Vietnam Journal of Mathematics **40**:2&3(2012) 255-274

Vietnam Journal of MATHEMATICS

On Inexact Tikhonov and Proximal Point Regularization Methods for Pseudomonotone Equilibrium Problems*

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

Received October 17, 2011
Revised April 4, 2012

Abstract. We investigate the inexact Tikhonov and proximal point regularization methods for pseudomonotone equilibrium problems. In this case, the regularized subproblems might not be strongly monotone, even not pseudomonotone. However, any iterative sequence of the regularized subproblems tends to the same solution, which, for the Tikhonov method, is the projection of the starting point onto the solution set of the original problem. This convergence result suggests algorithms for finding the limit point of the Tikhonov regularization method. Application to multivalued pseudomonotone variational inequalities is discussed.

2000 Mathematics Subject Classification. 49J40, 90C33, 47H17.

Key words. Pseudomonotone equilibrium problem, inexact Tikhonov and proximal point methods, multivalued pseudomonotone variational inequality.

1. Introduction

Throughout this article we assume that $\mathcal{H}$ is a real Hilbert space whose inner product and the norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. We say that

* This work is supported in part by NAFOSTED.

a sequence $\{x^k\} \subset \mathcal{H}$ weakly converges to a vector $x \in \mathcal{H}$ and write $x^k \rightharpoonup x$, if $\{x^k\}$ converges to x in the weak topology. If $\lim_{k\to\infty} \|x^k - x\| = 0$, then we say it strongly converges to x and we write $x^k \to x$. Assume that K is a nonempty, convex, closed subset in $\mathcal{H}$, and $f : K \times K \to \mathbb{R}$ is a bifunction satisfying $f(x,x) = 0$ for all $x \in K$; such a bifunction is often called an *equilibrium bifunction*.

We consider the following equilibrium problem, which is known also as the Ky Fan inequality due to his contribution to the field

$$\text{Find } x \in K \text{ such that } f(x,y) \geq 0 \quad \forall y \in K. \tag{1}$$

As usual, the *dual problem* of (1) is defined as

$$\text{Find } x \in K \ \text{ such that } f(y,x) \leq 0 \quad \forall y \in K. \tag{2}$$

We will refer to (1) and (2) as $EP(K,f)$ and $DEP(K,f)$, and their solution sets are denoted by $SEP(K,f)$ and $SDEP(K,f)$, respectively.

The equilibrium problem is very general since it includes, as special cases, optimization problem, Nash equilibria, Kakutani fixed points and variational inequality (see e.g. [4, 18]). The interest of the equilibrium problem is that it unifies all above mentioned problems in a convenient way. Moreover, many methods devoted to solving one of these problems can be extended, with suitable modifications, to the equilibrium problem.

The regularization is one of the most important techniques in handling ill-posed problems [25]. The Tikhonov and proximal point methods are the two most popular regularization methods. The main idea of these two methods for equilibrium problems is to construct regularized subproblems from the original problem by adding a strongly monotone bifunction depending on a parameter. Unlike the Tikhonov method, in the proximal point method the regularized subproblem is updated at each iteration. A solution of the original problem can be received as the limit of the iterate sequence obtained by solving the subproblems as the parameters tend to a suitable limit.

The Tikhonov and proximal point methods are widely used for solving various classes of applied mathematics such as convex optimization problem, variational inequality including maximal monotone operator, and others. Recently, these methods have been applied to monotone equilibrium problems [15, 20]. In this monotone case the regularized subproblems are uniquely solvable due to the fact that the sum of a monotone and a strongly monotone bifunction is strongly monotone. However, this uniqueness property no longer holds for pseudomonotone equilibrium problems, since the subproblems are in general not strongly monotone, even not pseudomonotone. Solving the subproblems becomes a difficult task, since almost all of the existing methods are applicable to equilibrium problems enjoying certain monotonicity property. In recent years some theoretical results on regularization methods for pseudomonotone variational inequalities and equilibrium problems have been established (see e.g. in [6–9, 12–14, 22–24] and the references therein), and in [11, 19, 21] for nonmonotone practical equilibrium models.

In our recent paper [8], we have studied the exact Tikhonov regularization method for pseudomonotone equilibrium problem in the Euclidean space $\mathbb{R}^n$. In this paper we continue our work in [8] by considering two inexact Tikhonov and proximal point methods for the pseudomonotone equilibrium problem in real Hilbert spaces. Precisely, we show that the regularized problems are solvable if and only if so are the original problems. Moreover, in both methods, any sequence of approximate iterates tends to the same limit which is a solution of the original problem. These results show that both the inexact Tikhonov and proximal point methods can be applied to the pseudomonotone equilibrium problems.

The paper is organized as follows. The next section presents some preliminaries on the existence of solutions to equilibrium problems. In the third section, we study an inexact Tikhonov and proximal regularization methods for the pseudomonotone equilibrium problem $EP(K, f)$. We show that each regularized problem admits a solution if and only if the original one does. Moreover any sequence of solutions of the regularized subproblems tends to the same solution of the original problem. This solution is also nearest to the guess solution chosen prior in the case of Tikhonov method. In the last section, we discuss an application to multivalued pseudomonotone variational inequalities.

2. Preliminaries

We recall the following well-known definitions on monotonicity (see e.g. [3, 10, 18]).

Definition 2.1. A bifunction $f : K \times K \to \mathbb{R}$ is said to be

a) *strongly monotone* on K with modulus $\gamma > 0$ if

$$f(x, y) + f(y, x) \le -\gamma \|x - y\|^2 \quad \forall x, y \in K;$$

b) *monotone* on K if

$$f(x, y) + f(y, x) \le 0 \quad \forall x, y \in K;$$

c) *pseudomonotone* on K if

$$f(x, y) \ge 0 \Rightarrow f(y, x) \le 0 \quad \forall x, y \in K.$$

The following implications are obvious from the definition

$$a) \Rightarrow b) \Rightarrow c).$$

In the sequel we make use of the following blanket assumptions:

(A_1) $f(\cdot, y)$ is weakly upper semicontinuous (shortly w.u.s.c.) for each $y \in K$;

(A_2) $f(x, \cdot)$ is weakly lower semicontinuous (shortly w.l.s.c.) and convex for each $x \in K$;

(A_3) there exist a closed ball $B \subset \mathcal{H}$ and a vector $y^0 \in B \cap K$ such that

$$f(x, y^0) < 0 \quad \forall x \in K \backslash B.$$

Assumption (A_3) is often called the coercivity property. Note that if Assumption (A_2) is satisfied, then by convexity of $f(x, \cdot)$, the lower level set

$$\{y \in K : f(x, y) \leq \alpha\}$$

is weakly closed and convex for every α (hence closed). Thus, Assumption (A_2) is equivalent to

(A_2') $f(x, \cdot)$ is lower semicontinuous and convex for each $x \in K$.

