at $\bar{x}$ and the second order tangent set $T^{''}(S;\bar{x};u)$ for every $u \in T(S;\bar{x})$ are locally compact;*

(ii) *A_f is relatively p-compact and bounded, B_f is abp-compact;*

(iii) $a(u) \geq 0 \quad \forall \ a \in A_f, \ u \in T(S;\bar{x})$;

(iv) *If $u \in T(S;\bar{x}) \cap S_X$ and $a(u) \leq 0$ for some $a \in$ p-clA_f, then*

(1) *for every $z \in T^2(S;\bar{x};u) \setminus \mathbb{R}_{++} u$,*

$$a(z) + \frac{1}{2} b(u,u) > 0 \quad \forall \ a \in \text{p-cl}A_f, \ b \in \text{p-cl}B_f;$$

(2) *for every* $z \in T_0^2(S;\bar{x};u) \setminus \mathbb{R}_+u$,

$$a(z) > 0 \quad \forall\ a \in \text{p-cl}A_f;$$

(3) *for every nonzero* $b_* \in \text{p-}(B_f)_\infty$, $b_*(u,u) > 0$.

Proof. Suppose to the contrary that $\bar{x}$ is not a strict local minimizer. Then there exists a sequence $x_n \in S \setminus \{\bar{x}\}$ converging to $\bar{x}$ such that

$$f(x_n) \leq f(\bar{x}). \tag{6}$$

Set $t_n = \|x_n - \bar{x}\| \to 0$. By (i) and a remark made after Definition 2.1, the set S is first order tangentiable at $\bar{x}$. By considering a subsequence if necessary we may assume that $x_n - \bar{x} \in [T(S;\bar{x})]_{\frac{1}{n}}$ for all $n = 1, 2, \ldots$ Then for each n, there exists $u_n \in T(S;\bar{x})$ such that

$$\left\| \frac{x_n - \bar{x}}{t_n} - u_n \right\| < \frac{1}{n}.$$

Since $T(S;\bar{x})$ is locally compact, we may assume the sequence $\{u_n\}$ converges to some $u \in T(S;\bar{x})$. It is clear that $\frac{x_n - \bar{x}}{t_n}$ converges to u and consequently $\|u\| = 1$. Hence $\{x_n\}$ converges to $\bar{x}$ in the direction u. Set

$$s_n = \left\| \frac{x_n - \bar{x}}{t_n} - u \right\| \to 0,$$

$$z_n = \begin{cases} \dfrac{1}{s_n}\left(\dfrac{x_n - \bar{x}}{t_n} - u\right), & s_n \neq 0, \\ 0, & s_n = 0. \end{cases}$$

If $s_n = 0$ for every n, the sequence $\{x_n\}$ converges to $\bar{x}$ in the direction (u, z) with $z = 0$. Otherwise we may assume that $s_n \neq 0$ for every n. By (i) we have

$$\frac{x_n - \bar{x}}{t_n} \in u + [T^{''}(S;\bar{x},u)]_\varepsilon$$

which implies

$$z_n \in [T^{''}(S;\bar{x},u)]_\varepsilon$$

for n sufficiently large. Since this is true for every $\varepsilon > 0$, we can find a subsequence $\{z_{k_n}\} \subset \{z_n\}$ and a sequence $\{w'_n\} \subset T^{''}(S;\bar{x};u)$ such that $\|z_{k_n} - w'_n\| < 1/n$ for all $n = 1, 2, \ldots$ Since $T^{''}(S;\bar{x};u)$ is locally compact, there exists $z \in T^{''}(S;x;u)$ such that $w'_n \to z$. Hence, $z_{k_n} \to z$ with $\|z\| = 1$ and $z \notin \mathbb{R}_+u$ (by Lemma 4.5). However, without loss of generality we may assume that $z_n \to z$. Thus $x_n \to \bar{x}$ in the direction (u, z) with $z = 0$ or $\|z\| = 1, z \notin \mathbb{R}_+u$.

By Definition 3.3 of the second order approximations, there exist real-valued functions r_u and $r_{(u,z)}$ defined on X such that for every n there exist $a_n, \bar{a}_n \in A_f$ and $b_n \in B_f$ satisfying

$$f(x_n) - f(\bar{x}) = \bar{a}_n(x_n - \bar{x}) + \|x_n - \bar{x}\| r_u(x_n - \bar{x}), \tag{7}$$

$$f(x_n) - f(\bar{x}) = a_n(x_n - \bar{x}) + \frac{1}{2} b_n(x_n - \bar{x}, x_n - \bar{x}) + \|x_n - \bar{x}\|^2 r_{(u,z)}(x_n - \bar{x}). \tag{8}$$

In view of (ii), $\{\bar{a}_n\}$ and $\{a_n\}$ pointwise converge to some $\bar{a}$ and $a \in$ p-clA_f, respectively. It follows from (6) and (7) that $\bar{a}(u) \leq 0$.

If $\{x_n\}$ converges to $\bar{x}$ in direction (u, z) with $z = 0$, for the case $s_n = 0$ for every n, the condition (8) yields

$$a_n(u) + \frac{1}{2} t_n b_n(u, u) + t_n r_{(u,z)}(t_n u) \leq 0.$$

By (iii), the above inequality implies

$$\frac{1}{2} b_n(u, u) + r_{(u,z)}(t_n u) \leq 0. \tag{9}$$

Since B_f is abp-compact, either the sequence $\{b_n\}$ is bounded and has a pointwise limit $b \in$ p-clB_f or it is unbounded with say p-$\lim_{n\to\infty} \frac{b_n}{\|b_n\|} = b_* \in$ p-$(B_f)_\infty \setminus \{0\}$. By (9), employing a similar argument as in the proof of Theorem 4.3, either

$$b(u, u) \leq 0 \quad \text{or} \quad b_*(u, u) \leq 0$$

which contradicts (iv) 1) and 3). Observe that in this scenario, $0 \in T^2(S; \bar{x}, u)$ since $x_n = \bar{x} + t_n u \in S$ for every n.

If $\{x_n\}$ converges to $\bar{x}$ in direction (u, z) with $\|z\| = 1$, $z \notin \mathbb{R}_+ u$, then consider the three possible cases concerning the sequence $\{t_n/s_n\}$.

