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Optimality Conditions of the *G*-Type in Locally Lipschitz Multiobjective Programming

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. In this paper, we extend the Antczak definition of G-invexity for differentiable functions to the case of locally Lipschitz functions. We establish G-Fritz John and G-Karush-Kuhn-Tucker necessary optimality conditions for locally Lipschitz multiobjective programming. Our optimality conditions extend the results of Antczak to the locally Lipschitz multiobjective programming.

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1. Introduction

In the theory of constrained extremum problems, optimality conditions results for differentiable nonlinear constrained problems are important theoretically as well as computationally. There are a large number of papers discussing optimality for optimization problems. But in most of such studies an assumption of convexity on the problems was made to prove the sufficiency of optimality conditions [6, 10, 11, 16]. However, in the recent years attempts have been made by several authors to define various classes of differentiable nonconvex functions and to study their optimality conditions [1, 7–9, 12, 13, 17]. Very recently, Antczak [2] introduced new necessary optimality conditions of G-Fritz John and G-Karush-Kuhn-Tucker type obtained for differentiable constrained mathematical programming problems. Moreover, Antczak [3, 4] proved G-Karush-Kuhn-Tucker necessary optimality conditions for the considered differentiable multiobjective programming problem and the duality between the problem and its nonconvex vector G-dual problems. Subsequently, Kim et al. [14] extend a class of nondifferentiable multiobjective programs with inequality and equality constraints in which each component of the objective function contains a term involving the support function of a compact convex set.

In this paper, we extend the Antczak definition of G-invexity for differentiable functions in [2] to the case of locally Lipschitz functions. We establish G-Fritz John and G-Karush-Kuhn-Tucker necessary optimality conditions for locally Lipschitz multiobjective programming. Our optimality conditions extend the results of Antczak [2, 3] to the locally Lipschitz multiobjective programming.

2. Preliminaries and notations

In general, a multiobjective programming problem is formulated as the following:

$$\text{(P)} \qquad \begin{array}{ll} \text{Minimize} & f(x) = (f_1(x), \ldots, f_p(x)) \\ \text{subject to} & x \in S, \end{array}$$

where S is a nonempty set of $\mathbb{R}^n$ and $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, p$ are locally Lipschitz functions.

For such optimization problems, minimization means in general obtaining (weak) Pareto optimal solutions in the following sense:

Definition 2.1. (1) A feasible point $\bar{x}$ is said to be a Pareto solution (an efficient solution) for (P) if and only if there exists no $x \in S$ such that

$$f(x) \leqq f(\bar{x}) \text{ and } f(x) \neq f(\bar{x}).$$

(2) A feasible point $\bar{x}$ is said to be a weak Pareto solution (a weakly efficient solution) for (P) if and only if there exists no $x \in S$ such that

$$f(x) < f(\bar{x}).$$

The following basic definitions can be found in [5].

Definition 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. The usual (one-sided) directional derivative of f at $x \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ is defined by

$$f'(x; v) := \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t},$$

if the limit exists.

Definition 2.3. (1) A function $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz at $x \in \mathbb{R}^n$ if there exist $K > 0$ and $\delta > 0$ such that for any $y, z \in B_\delta(x)$,

$$|f(y) - f(z)| \leq K\|y - z\|,$$

where $B_\delta(x) = \{z \in \mathbb{R}^n \mid \|z - x\| < \delta\}$.

(2) Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz at $x \in \mathbb{R}^n$. The generalized directional derivative of f at x in the direction of $v \in \mathbb{R}^n$ is defined by

$$f^o(x; v) := \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}.$$

(3) The Clarke generalized subgradient of a locally Lipschitz function f at x is denoted by

$$\partial f(x) = \{\xi \in \mathbb{R}^n \mid f^0(x; d) \geq \langle \xi, d \rangle \quad \forall d \in \mathbb{R}^n\}.$$

(4) f is said to be regular at x provided

(i) for all v, the usual one-sided directional derivative $f'(x; v)$ exists;
(ii) for all v, $f'(x; v) = f^0(x; v)$.

Now, in a natural way, we generalize the definition of a vector-valued G-invex function introduced by Antczak [3] to the locally Lipschitz vector-valued case. Let $f = (f_1, \ldots, f_p) : X \to \mathbb{R}^p$ be a locally Lipschitz vector-valued function defined on a nonempty open set $X \subset \mathbb{R}^n$, and $I_{f_i}(X), i = 1, \ldots, p$, be the range of f_i, that is, the image of X under f_i.