The following well-known propositions will be used in the next section.

Proposition 2.2. (See [3, Propositions 3.1, 3.2])

a) *If f satisfies Assumptions* $(A_1), (A_2)$ *and is strongly monotone on K, then $EP(K, f)$ has a unique solution;*

b) *If f satisfies Assumptions* $(A_1), (A_2)$ *and is pseudomonotone on K, then $SEP(K, f) = SDEP(K, f)$, and they are convex, weakly closed sets;*

c) *If f satisfies Assumptions* $(A_1), (A_2)$ *and* (A_3)*, then the solution set of $SEP(K, f)$ is nonempty. In addition, if f is pseudomonotone on K, then $SEP(K, f)$ is a convex, weakly compact set.*

The following proposition has been proved in finite dimensional Euclidean spaces (see [8, Lemma 3.1]). It can be extended to real Hilbert spaces as follows.

Proposition 2.3. *Suppose that f satisfies Assumptions* (A_1) *and* (A_2)*. Consider the following statements*

a) *There exists a vector $y^0 \in K$ such that the set*

$$L(y^0, f) := \{x \in K : f(x, y^0) \geq 0\}$$

is bounded;

b) *There exist a closed ball $B \subset \mathcal{H}$ and a vector $y^0 \in K \cap B$ such that*

$$f(x, y^0) < 0 \quad \forall x \in K \setminus B;$$

c) *The set $SEP(K, f)$ is nonempty and weakly compact.*

It holds that a) $\Rightarrow$ b) $\Rightarrow$ c). In addition, if f is pseudomonotone on K, then $SEP(K, f)$ is convex and the set

$$L_>(y^0, f) := \{x \in K : f(x, y^0) > 0\}$$

is empty for any $y^0 \in SEP(K, f)$.

Proof. a) ⇒ b): By the assumption a), we can take B as a closed ball containing $L(y^0, f)$. Then it is obvious that

$$\{x \in K \setminus B : f(x, y^0) \geq 0\} = \emptyset.$$

Hence b) holds.

b) ⇒ c): By Proposition 2.2.c) we have $SEP(K, f) \neq \emptyset$. Since K is weakly closed and $f(\cdot, y)$ is weakly upper semicontinuous on K, the set $SEP(K, f)$ is weakly closed. Moreover, from b) and the definition of $L(y^0, f)$ it follows that

$$SEP(K, f) \subseteq L(y^0, f) \subseteq K \cap B.$$

Thus $SEP(K, f)$ is weakly compact.

To see the last assertion, let $y^0 \in SEP(K, f)$. Then $f(y^0, x) \geq 0$ for every $x \in K$. By pseudomonotonicity, it follows that $f(x, y^0) \leq 0$ for every $x \in K$. Hence $L_>(y^0, f) = \emptyset$. The convexity of $SEP(K, f)$ follows from Proposition 2.2.c). ■

We recall [5, page 51] that a Banach space E is *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\forall x, y \in E : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| > \varepsilon \Rightarrow \left\| \frac{x + y}{2} \right\| < 1 - \delta.$$

All Hilbert spaces are uniformly convex (see [5, Proposition V.1]).

Proposition 2.4. (see [5, Proposition III.30]) *Let E be a uniformly convex Banach space. Suppose that the sequence $\{x^k\}$ in E weakly converges to x and*

$$\overline{\lim}_{k\to\infty} \|x^k\| \leq \|x\|.$$

Then $\{x^k\}$ strongly converges to x.

Proposition 2.5. (see [16, Lemma 2.2]) *Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences in $\mathbb{R}_+$ such that $\{c_n\} \in l^1$ and $a_{n+1} \leq a_n - b_n + c_n$. Then $\{a_n\}$ converges and $\{b_n\} \in l^1$.*

3. Main results

3.1. The inexact Tikhonov regularization

We associate with the equilibrium problem $EP(K, f)$ the regularized problem $EP(K, f_\varepsilon)$ defined as follows

$$\begin{cases} \text{Find } x \in K \text{ such that} \\ f_\varepsilon(x,y) := f(x,y) + \varepsilon\langle x - x^g, y - x\rangle \geq 0 \quad \forall y \in K, \end{cases} \tag{3}$$

where $x^g \in K$ is a given point which plays the role of a guess solution to problem (1) and $\varepsilon > 0$ is a given regularization parameter. Given $\delta \geq 0$, we call a point $x \in K$ that satisfies

$$f_\varepsilon(x,y) := f(x,y) + \varepsilon\langle x - x^g, y - x\rangle \geq -\delta \quad \forall y \in K$$

a *δ-solution* to the subproblem (3). By $S_\delta(K, f_\varepsilon)$ we denote the set of all δ-solutions to (3).

Remark 3.1. We note that

(i) $x = x^g$ is a solution of the equilibrium problem (1) if and only if it is a solution of the auxiliary problem (3).

The next lemma was proved by Noor [22] for monotone equilibrium problems (see also Tam-Yao-Yen [24] for variational inequalities). We now extend it to pseudomonotone equilibrium problems.

Lemma 3.2. *Suppose that f is pseudomonotone on K. Then for any $\varepsilon > 0$, $\delta \geq 0$, $\bar{x} \in SEP(K,f)$, $x(\varepsilon) \in S_\delta(K, f_\varepsilon)$ and $x^g \in K$, it holds that*

a) $\|x^g - x(\varepsilon)\|^2 + \|x(\varepsilon) - \bar{x}\|^2 \leq \|x^g - \bar{x}\|^2 + 2\frac{\delta}{\varepsilon}$;

b) $S_\delta(K, f_\varepsilon) \subset \overline{B}\left(0, \left\|\frac{\bar{x}+x^g}{2}\right\| + \sqrt{\left\|\frac{\bar{x}-x^g}{2}\right\|^2 + \frac{\delta}{\varepsilon}}\right) \cap K$;

c) $\|x(\varepsilon) - x^g\| \leq \left\|\frac{\bar{x}-x^g}{2}\right\| + \sqrt{\left\|\frac{\bar{x}-x^g}{2}\right\|^2 + \frac{\delta}{\varepsilon}}$,

where $\overline{B}(x,r)$ stands for the closed ball around x with radius r.