Case 1. $\lim_{n\to\infty} t_n/s_n = \infty$, or equivalently $\lim_{n\to\infty} s_n/t_n = 0$. Setting $w_n = (s_n/t_n) z_n$, we have

$$x_n = \bar{x} + t_n u + t_n^2 w_n = \bar{x} + t_n u + t_n s_n z_n \in S$$

with $\lim_{n\to\infty} w_n = 0$. Hence $0 \in T^2(S; x, u)$. By the conditions (6), (8) and (iii),

$$a_n(w_n) + \frac{1}{2} b_n(u + t_n w_n, u + t_n w_n) + \|u + t_n w_n\|^2 r_{(u,z)}(t_n u + t_n s_n z_n) \leq 0.$$

Since $\lim_{n\to\infty} a_n(w_n) = 0$ and B_f is abp-compact, as done in earlier proofs, there exists some element $b \in$ p-cl$B_f \cup$ [p-$(B_f)_\infty \setminus \{0\}$] such that $b(u, u) \leq 0$ which again contradicts (iv) 1) and 3).

Case 2. $\lim_{n\to\infty} t_n/s_n = \gamma > 0$. Then $z \in T^2_\gamma(S; x, u)$. By the conditions (8) and (iii),

$$\frac{s_n}{t_n} a_n(z_n) + \frac{1}{2} b_n(u + s_n z_n, u + s_n z_n) + \|u + s_n z_n\|^2 r_{(u,z)}(t_n u + t_n s_n z_n) \leq 0. \tag{10}$$

We have $\lim_{n\to\infty} a_n(z_n) = a(z)$. If the sequence $\{b_n\}$ is bounded, then it pointwise converges to some $b \in \text{p-cl}B_f$ and taking the limit as $n \to \infty$, (10) yields

$$a\left(\frac{z}{\gamma}\right) + \frac{1}{2}b(u,u) \le 0$$

which contradicts (iv) 1) as $z/\gamma \in T^2(S;\bar{x};u) \setminus \mathbb{R}_{++}u$. If the sequence $\{b_n\}$ is unbounded, $\text{p-}\lim_{n\to\infty} \dfrac{b_n}{\|b_n\|} = b_* \in \text{p-}(B)_f)_\infty \setminus \{0\}$. Then by dividing (10) by $\|b_n\|$ and taking the limit as $n \to \infty$, we have

$$b_*(u,u) \le 0$$

which contradicts (iv) 3).

Case 3). $\lim_{n\to\infty} t_n/s_n = 0$. Then $z \in T_0^2(S;x,u) \setminus \mathbb{R}_+u$. As the above cases, it follows from (8) and (iii) that

$$a_n(z_n)+\frac{1}{2}\frac{t_n}{s_n}b_n(u+s_nz_n,u+s_nz_n)+\frac{t_n}{s_n}\|u+s_nz_n\|^2 r_{(u,z)}(t_nu+t_ns_nz_n) \le 0. \quad (11)$$

If the sequence $\{b_n\}$ is bounded, working along the lines as before, (11) implies

$$a(z) \le 0$$

which contradicts (iv) 2). If the sequence $\{b_n\}$ is unbounded, then as before $\text{p-}\lim_{n\to\infty} \dfrac{b_n}{\|b_n\|} = b_* \in \text{p-}(B_f)_\infty \setminus \{0\}$. Since $a(z) > 0$ by iv)(2), $a_n(z_n) > 0$ for n sufficiently large. Then (11) leads to

$$\frac{1}{2}b_n(u+s_nz_n,u+s_nz_n) + \|u+s_nz_n\|^2 r_{(u,z)}(t_nu+t_ns_nz_n) \le 0$$

which as before implies that

$$b_*(u,u) \le 0,$$

thereby leading to a contradiction. Hence, the proof is established. ■

If the function f admits a second order approximation (A_f, B_f) at $\bar{x}$ such that A_f, B_f are singletons then by Theorem 4.3, we obtain the following necessary conditions for $\bar{x}$ to be a minimizer of f on S.

(i) For every $u \in T(S;\bar{x})$, $A_f(u) \ge 0$.

(ii) If $u \in T(S;\bar{x}) \cap S_X$ and $A_f(u) \le 0$ then

  (1) for every $z \in T^2(S;\bar{x};u) \setminus \mathbb{R}_{++}u$,

$$A_f(z) + \frac{1}{2}B_f(u,u) \ge 0;$$

(2) for every $z \in T_0^2(S;\bar{x};u) \setminus \mathbb{R}_+u$,

$$A_f(z) \geq 0.$$

In this case, we can see that the sufficient conditions only differ from the necessary conditions in the substitution of a nonstrict inequality by a strict inequality.

Example 4.7. Let H be a Hilbert space as in Example 3.2. Set $S := \{te_1 + |t|^\alpha e_2 \ : \ t \in \mathbb{R}\}, \alpha \in (1,2), \bar{x} = 0; u = e_1$. Then $T(S;\bar{x}) = \mathbb{R}e_1, u \in T(S;\bar{x}); e_2 \in T^2{}_0(S;\bar{x},u),\ \ T^2(S;\bar{x},u) = \emptyset$. Let f be a function on S which admits a second order approximation (A_f, B_f) at $\bar{x}$ such that A_f is a singleton and B_f is compact. Then by Theorem 4.6, sufficient conditions for $\bar{x}$ to be a strict minimizer of f are that $A_f(e_1) = 0$ and $A_f(e_2) > 0$.

By a similar proof of Theorem 4.6 with the note of Lemma 4.5 ii), when X is a Hilbert space, we have

Theorem 4.6' (Second order sufficient condition). *Assume that f admits a second order approximation (A_f, B_f) at $\bar{x} \in S$. Then the following conditions are sufficient for $\bar{x}$ to be a strict local minimizer of f on S.*

(i) *S is a second-order tangentiable at $\bar{x}$ and the second order tangent set $T''(S;\bar{x};u)$ for every $u \in T(S;\bar{x})$ are locally compact.*

(ii) *A_f is relatively* p-*compact and bounded, B_f is* abp-*compact.*

(iii) *$a(u) \geq 0 \quad \forall\ a \in A_f,\ u \in T(S;\bar{x})$;*

(iv) *If $u \in T(S;\bar{x}) \cap S_X$ and $a(u) \leq 0$ for some $a \in$ p-clA_f, then*

(1) *for every $z \in T^2(S;\bar{x};u) \cap u^\perp$,*

$$a(z) + \frac{1}{2}b(u,u) > 0 \quad \forall\ a \in \text{p-cl}A_f,\ b \in \text{p-cl}B_f;$$

(2) *for every $z \in T_0^2(S;\bar{x};u) \cap u^\perp \setminus \{0\}$,*

$$a(z) > 0 \quad \forall\ a \in \text{p-cl}A_f;$$

(3) *for every nonzero $b_* \in$ p-$(B_f)_\infty$, $b_*(u,u) > 0$.*

For $\bar{x} \in S,\ \delta > 0$, set

$$S_\delta(\bar{x}) := \{t(x - \bar{x}) :\ t \geq 0, x \in S, \|x - \bar{x}\| < \delta\}.$$

It can be seen that $T(S;\bar{x}) = \bigcap_{\delta>0} \operatorname{cl} S_\delta(\bar{x})$. Next we present an alternate theorem to establish second order sufficient optimality conditions under different set of conditions.