Definition 2.4. Let $f : X \to \mathbb{R}^p$ be a locally Lipschitz vector-valued function defined on a nonempty set $X \subset \mathbb{R}^n$ and $u \in X$. If there exists a differentiable vector-valued function $G_f = (G_{f_1}, \ldots, G_{f_p}) : \mathbb{R} \to \mathbb{R}^p$ such that any its component $G_{f_i} : I_{f_i}(X) \to \mathbb{R}$ is a strictly increasing function on its domain and a vector-valued function $\eta : X \times X \to \mathbb{R}^n$ such that for all $x \in X (x \neq u)$ and for any $\xi_i \in \partial f_i(u)$, $i = 1, \ldots, p$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u)) \geqq G'_{f_i}(f_i(u))\xi_i \eta(x, u) \quad (>),$$

then f is said to be a (strictly) vector G_f-invex function at u on X(with respect to η) (or shortly, G-invex function at u on X).

Remark 2.5. In the case when f is a differentiable function, we obtain the definition of a vector G-invex function introduced by Antczak [3].

Proposition 2.6. [5] *If $G_f : \mathbb{R} \to \mathbb{R}$ is strictly increasing and continuously differentiable, $f : X \to \mathbb{R}$ is locally Lipschitz, X is an open subset of $\mathbb{R}^n$, then*

(1) *$G_f \circ f : X \to \mathbb{R}$ is locally Lipschitz;*

(2) $\partial(G_f \circ f)(x_0) = G_f'(f(x_0))\partial f(x_0)$;

(3) *if* f *is regular, then* $G_f \circ f$ *is regular;*

(4) *if* f *is regular, then* $(G_f \circ f)'(x_0; d) = G_f'(f(x_0))f'(x_0; d)$.

Proof. (1) See in [5].

(2) See in [5, Theorem 2.3.9 (ii)].

(3) See in [5, Theorem 2.3.9 (i)].

(4) Suppose that f is regular. Then $G_f \circ f$ is regular (by (3)),

$$\begin{aligned}
(G_f \circ f)'(x_0; d) &= (G_f \circ f)^0(x_0; d) \\
&= \max\left\{\xi^T d \mid \xi \in \partial(G_f \circ f)(x_0)\right\} \\
&= \max\left\{\xi^T d \mid \xi \in G_f'(f(x_0))\partial f(x_0)\right\} \\
&= \max\left\{G_f'(f(x_0))\bar{\xi}^T d \mid \bar{\xi} \in \partial f(x_0)\right\} \\
&= G_f'(f(x_0)) \max\left\{\bar{\xi}^T d \mid \bar{\xi} \in \partial f(x_0)\right\} \\
&= G_f'(f(x_0)) f^0(x_0; d) \\
&= G_f'(f(x_0)) f'(x_0; d).
\end{aligned}$$

■

3. Optimality conditions

Now, we consider the following nonsmooth multiobjective optimization problem:

$$\begin{array}{lll}
\text{(P)} & \text{Minimize} & f(x) = (f_1(x), \ldots, f_p(x)) \\
& \text{subject to} & g_j(x) \leqq 0, j \in J = \{1, \ldots, m\},
\end{array}$$

where $f_i : X \to \mathbb{R}, i = 1, \ldots, p,\ g_j : X \to \mathbb{R}, j \in J$ are locally Lipschitz functions and X is a nonempty open subset of $\mathbb{R}^n$. We denote the set of all feasible points of (P) by

$$S := \{x \in X \mid g_j(x) \leqq 0,\ j \in J\}$$

and denote

$$J(\bar{x}) = \{j \in J \mid g_j(\bar{x}) = 0\}.$$

Now, we give the G-Fritz John necessary optimality conditions for problem (P).