Proof. Since $\bar{x} \in SEP(K,f)$, by the pseudomonotonicity of f, we have

$$f(\bar{x}, y) \geq 0 \Rightarrow f(y, \bar{x}) \leq 0 \quad \forall y \in K. \tag{4}$$

As $x(\varepsilon) \in S_\delta(K, f_\varepsilon)$, it holds that

$$f(x(\varepsilon), y) + \varepsilon\langle x(\varepsilon) - x^g, y - x(\varepsilon)\rangle \geq -\delta \quad \forall y \in K. \tag{5}$$

Substituting $y = x(\varepsilon)$ into the second inequality in (4) and $y = \bar{x}$ in (5) we obtain

$$f(x(\varepsilon), \bar{x}) \leq 0 \;\text{ and }\; f(x(\varepsilon), \bar{x}) + \varepsilon\langle x(\varepsilon) - x^g, \bar{x} - x(\varepsilon)\rangle \geq -\delta.$$

From which we deduce that

$$\frac{1}{2}\left[\|x^g - \bar{x}\|^2 - \|x^g - x(\varepsilon)\|^2 - \|x(\varepsilon) - \bar{x}\|^2\right] = \langle x(\varepsilon) - x^g, \bar{x} - x(\varepsilon)\rangle \geq -\frac{\delta}{\varepsilon}.$$

Hence, a) holds true. On the other hand

$$\|x(\varepsilon)-x^g\|^2+\|[x(\varepsilon)-x^g]-[\bar{x}-x^g]\|^2\le\|\bar{x}-x^g\|^2+2\frac{\delta}{\varepsilon},$$

which implies

$$\|x(\varepsilon)-x^g\|^2-\langle x(\varepsilon)-x^g,\bar{x}-x^g\rangle\le\frac{\delta}{\varepsilon}.$$

Thus

$$\begin{aligned}\left\|x(\varepsilon)-\frac{\bar{x}+x^g}{2}\right\|^2&=\left\|x(\varepsilon)-x^g-\frac{\bar{x}-x^g}{2}\right\|^2\\&=\|x(\varepsilon)-x^g\|^2-\langle x(\varepsilon)-x^g,\bar{x}-x^g\rangle+\left\|\frac{\bar{x}-x^g}{2}\right\|^2\\&\le\left\|\frac{\bar{x}-x^g}{2}\right\|^2+\frac{\delta}{\varepsilon}\end{aligned}$$

which proves b) and c). ■

Lemma 3.3. *Suppose that f is pseudomonotone on K and satisfies Assumptions* (A_1) *and* (A_2). *If the solution set of $EP(K,f)$ is nonempty, then for any $\varepsilon>0$, the δ-solution set $S_\delta(K,f_\varepsilon)$ is nonempty and weakly compact.*

Proof. According to Proposition 2.3, it is sufficient to find a vector $y^0\in K$ such that the set

$$L_\delta(y^0,f_\varepsilon):=\{x\in K: f_\varepsilon(x,y^0)=f(x,y^0)+\varepsilon\langle x-x^g,y^0-x\rangle\ge-\delta\}$$

is bounded. By the nonemptiness of $SEP(K,f)$, we can take $y^0\in SEP(K,f)$. For any $x\in L_\delta(y^0,f_\varepsilon)$, it holds that

$$f_\varepsilon(x,y^0)=f(x,y^0)+\varepsilon\langle x-x^g,y^0-x\rangle\ge-\delta.$$

Using the inequality in Lemma 3.2.a) with $x(\varepsilon)=x,\bar{x}=y^0$ we obtain

$$\|x^g-x\|^2+\|x-y^0\|^2\le\|x^g-y^0\|^2+2\frac{\delta}{\varepsilon}$$

which implies

$$\|x-x^g\|\le\sqrt{\|y^0-x^g\|^2+2\frac{\delta}{\varepsilon}}.$$

Thus

$$\|x\|\le\|x^g\|+\sqrt{\|y^0-x^g\|^2+2\frac{\delta}{\varepsilon}}\quad\forall x\in L_\delta(y^0,f_\varepsilon).$$

That means the set $L_\delta(y^0,f_\varepsilon)$ is bounded. ■

It is obvious from Lemma 3.3 that the exact solution set $SEP(K,f_\varepsilon)$ is weakly compact if it is nonempty.

As we have known, if f is monotone on K, then the bifunction f_ε of the regularized problem $EP(K, f_\varepsilon)$ is strongly monotone on K and the subproblem $EP(K, f_\varepsilon)$ has a unique solution. When f is pseudomonotone, the bifunction f_ε, in general, is nonpseudomonotone (see [8, Example 3.1]). Roughly speaking, the following theorem states that, for pseudomonotone equilibrium problems, the regularized problems are uniquely solvable at the infinity in the sense that the solution sets of the regularized subproblems tend to a singleton as the regularization parameter $\frac{1}{t}$ tends to infinity.

Theorem 3.4. *Suppose that f is pseudomonotone on K, that f satisfies Assumptions* $(A_1), (A_2)$*, and that the solution set of $EP(K, f)$ is nonempty. Let $\{\varepsilon_k\}$, $\{\delta_k\}$ be two decreasing sequences of positive numbers satisfying $\varepsilon_k \to 0$, $\delta_k \to 0$ and $\frac{\delta_k}{\varepsilon_k} \to 0$ as $k \to \infty$. Then the following hold*

a) *For any $k \in \mathbb{N}$, the δ_k-solution set $S_{\delta_k}(K, f_{\varepsilon_k})$ is nonempty, weakly compact, and it holds that*

$$\|x^g - x^k\|^2 + \|x^k - \bar{x}\|^2 \leq \|x^g - \bar{x}\|^2 + 2\frac{\delta_k}{\varepsilon_k}, \tag{6}$$

where $\bar{x} \in SEP(K, f)$, $x^k \in S_{\delta_k}(K, f_{\varepsilon_k})$ and $x^g \in K$;

b) *The sequence $\{x^k\}$, where x^k is arbitrarily chosen in $S_{\delta_k}(K, f_{\varepsilon_k})$, strongly converges to the unique solution x^* of the equilibrium problem $EP(\widetilde{K}, g)$ with $\widetilde{K} := SEP(K, f)$ and $g(x, y) := \langle x - x^g, y - x\rangle$. In addition, x^* is the element in the solution set of $EP(K, f)$ that is nearest in norm to x^g.*

Proof. a) can be obtained from Lemmas 3.2 and 3.3 with

$$x(\varepsilon) = x^k, \varepsilon = \varepsilon_k \text{ and } \delta = \delta_k.$$

b) As $EP(K, f)$ is nonempty, we can take any $\bar{x} \in SEP(K, f)$. By a), the δ_k-solution set of $EP(K, f_{\varepsilon_k})$ is nonempty for all $k \in \mathbb{N}$. Let $x^k \in S_{\delta_k}(K, f_{\varepsilon_k})$ be arbitrary. Then, for each $k \in \mathbb{N}$, we have

$$f(\bar{x}, x^k) \geq 0 \text{ and } f_{\varepsilon_k}(x^k, \bar{x}) := f(x^k, \bar{x}) + \varepsilon_k\langle x^k - x^g, \bar{x} - x^k\rangle \geq -\delta_k.$$