Theorem 4.8 (Second order sufficient condition). *Assume that f admits an* abp-*compact second order approximation (A_f, B_f) at $\bar{x} \in S$. Suppose that S*

is a first-order tangentiable at $\bar{x}$ and the tangent cone $T(S;\bar{x})$ is locally compact. Then each of the following conditions are sufficient for $\bar{x}$ to be a strict local minimizer of f on S.

(i) $a(u) > 0 \quad \forall\, a \in \text{p-cl}A_f \cup [\text{p-}(A_f)_\infty \setminus \{0\}],\ u \in T(S;\bar{x}) \setminus \{0\}$.

(ii) (a) *There exists $\delta > 0$ such that*

$$a(u) \geq 0 \quad \forall\, a \in A_f, u \in S_\delta(\bar{x});$$

(b) *If for nonzero $u \in T(S;\bar{x})$ and $a(u) \leq 0$ for some $a \in \text{p-cl}A_f \cup (\text{p-}(A_f)_\infty \setminus \{0\})$, then*

$$b(u,u) > 0 \quad \forall\, b \in \text{p-cl}B_f \cup [\text{p-}(B_f)_\infty \setminus \{0\}].$$

Proof. Suppose to the contrary that $\bar{x}$ is not a strict local minimizer. Then there exists a sequence $x_n \in S \setminus \{\bar{x}\}$ converging to $\bar{x}$ such that

$$f(x_n) \leq f(\bar{x}). \tag{12}$$

By repeating the argument which has been employed in the proof of Theorem 4.6, there exists $u \in T(S;\bar{x})$ with $\|u\| = 1$ such that $x_n \to \bar{x}$ in the direction u. Consequently, $x_n \to \bar{x}$ in the direction $(u,0)$ and there exist $t_n \downarrow 0,\ v_n \to u$ such that $x_n = \bar{x} + t_n v_n$. By Definition 3.3, there exist real-valued functions r_u and $r_{(u,0)}$ defined on X such that for every n there exist $a_n, \bar{a}_n \in A_f$ and $b_n \in B_f$ satisfying

$$f(x_n) - f(\bar{x}) = \bar{a}_n(x_n - \bar{x}) + \|x_n - \bar{x}\| r_u(x_n - \bar{x}), \tag{13}$$

$$f(x_n) - f(\bar{x}) = a_n(x_n - \bar{x}) + \frac{1}{2} b_n(x_n - \bar{x}, x_n - \bar{x}) + \|x_n - \bar{x}\|^2 r_{(u,0)}(x_n - \bar{x}). \tag{14}$$

By conditions (12), (13) and the abp-compactness of A_f, working along the lines of earlier proofs, there exists $\bar{a} \in \text{p-cl}A_f \cup (\text{p-}(A_f)_\infty \setminus \{0\})$ such that

$$\bar{a}(u) \leq 0$$

which contradicts (i).

Now suppose that (ii) holds. Then condition (14) yields that

$$a_n(v_n) + \frac{1}{2} t_n b_n(v_n, v_n) + t_n \|v_n\|^2 r_{(u,0)}(t_n v_n) \leq 0.$$

By (ii)(a), for n sufficiently large, the latter inequality implies that

$$\frac{1}{2} b_n(v_n, v_n) + \|v_n\|^2 r_{(u,0)}(t_n v_n) \leq 0.$$

By the above inequality along with the abp-compactness of B_f, as done above, there exists $b \in \text{p-cl}B_f \cup (\text{p-}(B_f)_\infty \setminus \{0\})$ such that

$$b(u,u) \leq 0$$

which contradicts (ii)(b). Hence completing the proof. ■

Corollary 4.9. *Assume that X is finite dimensional and f admits a second order approximation (A_f, B_f) at $\bar{x} \in S$. Then each of the following conditions is sufficient for $\bar{x}$ to be a strict local minimizer of f on S.*

(i) $a(u) > 0 \quad \forall\ a \in cl\ A_f \cup [(A_f)_\infty \setminus \{0\}], u \in T(S;\bar{x}) \setminus \{0\}.$

(ii) (a) *There exists $\delta > 0$ such that*

$$a(u) \geq 0 \quad \forall\ a \in A_f, u \in S_\delta(\bar{x});$$

(b) *If $u \in T(S;\bar{x}), u \neq 0$ and $a(u) \leq 0$ for some $a \in \mathrm{cl}\, A_f \cup [(A_f)_\infty \setminus \{0\}]$, then*

$$b(u,u) > 0 \quad \forall\ b \in \mathrm{cl}\, B_f \cup [(B_f)_\infty \setminus \{0\}].$$

Proof. If X is finite dimensional, then S is first order tangentiable, $T(S;\bar{x})$ is locally compact and (A_f, B_f) is abp-compact. Then applying Theorem 4.8, the corollary holds. ■

Remark 4.10. Theorem 4.3 generalizes Proposition 3.1 of [23] in which the objective function is assumed twice differentiabe and Theorem 3.1.1 in [1] since the concept of approximation in this work generalizes that concept defined by Allali and Amahroq and used in [1]. Theorem 4.6 generalizes Proposition 3.4 of [23]. With $Y = \mathbb{R}$, Theorem 4.3 implies [16, Theorem 4.1] in which an assumption of first order differentiability is used and Corollary 4.9 implies Theorem 4.2 of [16].