Theorem 3.1. *Let* $\bar{x}$ *be a weakly efficient solution of* (P). *Assume that* $G_{f_i}, i = 1, \ldots, p$ *is a continuously differentiable, real-valued and strictly increasing function defined on* $I_{f_i}(X)$, $G_{g_j},\ j \in J(\bar{x})$ *is a continuously differentiable, real-valued and strictly increasing function defined on* $I_{g_j}(X)$. *Assume that* $f_i, i = 1, \ldots, p$ *and* $g_j,\ j \in J(\bar{x})$ *are regular at* $\bar{x}$. *Then there exist* $\lambda_i \geqq 0,\ i = 1, \ldots, p$, $\mu_j \geqq 0, j \in J(\bar{x})$, *not all zero, such that*

$$0 \in \sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x}))\partial f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(\bar{x}))\partial g_j(\bar{x}) \tag{1}$$

$$\mu_j g_j(\bar{x}) = 0, \ j \in J. \tag{2}$$

Proof. Let $\bar{x}$ be a weakly efficient solution of (P). We show that the system of convex inequalities in $z \in \mathbb{R}^n$

$$\left\langle \begin{array}{l} G'_{f_i}(f_i(\bar{x}))f_i^0(\bar{x};z) < 0, \quad i = 1,\ldots,p \\ G'_{g_j}(g_j(\bar{x}))g_j^0(\bar{x};z) < 0, \quad j \in J(\bar{x}) \end{array} \right\rangle$$

has no solution. Suppose to the contrary that the system has a solution $z \in \mathbb{R}^n$. Then since $f_i, i = 1,\ldots,p$ and $g_j, j \in J(\bar{x})$ are regular at $\bar{x}$, the system

$$\left\langle \begin{array}{l} (G_{f_i} \circ f_i)'(\bar{x};z) < 0, \quad i = 1,\ldots,p \\ (G_{g_j} \circ g_j)'(\bar{x};z) < 0, \quad j \in J(\bar{x}) \end{array} \right\rangle$$

has a solution $z \in \mathbb{R}^n$. Here we may assume that $f'_i(\bar{x};z) \neq 0$, $g'_j(\bar{x};z) \neq 0$. Since the functions $f_i, i = 1,\ldots,p$, and $g_j, j \in J(\bar{x})$ are regular and $G_{f_i}, i = 1,\ldots,p$ and $G_{g_j}, j \in J(\bar{x})$ are continuously differentiable, we have, for small $\theta > 0$,

$$\begin{array}{l} G_{f_i}(f_i(\bar{x}+\theta z)) = G_{f_i}(f_i(\bar{x})) + \theta(G_{f_i} \circ f_i)'(\bar{x};z) + O(\theta), i = 1,\ldots,p, \\ G_{g_j}(g_j(\bar{x}+\theta z)) = G_{g_j}(g_j(\bar{x})) + \theta(G_{g_j} \circ g_j)'(\bar{x};z) + O(\theta), \quad j \in J(\bar{x}). \end{array}$$

Thus we get for small $\theta > 0$,

$$\begin{array}{l} G_{f_i}(f_i(\bar{x}+\theta z)) < G_{f_i}(f_i(\bar{x})), \quad i = 1,\ldots,p, \\ G_{g_j}(g_j(\bar{x}+\theta z)) < G_{g_j}(g_j(\bar{x})), \quad j \in J(\bar{x}). \end{array}$$

Since $G_{f_i}, i = 1,\ldots,p, G_{g_j}, j \in J(\bar{x})$ are increasing functions on $I_{f_i}(X)$ and $I_{g_j}(X)$, we obtain

$$\begin{array}{l} f_i(\bar{x}+\theta z) < f_i(\bar{x}), \quad i = 1,\ldots,p, \\ g_j(\bar{x}+\theta z) < g_j(\bar{x}) = 0, \quad j \in J(\bar{x}). \end{array}$$

Since $g_j(\bar{x}) < 0 \ \forall j \in J \setminus J(\bar{x})$ and g_j is continuous at $\bar{x}$, for sufficiently small $\theta > 0$, we have

$$\begin{array}{l} f_i(\bar{x}+\theta z) < f_i(\bar{x}), \quad i = 1,\ldots,p, \\ g_j(\bar{x}+\theta z) < 0, \quad j \in J. \end{array}$$