Since f is pseudomonotone, $f(x^k, \bar{x}) \leq 0$. Thus, from the last inequality it follows that

$$g(x^k, \bar{x}) := \langle x^k - x^g, \bar{x} - x^k\rangle \geq -\frac{\delta_k}{\varepsilon_k} \quad \forall k. \tag{7}$$

On the other hand, since $\frac{\delta_k}{\varepsilon_k} \to 0$, it is bounded, i.e.,

$$\exists M > 0 : 0 < \frac{\delta_k}{\varepsilon_k} \leq M \quad \forall k.$$

Applying assertions b) and c) in Lemma 3.2 with $x(\varepsilon) = x^k, \varepsilon = \varepsilon_k, \delta = \delta_k$, we have

$$x^k \in S_{\delta_k}(K, f_{\varepsilon_k}) \subset \overline{B}\left(0, \left\|\frac{\bar{x}+x^g}{2}\right\| + \sqrt{\left\|\frac{\bar{x}-x^g}{2}\right\|^2 + \frac{\delta_k}{\varepsilon_k}}\right) \cap K$$
$$\subset \overline{B}\left(0, \left\|\frac{\bar{x}+x^g}{2}\right\| + \sqrt{\left\|\frac{\bar{x}-x^g}{2}\right\|^2 + M}\right) \cap K \quad \forall k$$

and

$$\|x^k - x^g\| \le \left\|\frac{\bar{x}-x^g}{2}\right\| + \sqrt{\left\|\frac{\bar{x}-x^g}{2}\right\|^2 + \frac{\delta_k}{\varepsilon_k}} \quad \forall k. \tag{8}$$

Since $\overline{B}\left(0, \|\frac{\bar{x}+x^g}{2}\| + \sqrt{\|\frac{\bar{x}-x^g}{2}\|^2 + M}\right) \cap K$ is a weakly compact set, there exists a subsequence $\{x^{k_j}\} \subseteq \{x^k\}$ such that

$$x^{k_j} \rightharpoonup x^* \in \overline{B}\left(0, \left\|\frac{\bar{x}+x^g}{2}\right\| + \sqrt{\left\|\frac{\bar{x}-x^g}{2}\right\|^2 + M}\right) \cap K.$$

Since x^{k_j} is a δ_{k_j}-solution to the subproblem $EP(K, f_{\varepsilon_{k_j}})$, we have

$$f_{\varepsilon_{k_j}}(x^{k_j}, y) = f(x^{k_j}, y) + \varepsilon_{k_j}\langle x^{k_j} - x^g, y - x^{k_j}\rangle \ge -\delta_{k_j} \quad \forall y \in K.$$

Because $\delta_{k_j} \searrow 0$, $\varepsilon_{k_j} \to 0$ and $\{x_{k_j}\}$ is bounded, by weak upper semicontinuity of $f(\cdot, y)$, we get

$$0 \le \overline{\lim}_{k_j\to\infty} f_{\varepsilon_{k_j}}(x^{k_j}, y) \le \overline{\lim}_{k_j\to\infty} f(x^{k_j}, y) \le f(x^*, y) \quad \forall y \in K,$$

which implies that $x^* \in SEP(K, f) := \widetilde{K}$. Moreover, applying (7) with $k = k_j$ we have

$$g(x^{k_j}, \bar{x}) := \langle x^{k_j} - x^g, \bar{x} - x^{k_j}\rangle \ge -\frac{\delta_{k_j}}{\varepsilon_{k_j}} \quad \forall k_j.$$

Then

$$0 \ge \overline{\lim}\langle x^{k_j} - x^g, x^{k_j} - \bar{x}\rangle \ge \langle x^* - x^g, x^* - \bar{x}\rangle.$$

Thus

$$g(x^*, \bar{x}) := \langle x^* - x^g, \bar{x} - x^*\rangle \ge 0.$$

Since $\bar{x}$ is an arbitrary element in $\widetilde{K}$, we can deduce that x^* is a solution of $EP(\widetilde{K}, g)$. It is easy to verify that g is strongly monotone on K which contains $\widetilde{K}$, hence, problem $EP(\widetilde{K}, g)$ has a unique solution by Proposition 2.2.a). Thus, we have shown that $\{x^k\}$ is bounded and any its weak limit point is the unique solution x^* of $EP(\widetilde{K}, g)$. Therefore the whole sequence $\{x^k\}$ must weakly converge to x^*. Substituting $\bar{x} = x^*$ into the inequality (8) yields

$$\|x^k - x^g\| \le \left\|\frac{x^* - x^g}{2}\right\| + \sqrt{\left\|\frac{x^* - x^g}{2}\right\|^2 + \frac{\delta_k}{\varepsilon_k}} \quad \forall k.$$

As $\frac{\delta_k}{\varepsilon_k} \to 0$, we obtain in the limit that

$$\overline{\lim}_{k\to\infty}\|x^k - x^g\| \le \lim_{k\to\infty}\left(\left\|\frac{x^* - x^g}{2}\right\| + \sqrt{\left\|\frac{x^* - x^g}{2}\right\|^2 + \frac{\delta_k}{\varepsilon_k}}\right) = \|x^* - x^g\|.$$

By Proposition 2.4, the sequence $\{x^k - x^g\}$ strongly converges to $x^* - x^g$, and therefore, $\{x^k\}$ strongly converges to x^*. In addition, from (6) we have

$$\|x^k - x^g\|^2 \le \|\bar{x} - x^g\|^2 + 2\frac{\delta_k}{\varepsilon_k} \quad \forall k$$

which implies

$$\|x^* - x^g\| \le \|\bar{x} - x^g\|. \tag{9}$$

By Proposition 2.2.b), the solution set $SEP(K, f)$ is a nonempty, weakly closed, convex set. Thus the projection of x^g onto $SEP(K, f)$ is uniquely defined. We denote this element by x'. Substituting $\bar{x} = x'$ into (9) we obtain

$$\|x^* - x^g\| \le \|x' - x^g\|$$

which implies $x^* = x'$. The proof is complete. ∎

Remark 3.5. Theorem 3.4 says that a solution of an ill-posed pseudomonotone equilibrium problem can be obtained as the limit of any sequence of δ_k-solutions of the subproblems $EP(K, f_{\varepsilon_k})$. Although each bifunction f_ε may not be pseudomonotone, and thus the existing algorithms cannot be applied to the regularized subproblem $EP(K, f_{\varepsilon_k})$, such a limit can be obtained by solving the following bilevel convex program

$$\min\{\|x - x^g\|^2 : \ x \in \widetilde{K}\}, \tag{10}$$

where $\widetilde{K}$ is the solution set of the original problem $EP(K, f)$. It is important to emphasize that although this set is closed convex, it is not given explicitly as in a standard mathematical programming problem, and therefore the existing methods of convex programming cannot be applied directly as well. Algorithms for solving this bilevel convex program are far beyond the scope of this paper and will be discussed in our subsequent papers.