5. Problems with equality and inequality constraints

Consider the following problem with equality and inequality constraints.

$$\begin{aligned} (\mathrm{P}) \qquad & \min \quad f(x) \\ & \text{subject to } x \in X, \\ & \quad g_i(x) \leq 0, \quad i = 1, \ldots, p, \\ & \quad h_j(x) = 0, \quad j = 1, \ldots, q, \end{aligned}$$

where $f, g_i,\ i = 1, \ldots, p$ and $h_j,\ j = 1, \ldots, q$ are continuously differentiable real-valued functions defined on X. The Lagrangian function of (P) is given by

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{p} \lambda_i g_i(x) + \sum_{j=1}^{q} \mu_j h_j(x).$$

The set of Karush-Kuhn-Tucker (KKT) multipliers $\Lambda(x)$ is defined as

$$\Lambda(x) = \{(\lambda, \mu) \in \mathbb{R}^p_+ \times \mathbb{R}^q \ : \ DL(x, \lambda, \mu) = 0, \ \lambda_i g_i(x) = 0, \ i = 1, \ldots, p\},$$

where $DL(x, \lambda, \mu)$ is the derivative of the function $L(\cdot, \lambda, \mu)$ at x. It is well known that if $\bar{x}$ is a local minimum of (P) and X is finite dimensional then under a first order constraint qualification [8], $\Lambda(\bar{x})$ is nonempty. Let

$$C = \{x \in X : \ g_i(x) \leq 0, i = 1, \ldots, p; \ h_j(x) = 0, j = 1, \ldots, q\}$$

denote the feasible set and define

$$C(\lambda) = \{x \in C : \ \sum_{i=1}^{p} \lambda_i g_i(x) = 0\}.$$

Now we establish the second order optimality conditions for the problem (P).

Theorem 5.1 (Second order necessary condition). *Assume that $\bar{x} \in C$ is a local solution of the problem* (P) *and $\Lambda(\bar{x})$ is nonempty. Let $(\lambda^*, \mu^*) \in \Lambda(\bar{x})$. Suppose that the function $L(\cdot, \lambda^*, \mu^*)$ admits $(DL(\bar{x}, \lambda^*, \mu^*), B_{L(\cdot,\lambda^*,\mu^*)})$ as a second order approximation at $\bar{x}$, with $B_{L(\cdot,\lambda^*,\mu^*)}$ abp-compact. Then for every $u \in T(C(\lambda^*), \bar{x})$, there exists $b \in$ p-cl$B_{L(\cdot,\lambda^*,\mu^*)} \cup [$p-$(B_{L(\cdot,\lambda^*,\mu^*)})_\infty \setminus \{0\}]$ such that*

$$b(u, u) \geq 0.$$

Proof. Let a nonzero direction $u \in T(C(\lambda^*), \bar{x})$ be arbitrary. Then there exist sequences $t_n \to 0^+, u_n \to u$ such that $x_n := \bar{x} + t_n u_n \in C(\lambda^*)$ and hence $x_n \to_{(u,0)} \bar{x}$. By Definition 3.3, there exist a real-valued function $r_{(u,0)}$ converging to 0 as $x \to_{(u,0)} 0$ and a sequence $\{b_n\} \subset B_{L(\cdot,\lambda^*,\mu^*)}$ such that

$$L(x_n, \lambda^*, \mu^*) = L(\bar{x}, \lambda^*, \mu^*) + DL(\bar{x}, \lambda^*, \mu^*)(x_n - \bar{x}) + \frac{1}{2} b_n(x_n - \bar{x}, x_n - \bar{x}) + \|x_n - \bar{x}\|^2 r_{(u,0)}(x_n - \bar{x}). \tag{15}$$

From (15) we have

$$0 \leq \frac{2}{t_n^2}(f(x_n) - f(\bar{x})) = b_n(u_n, u_n) + 2\|u_n\|^2 r_{(u,0)}(x_n - \bar{x}). \tag{16}$$

By the abp-compactness of $B_{L(\cdot,\lambda^*,\mu^*)}$, either $\{b_n\}$ is bounded and converges pointwise to some $b \in$ p-cl$B_{L(\cdot,\lambda^*,\mu^*)}$ or $\{b_n\}$ is unbounded and $\{b_n/\|b_n\|\}$ converges pointwise to some $b \in$ p-$(B_{L(\cdot,\lambda^*,\mu^*)})_\infty \setminus \{0\}$, then by repeating an argument which has been employed several times previously, (16) yields

$$b(u, u) \geq 0.$$

Hence completing the proof. ■

Remark 5.2. A similar result can also be obtained if the inequality constraints are replaced by set inclusion constraint, that is,

$$\widehat{C} = \{x \in X : \ g(x) \in D; \ h_j(x) = 0, j = 1, \dots, q\},$$

where $g : X \to Y$ is a continuously differentiable function and D is a convex subset of Y. For the problem (P) with the constraint set $\widehat{C}$, the Lagrangian function is defined as

$$\widehat{L}(x, \lambda, \mu) = f(x) + (\lambda g)(x) + \sum_{j=1}^{q} \mu_j h_j(x),$$

where $\lambda \in Y^*$. The set of KKT multipliers are given by

$$\widehat{\Lambda}(x) = \{(\lambda, \mu) \in Y^* \times \mathbb{R}^q \ : \ \lambda \in N(g(x), D), \ D\widehat{L}(x, \lambda, \mu) = 0\},$$

where $N(\bar{y}, D)$ denotes the normal cone to the set $D \subset Y$ at $\bar{y}$ defined as

$$N(\bar{y}, D) := \{v \in Y^* : \langle v, y - \bar{y} \rangle \le 0 \quad \forall \ y \in D\}.$$

For $\bar{x} \in \widehat{C}$, define

$$\widehat{C}(\lambda, \bar{x}) = \{x \in \widehat{C} : \ \langle \lambda, g(x) - g(\bar{x}) \rangle = 0\}.$$

Then Theorem 5.1 holds for a local solution $\bar{x}$ of the problem (P) with $T(C(\lambda^*), \bar{x})$ replaced by $T(\widehat{C}(\lambda^*, \bar{x}), \bar{x})$. The proof can be worked out by applying Proposition 3.7. If the second order approximations of the functions are known explicitly, then the above theorem can also be obtained under the compactness of the second order approximation. Below we present the result for the constrained problem with constraint set C. A similar result can be established for the problem with constraint set $\widehat{C}$ by replacing the second order approximations of $g_i, \ i = 1, 2, \dots, p$ by that of g, that is, g admits second order approximation $(Dg(\bar{x}), B_g)$ at $\bar{x}$ with B_g compact.