So, $\bar{x}$ cannot be a weakly efficient solution. This is a contradiction. So,

$$\left\langle \begin{array}{l} G'_{f_i}(f_i(\bar{x}))f'_i(\bar{x};z) < 0, \quad i = 1,\ldots,p \\ G'_{g_j}(g_j(\bar{x}))g'_j(\bar{x};z) < 0, \quad j \in J(\bar{x}) \end{array} \right\rangle$$

has no solution. Since the functions $G_{f_i}, i = 1,\ldots,p$ are differentiable and strictly increasing, $G'_{f_i}(f_i(\bar{x})) > 0$. Since f_i is regular at $\bar{x}$, $f'_i(\bar{x};z) = f^0(\bar{x};z)$ for any $z \in \mathbb{R}^n$. Since $f_i^0(\bar{x};\cdot)$ is convex, $G'_{f_i}(f_i(\bar{x}))f'_i(\bar{x};\cdot)$ is convex. Similarly, $G'_{g_j}(g_j(\bar{x}))g'_j(\bar{x};\cdot), j \in J(\bar{x})$ are convex. By the Gordan theorem for convex function, there exist $\lambda_i \geqq 0$, $i = 1, \ldots, p$, $\mu_j \geqq 0$, $j \in J(\bar{x})$, not all zero, such that

$$\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) f'_i(\bar{x}; z) + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) g'_j(\bar{x}; z) \geqq 0 \quad \forall z \in \mathbb{R}^n. \tag{3}$$

Suppose that $0 \notin \sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) \partial f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) \partial g_j(\bar{x})$. Then by [5, Proposition 2.3.3],

$$0 \notin \partial \Big(\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) f_i + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) g_j \Big)(\bar{x}).$$

Since $\partial \left(\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) f_i + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) g_j \right)(\bar{x})$ is compact and convex, by separation theorem, there exists $z^* \in \mathbb{R}^m$ such that

$$\xi^T z^* < 0 \qquad \forall \xi \in \partial \left(\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) f_i + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) g_j \right)(\bar{x}).$$

This implies that

$$\Big(\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) f_i + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) g_j \Big)^0 (\bar{x}; z^*) < 0.$$

Since $\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) f_i + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) g_j$ is regular, by Theorem 3.2.2 in [15]

$$\sum_{i=1}^{p} \lambda_i G'_{f_i}(f'_i(\bar{x})) f'_i(\bar{x}; z^*) + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) g'_j(\bar{x}; z^*) < 0$$

which contradicts (3). Therefore,

$$0 \in \sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) \partial f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x})) \partial g_j(\bar{x}).$$

Since $g_j(\bar{x}) = 0$, $j \in J(\bar{x})$, then $\mu_j g_j(\bar{x}) = 0$, $j \in J(\bar{x})$. If $j \notin J(\bar{x})$, letting $\mu_j = 0$, we have

$$\begin{aligned} &0 \in \sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x})) \partial f_i(\bar{x}) + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(\bar{x})) \partial g_j(\bar{x}), \\ &\mu_j g_j(\bar{x}) = 0, \quad j \in J. \end{aligned}$$

■

Now, we give the G-Karush-Kuhn-Tucker optimality condition for problem (P).

Theorem 3.2. *Let $\bar{x}$ be a weakly efficient solution of* (P)*. Let the assumptions of Theorem 3.1 be satisfied. Assume that* $0 \notin \mathrm{co}\{G'_{g_j}(g_j(\bar{x}))\partial g_j(\bar{x}) \mid j \in J(\bar{x})\}$*, where* $\mathrm{co}\, A$ *is the convex hull of* $A \subset X$*. Then there exist* $\lambda_i \geqq 0,\ i = 1,\dots,p, (\lambda_1,\dots,\lambda_p) \neq 0$ *and* $\mu_j \geqq 0,\ j \in J$*, such that* (1) *and* (2) *hold.*

Proof. Let $\bar{x}$ be a weakly efficient solution of (P). By Theorem 3.1, there exist $\lambda_i \geqq 0,\ i = 1,\dots,p,\ \mu_j \geqq 0,\ j \in J$, not all zero, such that (1) and (2) hold. Suppose that $(\lambda_1,\dots,\lambda_p) = 0$. From (1), $\mu_j \geqq 0, j \in J(\bar{x})$, not all zero, such that $0 \in \sum_{j\in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x}))\partial g_j(\bar{x})$. So, $0 \in \mathrm{co}\{G'_{g_j}(g_j(\bar{x}))\partial g_j(\bar{x}) \mid j \in J(\bar{x})\}$, which contradicts the assumption. Thus $(\lambda_1,\dots,\lambda_p) \neq 0$. Hence the result holds. ■

We give an example showing the efficient application of Theorem 3.2 to locally Lipschitz multiobjective programming.