Now we want to give an example showing that the Tikhonov regularization method can be applied to handle ill-posed pseudomonotone equilibrium problems. For this purpose we show that problem (10) is well-posed in the sense that, it has a unique solution and the solution depends continuously on the data of the original problem. To this end, we assume that the original problem depends on a parameter v in a complete metric space Y. Then the solution set $\widetilde{K}$ depends on v. In this case we have to consider the problem

$$\min\{\|x - x^g\|^2 : \; x \in \widetilde{K}(v)\}. \tag{11}$$

Suppose that $\widetilde{K} = \widetilde{K}(0)$. Under Assumptions (A_1), (A_2) and (A_3), the solution set $\widetilde{K}(0)$ is nonempty closed and convex. Suppose, in addition to Assumptions (A_1), (A_2) and (A_3), that the mapping $\widetilde{K}(\cdot)$ has nonempty closed, convex values and is upper semicontinuous in a neighborhood of 0. Then, by the well-known Berge maximum theorem [2], the unique solution $x(v)$ of the problem (11) is continuous at 0.

Below we give a particular case, where the solution set mapping $\widetilde{K}(\cdot)$ is upper semicontinuous. First we recall [2] that, a multivalued mapping F from a Banach space X to a Banach space Y is closed (resp. convex) if its graph is closed (resp. convex) in $X \times Y$.

Now we suppose that the feasible domain $K := F^{-1}(0)$ and we consider the parametric problem

$$\begin{cases} \text{Find } \; x(v) \in F^{-1}(v) \; \text{ such that} \\ f(x(v), x) \geq 0 \quad \forall x \in F^{-1}(v). \end{cases} \tag{12}$$

Let $\widetilde{K}(v)$ denote the solution set of this problem. We need the following lemma:

Lemma 3.6. [17, Lemma 1]*Suppose that F is a multivalued mapping from X into Y satisfying*

a) *F is convex and closed;*
b) *$F(X) = Y$;*
c) *$F^{-1}(0)$ is bounded.*

Then for each bounded neighborhood V_0 of $0 \in Y$ there is a bounded closed set $B \subset X$ such that $F^{-1}(v) \subset B$ for all $v \in V_0$ and F^{-1} is upper semicontinuous in V_0.

The following proposition on the upper semicontinuity of the solution set mapping has been proved in [17] for monotone bifunctions. Now we extend it to Problem (12) with f being an equilibrium pseudomonotone bifunction on $X = \mathcal{H}$.

Proposition 3.7. *Suppose that f is pseudomonotone on X. Then under Assumptions $(A_1), (A_2)$ and the conditions specified in Lemma 3.6, there exists a neighborhood V_0 of $0 \in Y$ such that problem (12) has a unique solution for every $v \in V_0$ and the mapping $\widetilde{K}(\cdot)$ is upper semicontinuous at 0.*

Proof. We outline the proof because it can be done in the same way as in the proof of Theorem 1 in [17]. Since F^{-1} is convex and closed, $F^{-1}(v)$ is convex and closed for every $v \in V_0$. Moreover, by Lemma 3.6, $F^{-1}(V_0)$ is contained in a bounded closed set. Then from Assumptions (A_1) and (A_2) it follows that (12) has a solution for every $v \in V_0$ [4]. In addition, the pseudomonotonicity of f implies that, $x(v)$ is a solution of (12) if and only if it is a solution of the dual problem (see e.g. [10, Proposition 2.1.15]), that is

$$x(v) \in F^{-1}(v): \ f(x(v), x) \geq 0 \quad \forall v \in F^{-1}(v)$$

if and only if

$$x(v) \in F^{-1}(v): \ f(x, x(v)) \leq 0 \quad \forall v \in F^{-1}(v).$$

Now take $h(v, x') := \max\{f(x, x') : x \in F^{-1}(v)\}$. Then by Lemma 3.6 and the well-known Berge maximum theorem, h is lower semicontinuous in $X \times V_0$. As $\widetilde{K}(v)$ is contained in a bounded set, to see the upper semicontinuity of the solution mapping $\widetilde{K}(\cdot)$, we need to show the closedness of its graph. Indeed, let $(v^0, x^0) \notin \text{graph}\widetilde{K}$. Then

$$x^0 \notin F^{-1}(v^0) \text{ or } h(v^0, x^0) > 0 \text{ or both.}$$

Then, by the closedness of $F^{-1}(v^0)$ and lower semicontinuity of h, there exists a neighborhood $V \times U$ of (v^0, x^0) such that

$$x \notin F^{-1}(v) \text{ or } h(v, x) > 0 \text{ or both,}$$

which implies that $(U \times V) \cap \text{graph}\widetilde{K} = \emptyset$. ■

To illustrate the result, let us consider an example, where $F(x) := M - G(x)$ with G being a mapping from X into Y and M a closed convex cone in Y. We suppose that

(i) G is continuous and $-G(X) + M = Y$;

(ii) G is M-convex on X, i.e.,

$$G(tx + (1-t)y) \in tG(x) + (1-t)G(y) + M \quad \forall x, y, \ \forall t \in [0, 1].$$

Then it is not hard to verify that $F^{-1}(v) = \{x \in X : G(x) + v \in M\}$ and that all assumptions imposed on F are satisfied.

General conditions ensuring the upper semicontinuity of the solution mappings of parametric equilibrium problems can be found, for example, in [1] and the references therein.

3.2. An inexact proximal point algorithm

In [15], Moudafi extended the proximal point method to a class of monotone equilibrium problems. In this monotonicity case the regularized subproblems are strongly monotone, and therefore they are solvable uniquely. For pseudomonotone equilibrium problems, in general, the subproblems do not inherit any monotonicity property. The existing methods that require any monotonicity properties can not be applied to solve the subproblems.