Example 5.3. Given the problem

$$\begin{array}{lrl} \text{(P')} & \min & f(x, y) \\ & \text{subject to} & (x, y) \in \mathbb{R}^2, \\ & & g(x, y) \le 0, \end{array}$$

where $f(x, y) = x^2 - y^2 + 2y$ and $g(x, y) = x^2 - y$. Consider the point $(0, 0)$. It is not difficult to see that $(0, 0)$ is a local solution of (P'). By simple computing we obtain $\lambda^* = 2 \in \Lambda((0, 0)), \ C = \{(x, y) \in \mathbb{R}^2 \ : \ y \ge x^2\}, \ C(\lambda^*) = \{(x, y) \ : \ y = x^2\}, \ T(C, (0, 0)) = \{(x, y) \ : \ y \ge 0\}, \ T(C(\lambda^*), (0, 0)) = \{(x, 0) \ : \ x \in \mathbb{R}\}$. Since f and g are twice differentiable at $(0, 0)$, the Lagragian $L((x, y), \lambda^*)$ is twice differentiable too. Hence it admits $(DL((0, 0), \lambda^*), D^2L((0, 0), \lambda^*))$ as a compact second order approximation at $(0, 0)$, in which $D^2L((0, 0), \lambda^*) = \begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix}$. Then for every $u = (a, 0) \in T(C(\lambda^*), (0, 0)), \ D^2L((0, 0), \lambda^*)(u, u) = 6a^2 \ge 0$. On the other hand, for $\bar{u} = (0, 1) \in T(C, (0, 0))$ one has $D^2L((0, 0), \lambda^*)(\bar{u}, \bar{u}) = -2 < 0$, which shows that the necessary condition of Theorem 5.1 is not true for $u \in T(C, \bar{x})$ that is Theorem 5.1 is not true when $C(\lambda^*)$ is replaced by C.

Corollary 5.4. *Assume that* $\bar{x} \in C$ *is a local solution of the problem* (P) *and* $\Lambda(\bar{x})$ *is nonempty. Suppose that* $f,\ g_i,\ i = 1, \ldots, p$ *and* $h_j,\ j = 1, \ldots, q$ *admit* $(Df(\bar{x}), B_f)$, $(Dg_i(\bar{x}), B_{g_i})$, $i = 1, \ldots, p$ *and* $(Dh_j(\bar{x}), B_{h_j})$, $j = 1, \ldots, q$ *respectively as the second order approximations at* $\bar{x}$ *with* $B_f,\ B_{g_i},\ i = 1, \ldots, p$ *and* $B_{h_j},\ j = 1, \ldots, q$ *compact. Let* $(\lambda^*, \mu^*) \in \Lambda(\bar{x})$. *Then for every* $u \in T(C(\lambda^*), \bar{x})$, *there exist* $b_f \in B_f$, $b_{g_i} \in B_{g_i}$, $i = 1, \ldots, p$ *and* $b_{h_j} \in B_{h_j}$, $j = 1, \ldots, q$ *such that*

$$b_f(u, u) + \sum_{i=1}^{p} \lambda_i^* b_{g_i}(u, u) + \sum_{j=1}^{q} \mu_j^* b_{h_j}(u, u) \geq 0.$$

Proof. Set $B_{L(\cdot,\lambda^*,\mu^*)} = B_f + \sum_{i=1}^{p} \lambda_i^* B_{g_i} + \sum_{j=1}^{q} \mu_j^* B_{h_j}$. Then $B_{L(\cdot,\lambda^*,\mu^*)}$ is compact and by Proposition 3.7, the Lagrangian function $L(\cdot, \lambda^*, \mu^*)$ admits $(DL(\bar{x}, \lambda^*, \mu^*), B_{L(\cdot,\lambda^*,\mu^*)})$ as a second order approximation at $\bar{x}$. By the compactness of $B_{L(\cdot,\lambda^*,\mu^*)}$ and by Theorem 5.1, there exists $b \in B_{L(\cdot,\lambda^*,\mu^*)}$ such that $b(u, u) \geq 0$. Then we can find $b_f \in B_f$, $b_{g_i} \in B_{g_i}$, $i = 1, \ldots, p$ and $b_{h_j} \in B_{h_j}$, $j = 1, \ldots, q$ such that

$$b_f(u, u) + \sum_{i=1}^{p} \lambda_i^* b_{g_i}(u, u) + \sum_{j=1}^{q} \mu_j^* b_{h_j}(u, u) = b(u, u) \geq 0,$$

thereby establishing the proof. ■

Below we present the results from [11] and [14] which can also be established using the above second order necessary optimality conditions.

Corollary 5.5. *Assume that* $X = \mathbb{R}^n$ *and the problem* (P) *attains a local minimum at* $\bar{x}$. *Suppose that* $(\lambda^*, \mu^*) \in \Lambda(\bar{x})$. *If* $L(\cdot, \lambda^*, \mu^*)$ *admits a pseudo-Hessian* $\partial_*^2 L(\bar{x}, \lambda^*, \mu^*)$ *at* $\bar{x}$ *and the set-valued map* $\partial_*^2 L(\cdot, \lambda^*, \mu^*)$ *is upper semicontinuous at* $\bar{x}$, *then for every* $u \in T(C(\lambda^*), \bar{x})$ *there exists* $M \in \partial_*^2 L(\bar{x}, \lambda^*, \mu^*)$ *such that*

$$\langle Mu, u \rangle \geq 0.$$

Proof. According to a note after Defnition 3.6, $L(\cdot, \lambda^*, \mu^*)$ admits $(DL(\bar{x}, \lambda^*, \mu^*), \operatorname{co} \partial_*^2 L(\bar{x}, \lambda^*, \mu^*))$ as a second order approximation at $\bar{x}$. By the compactness of $\partial_*^2 L(\bar{x}, \lambda^*, \mu^*)$ and by Theorem 5.1, for every $u \in T(C(\lambda^*), \bar{x})$ there exists $N \in \operatorname{co} \partial_*^2 L(\bar{x}, \lambda^*, \mu^*)$ such that

$$\langle Nu, u \rangle \geq 0.$$

Hence, there exist $\alpha_1, \ldots, \alpha_k > 0$, $\sum_{i=1}^{k} \alpha_i = 1$ and $N_1, \ldots, N_k \in \partial_*^2 L(\bar{x}, \lambda^*, \mu^*)$ such that

$$\sum_{i=1}^{k} \alpha_i \langle N_i u, u\rangle = \langle Nu, u\rangle \geq 0.$$

This relation implies the existence of some $N_{i_0}, i_0 \in \{1, \ldots, k\}$ which satisfies

$$\langle N_{i_0} u, u\rangle \geq 0.$$

By setting $M = N_{i_0}$, the proof of the corollary is complete. ■

Corollary 5.6. *Assume that $X = \mathbb{R}^n$, the functions $f,\ g_i,\ i = 1, \ldots, p$ and h_j, $j = 1, \ldots, q$ are $C^{1,1}$ and that the problem* (P) *attains a local minimum at $\bar{x}$. If $(\lambda^*, \mu^*) \in \Lambda(\bar{x})$, then for every $u \in T(C(\lambda^*), \bar{x})$ there exists $M \in \partial_C^2 L(\bar{x}, \lambda^*, \mu^*)$ such that*

$$\langle Mu, u\rangle \geq 0.$$

Proof. The proof is straightforward from a note after Definition 3.6, Theorem 5.1 and from the compactness of $\partial_C^2 L(\bar{x}, \lambda^*, \mu^*)$. ■

Theorem 5.7 (Second order sufficient condition). *Assume that $\bar{x} \in C$ is a feasible point of the problem* (P) *and $\Lambda(\bar{x}) \neq \emptyset$. Let $(\lambda^*, \mu^*) \in \mathit{\Lambda}(\bar{x})$. The following conditions are sufficient for $\bar{x}$ to be a local strict minimizer of* (P).