Example 3.3. Let $f_1(x) = \ln(x^2+1),\ f_2(x) = e^{|x|}$ and $D = \{x \in \mathbb{R} : g(x) := x^2 - 1 \leq 0\} = [-1,1]$. Consider the following multiobjective programming problem:

$$\text{(P)} \qquad \text{Minimize} \quad \left(\ln(x^2+1), e^{|x|}\right)$$
$$\text{subject to} \quad x \in D.$$

Let $\bar{x} = 0$ be a weakly efficient solution of (P). Then (P) satisfies the Kuhn-Tucker constraint qualification at $\bar{x} = 0$. We assume that $G_{f_1}(x) = e^x,\ G_{f_2}(x) = \ln x$ and $G_g(x) = x$. Notice that $G'_{f_1}(f_1(\bar{x}))\partial f_1(\bar{x}) = 0,\ G'_{f_2}(f_2(\bar{x}))\partial f_2(\bar{x}) = [-1,1],\ G'_g(g(\bar{x}))\partial g(\bar{x}) = 0$. Then by Theorem 3.2, there exist $\lambda_i \in \mathbb{R},\ i = 1,2$, $\mu \in \mathbb{R}$ and $\xi \in [-1,1]$ such that

$$\lambda_2 \xi = 0, \tag{4}$$
$$\mu(-1) = 0, \tag{5}$$
$$\lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ (\lambda_1,\lambda_2) \neq 0,\ \mu \geqq 0. \tag{6}$$

It is not difficult to solve the system (4)–(6). From (4)–(6),

$$\begin{cases} \lambda_1 \geqq 0,\ \lambda_2 \geqq 0,\ (\lambda_1,\lambda_2) \neq 0,\ \mu = 0 & \text{if } \xi = 0, \\ \lambda_1 > 0,\ \lambda_2 = 0,\ \mu = 0 & \text{if } \xi \in [-1,0) \cup (0,1]. \end{cases}$$

Therefore, $\bar{x} = 0$ has $\lambda_1,\ \lambda_2,\ \mu$ satisfying (4)–(6).

From the above example, we prove optimality of an arbitrary feasible point in the nonlinear optimization problem considered in an easier way, by using the G-Karush-Kuhn-Tucker necessary optimality conditions, than by using the classical Karush-Kuhn-Tucker necessary optimality conditions.

Now, we give the G-Fritz John sufficient optimality condition for problem (P).

Theorem 3.4. *Let $\bar{x}$ be feasible for* (P)*. Let the assumptions of Theorem 3.1 be satisfied. Assume that $f_i, i = 1, \ldots, p$ is G_{f_i}-invex, $g_j, j \in J(\bar{x})$ is strictly G_{g_j}-invex at $\bar{x}$ on X and $\mu_j > 0$ for some $j \in J(\bar{x})$. Then $\bar{x}$ is a (weakly) efficient solution of* (P)*.*

Proof. Suppose that $\bar{x}$ is not a weakly efficient solution of (P). Then there exists a feasible solution x such that

$$f_i(x) < f_i(\bar{x}) \qquad \forall i = 1, \ldots, p.$$

Since G_{f_i} is a strictly increasing function defined on $I_{f_i}(X)$ then

$$G_{f_i}(f_i(x)) < G_{f_i}(f_i(\bar{x})) \quad \forall i = 1, \ldots, p. \tag{7}$$

By assumption, $f_i, i = 1 \ldots, p$ is a G_{f_i}-invex function at $\bar{x}$ with respect to η on X,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(\bar{x})) \geqq G'_{f_i}(f_i(\bar{x}))\xi_i\eta(x, \bar{x}) \tag{8}$$

for any $\xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$. Thus, using (7) together with (8) we get

$$G'_{f_i}(f_i(\bar{x}))\xi_i\eta(x, \bar{x}) < 0 \tag{9}$$

for any $\xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$. Thus, for the given $\lambda_i \geqq 0, i = 1, \ldots, p$,

$$\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x}))\xi_i\eta(x, \bar{x}) \leqq 0 \tag{10}$$

for any $\xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$. Since g_j is a strictly G_{g_j}-invex function at $\bar{x}$, one has

$$G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x})) > G'_{g_j}(g_j(\bar{x}))\zeta_j\eta(x, \bar{x})$$

for any $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$. Since $\mu_j > 0$ for some $j \in J(\bar{x})$,

$$\sum_{j \in J(\bar{x})} \mu_j G_{g_j}(g_j(x)) - \sum_{j \in J(\bar{x})} \mu_j G_{g_j}(g_j(\bar{x})) > \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x}))\zeta_j\eta(x, \bar{x})$$