In this section we investigate an inexact proximal point algorithm for pseudomonotone equilibrium problems. A similar convergence result as in the last subsection is obtained for this algorithm. Unlike the Tikhonov regularization method, in the proximal point method, at each iteration, the subproblem depends on the last iterate which plays the role of a current guess solution. Namely, starting from a given point $x^0 := x^g$, at each iteration $k = 1, 2, \ldots,$ we consider the problem $EP(K, f_k)$ given as

$$\begin{cases} \text{Find } x^k \in K \text{ such that} \\ f_k(x^k, y) := f(x^k, y) + c_k\langle x^k - x^{k-1}, y - x^k\rangle \geq -\delta_k \quad \forall y \in K, \end{cases} \tag{13}$$

where the regularization parameter $c_k > 0$ and the tolerance $\delta_k \geq 0$ are given. As usual, we call a solution of (13) a δ_k-solution to $EP(K, f_k)$ and we denote the set of all δ_k-solutions by $S_{\delta_k}(K, f_k)$. We have the following convergence results whose proofs rely on some techniques in [24].

Theorem 3.8. *Suppose that f is pseudomonotone on K, satisfies Assumptions $(\mathrm{A}_1), (\mathrm{A}_2)$, and that the problem $EP(K, f)$ admits a solution. Let $\{c_k\}$ and $\{\delta_k\}$ be two sequences of positive numbers such that $c_k \leq c < +\infty$ for every k, $\delta_k \to 0^+$ and $\sum_{k=1}^{\infty} \frac{\delta_k}{c_k} < +\infty$. Then*

a) *for any $k \in \mathbb{N}$, the set $SEP(K, f_k)$ is nonempty, weakly compact, and it holds that*

$$\|x^{k-1} - x^k\|^2 + \|x^k - \bar{x}\|^2 \leq \|x^{k-1} - \bar{x}\|^2 + 2\frac{\delta_k}{c_k}, \tag{14}$$

where $\bar{x} \in SEP(K, f)$ and $x^k \in S_{\delta_k}(K, f_{\varepsilon_k})$;

b) *the sequence $\{x^k\}$, where x^k is arbitrarily chosen in $S_{\delta_k}(K, f_{\varepsilon_k})$, weakly converges to a solution of $SEP(K, f)$. In addition, if $\{x^k\}$ has a subsequence $\{x^{k_j}\}$ strongly converging to some $x^* \in \mathcal{H}$, then $x^* \in SEP(K, f)$ and x^k strongly converges to x^*.*

Proof. a) Using Lemma 3.3 with $x^g = x^{k-1} \in K$ and $\varepsilon = c_k > 0$, we see that the solution set of Problem $EP(K, f_k)$ is nonempty and weakly compact for all $k = 1, 2, \ldots$. To prove the inequality (14), we just apply (4) in Lemma 3.2 with

$$\varepsilon = c_k, x^g = x^{k-1}, x(\varepsilon) = x^k, \delta = \delta_k.$$

b) Fix any point $\bar{x}$ in the solution set of Problem $EP(K, f)$. Let $x^k \in S_{\delta_k}(K, f_k)$ with $k \geq 1$. From (14), we have

$$\|x^k - \bar{x}\|^2 \leq \|x^{k-1} - \bar{x}\|^2 + 2\frac{\delta_k}{c_k}. \tag{15}$$

Since $\sum_{k=1}^{+\infty} \frac{\delta_k}{c_k} < +\infty$, applying Proposition 2.5 with

$$a_n = \|x^k - \bar{x}\|, b_n = 0, c_n = 2\frac{\delta_k}{c_k},$$

we obtain

$$\lim_{k\to\infty} \|x^k - \bar{x}\| = \mu < \infty. \tag{16}$$

Using again the inequality (14), we can write

$$\|x^k - x^{k-1}\|^2 \le \|x^{k-1} - \bar{x}\|^2 - \|x^k - \bar{x}\|^2 + 2\frac{\delta_k}{c_k}.$$

By the properties (16) and $\frac{\delta_k}{c_k} \to 0$ as $k \to \infty$, we obtain

$$\lim_{k\to\infty} \|x^k - x^{k-1}\| = 0. \tag{17}$$

On the other hand, it is easy to see that the sequence $\left\{\sum_{j=1}^{k} \frac{\delta_j}{c_j}\right\}$ is bounded, that is

$$\exists M > 0 : 0 < 2\sum_{j=1}^{k} \frac{\delta_j}{c_j} \le M \quad \forall k.$$

It is also a consequence of (15) that

$$\begin{aligned} &\|x^k - \bar{x}\|^2 \le \|x^g - \bar{x}\|^2 + 2\sum_{j=1}^{k} \frac{\delta_j}{c_j} \le \|x^g - \bar{x}\|^2 + M \quad \forall k \\ \Rightarrow\ &\|x^k - \bar{x}\| \le \sqrt{\|x^g - \bar{x}\|^2 + M} \quad \forall k \\ \Rightarrow\ &\|x^k\| \le \|\bar{x}\| + \sqrt{\|x^g - \bar{x}\|^2 + M} \quad \forall k \\ \Rightarrow\ &x^k \in S_{\delta_k}(K, f_k) \subset \overline{B}\left(0, \|\bar{x}\| + \sqrt{\|x^g - \bar{x}\|^2 + M}\right) \cap K \quad \forall k. \end{aligned}$$

So $\{x^k\}$ is bounded, and therefore there exists a subsequence $\{x^{k_j}\} \subseteq \{x^k\}$ such that

$$x^{k_j} \rightharpoonup x^* \in \overline{B}\left(0, \|\bar{x}\| + \sqrt{\|x^g - \bar{x}\|^2 + M}\right) \cap K.$$

Since x^{k_j} is a δ_{k_j}-solution of $EP(K, f_{k_j})$ for every k_j, we have

$$f_{k_j}(x^{k_j}, y) = f(x^{k_j}, y) + c_{k_j}\langle x^{k_j} - x^{k_j - 1}, y - x^{k_j}\rangle \ge -\delta_{k_j} \quad \forall y \in K. \tag{18}$$

Taking account of (17), the weak upper semicontinuity of f and the conditions $0 < c_{k_j} < c < +\infty$, $\delta_{k_j} \to 0^+$, we obtain from (18) in the limit that

$$0 \le \overline{\lim}_{k_j\to\infty} f_{k_j}(x^{k_j}, y) \le \overline{\lim}_{k_j\to\infty} f(x^{k_j}, y) \le f(x^*, y) \quad \forall y \in K.$$

This shows that $x^* \in SEP(K, f)$. Now we show that x^* is the unique weakly cluster point of $\{x^k\}$. Indeed, suppose that x_1^* and x_2^* are two distinct weakly cluster points of $\{x^k\}$. Then $x_1^*, x_2^* \in SEP(K, f)$, as just we have seen. Then one can apply (16) with x_i^* $(i = 1, 2)$ playing the role of $\bar{x}$ to obtain

$$\lim_{k\to\infty} \|x^k - x_i^*\| = \mu_i, \ i = 1, 2. \tag{19}$$

Clearly,

$$2\langle x^k - x_1^*, x_1^* - x_2^*\rangle = \|x^k - x_2^*\|^2 - \|x^k - x_1^*\|^2 - \|x_1^* - x_2^*\|^2. \tag{20}$$

As x_1^* is a weakly cluster point of $\{x^k\}$, from (19) and (20) it follows that

$$0 = 2\lim_{k\to\infty}\langle x^k - x_1^*, x_1^* - x_2^*\rangle = \mu_2^2 - \mu_1^2 - \|x_1^* - x_2^*\|^2.$$

Thus

$$\mu_2^2 - \mu_1^2 = \|x_1^* - x_2^*\|^2 > 0.$$

Changing the roles of x_1^* and x_2^* to each other, we also have $\mu_1^2 - \mu_2^2 > 0$. This contradiction asserts the uniqueness of x^*.