(i) *The feasible set C is first order tangentiable at $\bar{x}$ and the tangent cone $T(C, \bar{x})$ is locally compact;*

(ii) *The function $L(\cdot, \lambda^*, \mu^*)$ admits $(DL(\bar{x}, \lambda^*, \mu^*), B_{L(\cdot,\lambda^*,\mu^*)})$ as a second order approximation at $\bar{x}$ with $B_{L(\cdot,\lambda^*,\mu^*)}$ being* abp-*compact;*

(iii) *For every $u \in T(C, \bar{x}) \setminus \{0\}$, $b \in$ p-cl$B_{L(\cdot,\lambda^*,\mu^*)} \cup [$p-$(B_{L(\cdot,\lambda^*,\mu^*)})_\infty \setminus \{0\}]$,*

$$b(u, u) > 0.$$

Proof. Suppose to the contrary that $\bar{x}$ is not a local strict minimizer. Then there exists a sequence $x_n \in C \setminus \{\bar{x}\}$ converging to $\bar{x}$ such that

$$f(x_n) \leq f(\bar{x}).$$

By repeating an argument which has been employed in the proof of Theorem 4.6, there exists $u \in T(C; \bar{x})$ with $\|u\| = 1$ such that $x_n \to \bar{x}$ in the direction u. Hence consequently, $x_n \to \bar{x}$ in the direction $(u, 0)$ and there exist $t_n \to 0^+, v_n \to u$ such that $x_n := \bar{x} + t_n v_n$. By Definition 3.3 of the second order approximation, there exists a real-valued function $r_{(u,0)}$ defined on X such that for each n there exists $b_n \in B_{L(\cdot,\lambda^*,\mu^*)}$ satisfying

$$L(x_n, \lambda^*, \mu^*) = L(\bar{x}, \lambda^*, \mu^*) + DL(\bar{x}, \lambda^*, \mu^*)(x_n - \bar{x}) + \frac{1}{2} b_n(x_n - \bar{x}, x_n - \bar{x}) \\ + \|x_n - \bar{x}\|^2 r_{(u,0)}(x_n - \bar{x}).$$

Then

$$\begin{aligned}
f(\bar{x}) &\geq f(x_n) \\
&\geq f(x_n) + \sum_{i=1}^{p} \lambda_i^* g_i(x_n) + \sum_{j=1}^{q} \mu_j^* h_j(x_n) \\
&= L(x_n, \lambda^*, \mu^*) \\
&= L(\bar{x}, \lambda^*, \mu^*) + DL(\bar{x}, \lambda^*, \mu^*)(x_n - \bar{x}) + \frac{1}{2} b_n(x_n - \bar{x}, x_n - \bar{x}) \\
&\quad + \|x_n - \bar{x}\|^2 r_{(u,0)}(x_n - \bar{x}) \\
&= f(\bar{x}) + \frac{1}{2} t_n^2 b_n(v_n, v_n) + t_n^2 \|v_n\|^2 r_{(u,0)}(x_n - \bar{x}).
\end{aligned}$$

Hence

$$\frac{1}{2} b_n(v_n, v_n) + \|v_n\|^2 r_{(u,0)}(x_n - \bar{x}) \leq 0. \tag{17}$$

By the abp-compactness of $B_{L(\cdot,\lambda^*,\mu^*)}$, either $\{b_n\}$ is bounded and pointwise converges to some $b \in$ p-cl$B_{L(\cdot,\lambda^*,\mu^*)}$ or $\{b_n\}$ is unbounded and $\{b_n/\|b_n\|\}$ pointwise converges to some $b_* \in$ p-$(B_{L(\cdot,\lambda^*,\mu^*)})_\infty \setminus \{0\}$. Thus taking limit as $n \to \infty$ in (17), either $b(u, u) \leq 0$ or $b_*(u, u) \leq 0$, which contradicts (iii), thereby completing the proof. ■

Note that an example similar to Example 5.3 can be given which shows that Theorem 5.7 is not true if in condition (iii) C is replaced by $C(\lambda^*)$.

Corollary 5.8 (Second order sufficient conditions). *Assume that $\bar{x}$ is a feasible point of the problem* (P) *and $\Lambda(\bar{x}) \neq \emptyset$. Let $(\lambda^*, \mu^*) \in \Lambda(\bar{x})$. The following conditions are sufficient for $\bar{x}$ to be a local strict minimizer of* (P).

(i) *The feasible set C is first order tangentiable at $\bar{x}$ and the tangent cone $T(C, \bar{x})$ is locally compact;*

(ii) *The functions f, g_i, $i = 1, \ldots, p$ and h_j, $j = 1, \ldots, q$ admit $(Df(\bar{x}), B_f)$, $(Dg_i(\bar{x}), B_{g_i})$, $i = 1, \ldots, p$ and $(Dh_j(\bar{x}), B_{h_j})$, $j = 1, \ldots, q$ as second order approximations at $\bar{x}$, respectively, where B_f, B_{g_i}, $i = 1, \ldots, p$ and B_{h_j}, $j = 1, \ldots, q$ are compact;*

(iii) *For every $u \in T(C, \bar{x}) \setminus \{0\}$, $b_f \in B_f$, $b_{g_i} \in B_{g_i}$, $i = 1, \ldots, p$, $b_{h_j} \in B_{h_j}$, $j = 1, \ldots, q$,*

$$b_f(u, u) + \sum_{i=1}^{p} \lambda_i^* b_{g_i}(u, u) + \sum_{j=1}^{q} \mu_j^* b_{h_j}(u, u) > 0.$$

Proof. As done in Corollary 5.4, set $B_{L(\cdot,\lambda^*,\mu^*)} = B_f + \sum_{i=1}^{p} \lambda_i^* B_{g_i} + \sum_{j=1}^{q} \mu_j^* B_{h_j}$. Then corresponding to $b_n \in B_{L(\cdot,\lambda^*,\mu^*)}$ in (16), there exist $b_f^n \in B_f$, $b_{g_i}^n \in B_{g_i}$, $i = 1, \ldots, p$ and $b_{h_j}^n \in B_{h_j}$, $j = 1, \ldots, q$ such that

$$\frac{1}{2}(b_f^n(v_n, v_n) + \sum_{i=1}^{p}\lambda_i^* b_{g_i}^n(v_n, v_n) + \sum_{j=1}^{q}\mu_j^* b_{h_j}^n(v_n, v_n)) + \|v_n\|^2 r_{(u,0)}(x_n - \bar{x}) \leq 0. \tag{18}$$