for any $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$. Since $g_j(x) \leqq g_j(\bar{x})$ and G_{g_j}, $j \in J(\bar{x})$ is a strictly increasing function, $G_{g_j}(g_j(x)) \leqq G_{g_j}(g_j(\bar{x}))$,

$$\sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x}))\zeta_j\eta(x, \bar{x}) < 0 \tag{11}$$

for any $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$. Since $\mu_j = 0$ for $j \in J \setminus J(\bar{x})$, by using (10) and (11),

$$\Big(\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x}))\xi_i + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(\bar{x}))\zeta_j\Big)\eta(x, \bar{x}) < 0,$$

which contradicts (1). Hence the result holds.

On the other hand, suppose that $\bar{x}$ is not an efficient solution of (P). Then there exists a feasible solution x such that

$$f_i(x) \leqq f_i(\bar{x}) \quad \forall i = 1, \ldots, p \text{ and } \; f_j(x) \neq f_j(\bar{x}) \; \text{ for some } j \neq i.$$

By a similar method, we get the inequality (10) and the same results as above. ∎

Remark 3.5. In Theorem 3.4, if $f_i, i = 1, \ldots, p$ is G_{f_i}-quasiinvex, $g_j, j \in J(\bar{x})$ is strictly G_{g_j}-invex at $\bar{x}$ on X and $\mu_j > 0$ for some $j \in J(\bar{x})$, then $\bar{x}$ is a (weakly) efficient solution of problem (P).

Now, we give the G-Karush-Kuhn-Tucker sufficient optimality condition for problem (P).

Theorem 3.6. *Let $\bar{x}$ be feasible for* (P)*. Let the assumptions of Theorem 3.1 be satisfied. Assume that $f_i, i = 1, \ldots, p$ is G_{f_i}-invex and $g_j, j \in J(\bar{x})$ is strictly G_{g_j}-invex at $\bar{x}$ on S. Then $\bar{x}$ is a weakly efficient solution of problem* (P)*.*

Proof. Suppose that $\bar{x}$ is not a weakly efficient solution of (P). Then we get the inequlity (9) in the proof of Theorem 3.4. Thus, for the given $\lambda_i \geqq 0, i = 1, \cdots, p$, not all zero,

$$\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x}))\xi_i \eta(x, \bar{x}) < 0 \tag{12}$$

for any $\xi_i \in \partial f_i(\bar{x})$, $i = 1, \ldots, p$. Since g_j is a G_{g_j}-invex function at $\bar{x}$, one has

$$G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x})) \geqq G'_{g_j}(g_j(\bar{x}))\zeta_j \eta(x, \bar{x}) \tag{13}$$

for any $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$. Multiplying (13) by the given $\mu_j \geqq 0$ and summing on $j \in J(\bar{x})$, one gets

$$\sum_{j \in J(\bar{x})} \mu_j \left[G_{g_j}(g_j(x)) - G_{g_j}(g_j(\bar{x}))\right] \geqq \sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x}))\zeta_j \eta(x, \bar{x})$$

for any $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$. Since $g_j(x) \leqq g_j(\bar{x})$ and G_{g_j}, $j \in J(\bar{x})$ is a strictly increasing function, $G_{g_j}(g_j(x)) \leqq G_{g_j}(g_j(\bar{x}))$,

$$\sum_{j \in J(\bar{x})} \mu_j G'_{g_j}(g_j(\bar{x}))\zeta_j \eta(x, \bar{x}) \leqq 0 \tag{14}$$

for any $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$. Since $\mu_j = 0$ for $j \in J \setminus J(\bar{x})$, by using (12) and (14),

$$\Big(\sum_{i=1}^{p} \lambda_i G'_{f_i}(f_i(\bar{x}))\xi_i + \sum_{j=1}^{m} \mu_j G'_{g_j}(g_j(\bar{x}))\zeta_j\Big)\eta(x, \bar{x}) < 0,$$

which contradicts (1). Hence the result holds. ■

Remark 3.7. In Theorem 3.6, if $f_i, i = 1, \ldots, p$ is G_{f_i}-quasiinvex and $g_j, j \in J(\bar{x})$ is strictly G_{g_j}-invex at $\bar{x}$ on X and $\mu_j > 0$ for some $j \in J(\bar{x})$, then $\bar{x}$ is a weakly efficient solution of problem (P).

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