Now, suppose that the subsequence $\{x^{k_j}\} \subseteq \{x^k\}$ strongly converges to some $x^* \in \mathcal{H}$. Using the same arguments as above, we have $x^* \in SEP(K, f)$. Applying (15) to $\bar{x} = x^*$, we obtain

$$\|x^k - x^*\|^2 \le \|x^{k-1} - x^*\|^2 + 2\frac{\delta_k}{c_k} \quad \forall k \in \mathbb{N}. \tag{21}$$

For any $\gamma > 0$, as $\lim_{k_j\to\infty} \|x^{k_j} - x^*\| = 0$ and $\sum_{k=1}^{\infty} \frac{\delta_k}{c_k} < +\infty$, we can find some $l \in \mathbb{N}$ such that

$$\|x^{k_l} - x^*\| \le \frac{\gamma}{\sqrt{2}} \ \text{ and } \sum_{i=k_l+1}^{\infty} \frac{\delta_i}{c_i} \le \frac{\gamma^2}{4}.$$

Hence, for $k > k_l + 1$, from (21), it holds that

$$\begin{aligned}
\|x^k - x^*\|^2 &\le \|x^{k-1} - x^*\|^2 + 2\frac{\delta_k}{c_k} \\
&\le \|x^{k-2} - x^*\|^2 + 2\left(\frac{\delta_k}{c_k} + \frac{\delta_{k-1}}{c_{k-1}}\right) \\
&\le \dots \\
&\le \|x^{k_l} - x^*\|^2 + 2\left(\frac{\delta_k}{c_k} + \frac{\delta_{k-1}}{c_{k-1}} + \dots + \frac{\delta_{k_l+1}}{c_{k_l+1}}\right) \\
&\le \frac{\gamma^2}{2} + \frac{\gamma^2}{2} = \gamma^2.
\end{aligned}$$

Thus

$$\|x^k - x^*\| \le \gamma \quad \forall k > k_l + 1.$$

Since $\gamma > 0$ is arbitrary, we can conclude that $\lim_{k\to\infty} \|x^k - x^*\| = 0$ as required.

■

Theorem 3.9. *Suppose that K is a nonempty closed convex subset of $\mathbb{R}^n$, that f is pseudomonotone on K, $f(\cdot, y)$ is upper semicontinuous for each $y \in K$, $f(x, \cdot)$ is lower semicontinuous and convex for each $x \in K$, and that the problem $EP(K, f)$ admits a solution. Let $\{c_k\}$ and $\{\delta_k\}$ be two sequences of positive numbers such that $c_k < c < +\infty$, $\delta_k \to 0^+$ and $\sum_{k=1}^{\infty} \frac{\delta_k}{c_k} < +\infty$. Then*

a) for any k, the δ_k-solution set of Problem $EP(K, f_k)$ is nonempty and compact.

b) the sequence $\{x^k\}$, with x^k being any δ_k-solution of Problem $EP(K, f_k)$, strongly converges to some solution of $EP(K, f)$.

Proof. Applying Theorem 3.8 and the property that, any bounded sequence in the space $\mathbb{R}^n$ must have a strongly convergent subsequence, we obtain the desired result. ■

3.3. Application to multivalued pseudomonotone variational inequality

In this subsection we apply the results obtained in the preceding sections to pseudomonotone multivalued variational inequalities. First we recall some well-known definitions on monotonicity for an operator.

Definition 3.10. A multivalued operator $\phi : \mathcal{H} \to 2^{\mathcal{H}}$ with $K \subseteq \mathrm{dom}\phi$ is said to be

a) *strongly monotone* on K with modulus γ if

$$\langle u - v, x - y\rangle \geq \gamma \|x - y\|^2 \quad \forall x, y \in K,\ \forall u \in \phi(x),\ \forall v \in \phi(y);$$

b) *monotone* on K if

$$\langle u - v, x - y\rangle \geq 0 \quad \forall x, y \in K,\ \forall u \in \phi(x),\ \forall v \in \phi(y);$$

c) *pseudomonotone* on K if

$$\langle u, x - y\rangle \leq 0 \Rightarrow \langle v, y - x\rangle \geq 0 \quad \forall x, y \in K,\ \forall u \in \phi(x),\ \forall v \in \phi(y).$$

Consider the multivalued variational inequality

$$\mathrm{VI}(K, F) \begin{cases} \text{Find } x^* \in K \text{ and } v^* \in F(x^*) \text{ such that} \\ \langle v^*, x - x^*\rangle \geq 0 \quad \forall x \in K, \end{cases}$$

where $F : \mathcal{H} \to 2^{\mathcal{H}}$ with $K \subseteq \mathrm{dom}F$. As usual, we say that Problem $\mathrm{VI}(K, F)$ satisfies the coercivity property if

$$(CO) \begin{cases} \text{There exist a closed ball } B \subset \mathcal{H} \text{ and } y^0 \in K \cap B \text{ such that} \\ \sup_{u \in F(x)} \langle u, y^0 - x\rangle < 0 \quad \forall x \in K \setminus B. \end{cases}$$

Let

$$f(x,y) := \sup_{u \in F(x)} \langle u, y - x \rangle. \tag{22}$$

It has been shown in [18] that, any solution of Problem VI(K, F) is a solution of Problem $EP(K, f)$, with f defined by (22). The reverse assertion is true if F has convex, compact values.