By the compactness of B_f, B_{g_i}, $i = 1, \ldots, p$ and B_{h_j}, $j = 1, \ldots, q$, the sequences $\{b_f^n\}$, $\{b_{g_i}^n\}$ and $\{b_{h_j}^n\}$ converge to some $b_f \in B_{f_l}$, $b_{g_i} \in B_{g_i}$ and $b_{h_j} \in B_{h_j}$. Thus taking the limit as $n \to \infty$, (18) yields

$$b_f(u, u) + \sum_{i=1}^{p}\lambda_i^* b_{g_i}(u, u) + \sum_{j=1}^{q}\mu_j^* b_{h_j}(u, u) \leq 0$$

which contradicts (iii), thereby completing the proof. ■

Remark 5.9. The above results can also be obtained for the problem (P) involving set inclusion constraint as in $\widehat{C}$ with the Lagrangian $\widehat{L}$ defined in Remark 5.2.

6. Minimax programming

Consider the following minimax programming problem

$$\begin{array}{lrl} \text{(MP)} & \min & \max\{f_1(x), f_2(x), \ldots, f_k(x)\} \\ & \text{subject to} & x \in X, \\ & & g_i(x) \leq 0, \quad i = 1, \ldots, p, \\ & & h_j(x) = 0, \quad j = 1, \ldots, q, \end{array}$$

where f_l, $l = 1, 2, \ldots, k$, g_i, $i = 1, \ldots, p$ and h_j, $j = 1, \ldots, q$ are continuously differentiable functions on X. Equivalently, (MP) can be rewritten in the form of (P) as

$$\begin{array}{lrl} (\mathrm{P_{eq}}) & \min & r \\ & \text{subject to} & f_l(x) - r \leq 0, \quad l = 1, \ldots, k, \\ & & x \in X, \\ & & g_i(x) \leq 0, \quad i = 1, \ldots, p, \\ & & h_j(x) = 0, \quad j = 1, \ldots, q, \\ & & r \in \mathbb{R}. \end{array}$$

Observe that $(\bar{x}, \bar{r})$ with $\bar{r} = \max\ \{f_1(\bar{x}), f_2(\bar{x}), \ldots, f_k(\bar{x})\}$ is a solution of $(\mathrm{P_{eq}})$ if and only if $\bar{x}$ is a solution of (MP). Let C_{eq} denote the constraint set of $(\mathrm{P_{eq}})$. Define the set of KKT multipliers as

$$\Lambda(x)$$
$$=\Big\{(\eta,\lambda,\mu)\in\mathbb{R}^k_+\times\mathbb{R}^p_+\times\mathbb{R}^q:\sum_{l\in I(x)}\eta_l Df_l(x)+\sum_{i=1}^p\lambda_i Dg_i(x)+\sum_{j=1}^q\mu_j Dh_j(x)=0,$$
$$\sum_{l\in I(x)}\eta_l=1,\ \eta_l=0,\ l\notin I(\bar{x}),\ \lambda_i g_i(x)=0, i=1,\dots,p\Big\},$$

where $I(x)\subset\{1,2,\dots,k\}$ is the set of indices where maximum is attained at x. Define

$$C(\eta,\lambda)=\{(x,r)\in C_{eq}:\sum_{l=1}^k\eta_l(f_l(x)-r)=0,\sum_{i=1}^p\lambda_i g_i(x)=0\}.$$

Now we establish the second order optimality conditions for the problem (MP).

Theorem 6.1 (Second order necessary conditions). *Assume that $\bar{x}$ is a local solution of the problem* (MP) *and $\Lambda(\bar{x})$ is nonempty. Suppose that $f_l,\ l=1,\dots,k,\ g_i,\ i=1,\dots,p$ and $h_j,\ j=1,\dots,q$ admit $(Df_l(\bar{x}),B_{f_l})$, $l=1,\dots,k$,$(Dg_i(\bar{x}),B_{g_i}),\ i=1,\dots,p$ and $(Dh_j(\bar{x}),B_{h_j}),\ j=1,\dots,q$ as second order approximations at $\bar{x}$, respectively, where $B_{f_l},\ l=1,\dots,k,\ B_{g_i},\ i=1,\dots,p$ and $B_{h_j},\ j=1,\dots,q$ are compact. Let $(\eta^*,\lambda^*,\mu^*)\in\Lambda(\bar{x})$. Then for every $u\in\mathbb{R}^n$ corresponding to which there exists $s\in\mathbb{R}$ satisfying $(u,s)\in T(C(\eta^*,\lambda^*),(\bar{x},\bar{r}))$, there exist $b_{f_l}\in B_{f_l},\ l=1,\dots,k,\ b_{g_i}\in B_{g_i},\ i=1,\dots,p$ and $b_{h_j}\in B_{h_j},\ j=1,\dots,q$ such that*

$$\sum_{l\in I(\bar{x})}\eta^*_l b_{f_l}(u,u)+\sum_{i=1}^p\lambda^*_i b_{g_i}(u,u)+\sum_{j=1}^q\mu^*_j b_{h_j}(u,u)\geq 0.$$

Proof. Since $\bar{x}$ is a minimizer of (MP), then $(\bar{x},\bar{r})$ is the minimizer of $(\mathrm{P_{eq}})$. Note that the objective function for the equivalent problem $(\mathrm{P_{eq}})$ is twice continuously differentiable. Therefore by Corollary 5.4, for every $(u,s)\in T(C(\eta^*,\lambda^*),(\bar{x},\bar{r}))$, there exist $b_{f_l}\in B_{f_l},\ l=1,\dots,k,\ b_{g_i}\in B_{g_i},\ i=1,\dots,p$ and $b_{h_j}\in B_{h_j}$, $j=1,\dots,q$ such that

$$(0,1)+\sum_{l=1}^k\eta^*_l(Df_l(\bar{x}),-1)+\sum_{i=1}^p\lambda^*_i(Dg_i(\bar{x}),0)+\sum_{j=1}^q\mu^*_j(Dh_j(\bar{x}),0)=(0,0),$$
$$\sum_{l=1}^k\eta^*_l b_{f_l}(u,u)+\sum_{i=1}^p\lambda^*_i b_{g_i}(u,u)+\sum_{j=1}^q\mu^*_j b_{h_j}(u,u)\geq 0,$$
$$\eta^*_l(f_l(\bar{x})-\bar{r})=0,\ l=1,\dots,k,\ \lambda^*_i g_i(\bar{x})=0,\ i=1,\dots,p.$$