Suppose that F is weakly upper semicontinuous (shortly w.u.s.c.) on K and $F(x)$ is nonempty, convex, weakly compact for every $x \in K$. Note that when F is w.u.s.c. with convex and weakly compact values, then, from the well-known Berge maximum theorem, it follows that $f(\cdot, y)$ is w.u.s.c.. Moreover, since for any fixed $x \in K$ the function $f(x, \cdot)$ is the maximum of a family of affine functions, it is a lower semicontinuous convex function. In addition, f is monotone (resp. strongly monotone, pseudomonotone) on K if and only if F is monotone (resp. strongly monotone, pseudomonotone) on K. Indeed, since

$$f(x,y) = \max_{u \in F(x)} \langle u, y - x \rangle$$

and

$$f(y,x) = \max_{v \in F(y)} \langle v, x - y \rangle,$$

by the weak compactness of $F(x)$ and $F(y)$, we have

$$f(x,y) = \langle u, y - x \rangle \text{ with } u \in F(x)$$

and

$$f(y,x) = \langle v, x - y \rangle \text{ with } v \in F(y).$$

Thus,

$$f(x,y) + f(y,x) = \langle u - v, y - x \rangle \quad \forall x, y \in K.$$

Hence

$$f(x,y) + f(y,x) \le 0 \quad \Leftrightarrow \quad \langle u - v, x - y \rangle \ge 0 \quad \forall x, y \in K.$$

In the same way one can see that f is strongly monotone (resp. pseudomonotone) if and only if F is strongly monotone (resp. pseudomonotone).

Now, to obtain the Tikhonov regularization with the regularization bifunction $g(x,y) = \langle x - x^g, y - x \rangle$, for $\varepsilon > 0$, we take

$$F_\varepsilon(x) := F(x) + \varepsilon(x - x^g)$$

and

$$f_\varepsilon(x,y) := \max_{u \in F_\varepsilon(x)} \langle u, y - x \rangle,$$

where x^g is given (a guess solution) as used earlier in this paper. Since $F(x)$ is weakly compact, we can write

$$f_\varepsilon(x,y) := \max_{u \in F_\varepsilon(x)} \langle u, y - x \rangle$$

$$= \max_{v \in F(x)} \langle v, y - x\rangle + \varepsilon\langle x - x^g, y - x\rangle$$
$$= w + \varepsilon\langle x - x^g, y - x\rangle,$$

where $w \in F(x)$ such that $\max_{v\in F(x)}\langle v, y - x\rangle = \langle w, y - x\rangle$. Thus

$$f_\varepsilon(x, y) = f(x, y) + \varepsilon\langle x - x^g, y - x\rangle$$

with

$$f(x, y) := \max_{v\in F(x)} \langle v, y - x\rangle.$$

In this case the regularized variational inequality takes the form

$$\mathrm{VI}(K, F_\varepsilon) \begin{cases} \text{Find } x^* \in K \ \text{ and } v^* \in F_\varepsilon(x^*) \text{ such that} \\ \langle v^*, x - x^*\rangle \geq 0 \quad \forall x \in K. \end{cases}$$

For this case, a point $x^k \in K$ is a δ_k-solution to Problem $\mathrm{VI}(K, F_{\varepsilon_k})$ with $\delta_k \geq 0$ if

$$\exists v^k \in F_{\varepsilon_k}(x^k): \ \langle v^k, x - x^k\rangle \geq -\delta_k \quad \forall x \in K.$$

From the above analysis, we can apply Theorem 3.4 to Problem $\mathrm{VI}(K, F)$ to obtain the following Tikhonov regularization for multivalued variational inequality problems.

Corollary 3.11. *Suppose that F is a w.u.s.c., pseudomonotone operator with nonempty convex, weakly compact values on K, and that Problem $VI(K, F)$ is solvable. Let $\{\delta_k\}$ and $\{\varepsilon_k\}$ be two sequences of positive numbers both tending decreasingly to 0 and $\frac{\delta_k}{\varepsilon_k} \to 0$. Then*

a) *for any k, the δ_k-solution set of Problem $VI(K, F_{\varepsilon_k})$ is nonempty and weakly compact;*

b) *the sequence $\{x^k\}$ with x^k being any δ_k-solution of Problem $VI(K, F_{\varepsilon_k})$ strongly converges to the solution of $VI(K, F)$ that is nearest to the guess solution x^g.*

Similarly, one can apply the inexact proximal algorithm described above to the variational inequality problem $\mathrm{VI}(K, F)$. In this case, the approximate regularized subproblem at iteration k is defined as

$$\mathrm{VI}(K, F_k) \begin{cases} \text{Find } x^k \in K \ \text{ and } v^k \in F_k(x^k) \text{ such that} \\ \langle v^k, x - x^k\rangle \geq -\delta_k \quad \forall x \in K, \end{cases}$$

where

$$F_k(x) := F(x) + c_k(x - x^{k-1}).$$

The following corollaries are immediate from Theorems 3.8 and 3.9.

Corollary 3.12. *Suppose that F is w.u.s.c., pseudomonotone with nonempty, convex, weakly compact values on K and that Problem $VI(K, F)$ is solvable.*

Let $\{\delta_k\}$ *and* $\{c_k\}$ *be two sequences of positive numbers such that* $\delta_k \to 0^+$, $c_k < c < +\infty$ *and* $\sum_{k=1^\infty} \frac{\delta_k}{c_k} < +\infty$. *Then*

a) *for any* k, *the* δ_k*-solution set of Problem* $VI(K, F_k)$ *is nonempty and weakly compact;*

b) *the sequence* $\{x^k\}$, *with* x^k *being any* δ_k*-solution of Problem* $VI(K, F_k)$, *weakly converges to a solution of* $VI(K, F)$. *In addition, if* $\{x^k\}$ *has a subsequence* $\{x^{k_j}\}$ *that strongly converges to some* $x^* \in \mathcal{H}$, *then* x^* *is a solution of* $VIP(K, F)$ *and the whole sequence* $\{x^k\}$ *strongly converges to* x^*.

Corollary 3.13. *Suppose that* K *is a nonempty closed convex subset of the space* $\mathbb{R}^n$, F *is an upper semicontinuous, pseudomonotone operator with nonempty, convex, compact values on* K *and that Problem* $VI(K, F)$ *is solvable. Let* $\{\delta_k\}$ *and* $\{c_k\}$ *be two sequences of positive numbers such that* $\delta_k \to 0^+$, $c_k < c < +\infty$ *and* $\sum_{k=1^\infty} \frac{\delta_k}{c_k} < +\infty$. *Then*

a) *for any* k, *the* δ_k*-solution set of Problem* $VI(K, F_k)$ *is nonempty and compact;*

b) *the sequence* $\{x^k\}$, *with* x^k *being any* δ_k*-solution of Problem* $VI(K, F_k)$, *strongly converges to some solution of* $VI(K, F)$.

Acknownledgements. We would like to thank the referees for their remarks that helped us very much in revising the paper.

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