The above conditions yield that

$$\sum_{l=1}^{k}\eta_l^* Df_l(\bar{x}) + \sum_{i=1}^{p}\lambda_i^* Dg_i(\bar{x}) + \sum_{j=1}^{q}\mu_j^* Dh_j(\bar{x}) = 0,$$

$$\sum_{l=1}^{k}\eta_l^* b_{f_l}(u,u) + \sum_{i=1}^{p}\lambda_i^* b_{g_i}(u,u) + \sum_{j=1}^{q}\mu_j^* b_{h_j}(u,u) \geq 0,$$

$$\sum_{l=1}^{k}\eta_l^* = 1,\ \eta_l^*(f_l(\bar{x}) - \bar{r}) = 0,\ l = 1,2,\ldots,k,\ \lambda_i^* g_i(\bar{x}) = 0,\ i = 1,2,\ldots,p.$$

From the latter condition, for $l \notin I(\bar{x})$, $\eta_l^* = 0$, thereby implying $(\eta^*, \lambda^*, \mu^*) \in \Lambda(\bar{x})$ and hence proving the result. ■

Theorem 6.2 (Second order sufficient conditions). *Assume that $\bar{x}$ is a feasible point of the problem* (MP) *and $\Lambda(\bar{x})$ is nonempty. Let $(\eta^*, \lambda^*, \mu^*) \in \Lambda(\bar{x})$. The following conditions are sufficient for $\bar{x}$ to be a local strict minimizer of* (MP).

(i) *The feasible set C is first-order tangentiable at $\bar{x}$ and the tangent cone $T(C,\bar{x})$ is locally compact;*

(ii) *The functions $f_l,\ l = 1,\ldots,k,\ g_i,\ i = 1,\ldots,p$ and $h_j,\ j = 1,\ldots,q$ admit $(Df_l(\bar{x}), B_{f_l}),\ l = 1,\ldots,k,\ (Dg_i(\bar{x}), B_{g_i}),\ i = 1,\ldots,p$ and $(Dh_j(\bar{x}), B_{h_j})$, $j = 1,\ldots,q$ as second order approximations at $\bar{x}$, respectively, where $B_{f_l},\ l = 1,\ldots,k,\ B_{g_i},\ i = 1,\ldots,p$ and $B_{h_j},\ j = 1,\ldots,q$ are compact;*

(iii) *For every $u \in T(C,\bar{x})\setminus\{0\}$, $b_{f_l} \in B_{f_l},\ l = 1,\ldots,k$, $b_{g_i} \in B_{g_i},\ i = 1,\ldots,p$, $b_{h_j} \in B_{h_j},\ j = 1,\ldots,q$,*

$$\sum_{l\in I(\bar{x})}\eta_l^* b_{f_l}(u,u) + \sum_{i=1}^{p}\lambda_i^* b_{g_i}(u,u) + \sum_{j=1}^{q}\mu_j^* b_{h_j}(u,u) > 0.$$

Proof. On the contrary, suppose that $\bar{x}$ is not a local strict minimizer. Then there exists a sequence $x_n \in C\setminus\{\bar{x}\}$ converging to $\bar{x}$ such that

$$f_l(x_n) \leq \max_{l=1,\ldots,k}\{f_1(x_n),\ldots,f_k(x_n)\} \leq \max_{l=1,\ldots,k}\{f_1(\bar{x}),\ldots,f_k(\bar{x})\} = f_l(\bar{x}),\ l \in I(\bar{x}).$$

Replacing the objective function f by $\sum_{l\in I(\bar{x})}\eta_l^* f_l$ in Corollary 5.8, condition (18) becomes

$$\frac{1}{2}\Big(\sum_{l\in I(\bar{x})}\eta_l^* b_{f_l}^n(v_n,v_n) + \sum_{i=1}^{p}\lambda_i^* b_{g_i}^n(v_n,v_n) + \sum_{j=1}^{q}\mu_j^* b_{h_j}^n(v_n,v_n)\Big) + \|v_n\|^2 r_{(u,0)}(x_n - \bar{x}) \leq 0. \tag{19}$$

By the compactness of $B_{f_l},\ l = 1,\ldots,k$, $B_{g_i},\ i = 1,\ldots,p$ and $B_{h_j},\ j = 1,\ldots,q$, the sequences $\{b_{f_l}^n\}$, $\{b_{g_i}^n\}$ and $\{b_{h_j}^n\}$ converge to some $b_{f_l} \in B_{f_l}$, $b_{g_i} \in B_{g_i}$ and $b_{h_j} \in B_{h_j}$. Thus taking the limit as $n \to \infty$, (19) yields

$$\sum_{l\in I(\bar{x})} \eta_l^* b_{f_l}(u,u) + \sum_{i=1}^{p} \lambda_i^* b_{g_i}(u,u) + \sum_{j=1}^{q} \mu_j^* b_{h_j}(u,u) \leq 0$$

which contradicts (iii), thereby completing the proof. ■

Remark 6.3. In [7], the minimax problem considered was

$$\begin{array}{lrl} \text{(MP)} & \min & \max_{y\in Y} f(x,y) \\ & \text{subject to} & x \in X, \\ & & g(x) \in D, \\ & & h_j(x) = 0, \quad j = 1,\dots,q. \end{array}$$

The second order necessary and sufficiency conditions as above can be worked out for this problem by considering the equivalent problem

$$\begin{array}{lrl} \text{(MP)} & \min & r \\ & \text{subject to} & \hat{f}(x) - re \in -C_+(Y), \\ & & r \in \mathbb{R}, x \in X, \\ & & g(x) \in D, \\ & & h_j(x) = 0, \quad j = 1,\dots,q, \end{array}$$

where $\hat{f} : X \to C(Y)$ and $e : Y \to \mathbb{R}$ are defined as $\hat{f}(x) = f(x,y)$ and $re(y) = r$ for every $y \in Y$, respectively. Here $C(Y)$ denotes the Banach space of real-valued continuous functions and $C_+(Y) = \{u \in C(Y) : u(y) \geq 0 \ \forall\, y \in Y\}$ is the space of positive functionals on Y. The second order necessary and sufficient optimality conditions can be worked out by considering sets defined in Remark 5.2 and applying the Riesz representation theorem.

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