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On Robust Multiobjective Optimization*

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. The robust approach (the worst-case approach) for a multiobjective optimization problem (MP) with uncertainty data is considered. Using the robust approach, we define three kinds of robust efficient solutions for an uncertain multiobjective optimization problem (UMP) which consists of more than two objective functions with uncertainty data and constraint functions with uncertainty data. We give scalarizing methods for properly robust efficient solutions and weakly robust efficient solution of (UMP), and establish necessary optimality theorems for weakly and properly robust efficient solutions for (UMP).

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1. Introduction and preliminaries

Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space of dimension n. The inner product on $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. For a set A in $\mathbb{R}^n$, the closure (resp. convex hull) of A is denoted by $\mathrm{cl}(A)$ (resp. $\mathrm{co}(A)$). We

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say that A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$.

For an extended real-valued function f on $\mathbb{R}^n$, the effective domain and the epigraph are respectively defined by $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\operatorname{epi} f := \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$. We say that f is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\operatorname{dom} f \neq \emptyset$. Moreover, if $\liminf_{x' \to x} f(x') \geq f(x)$ for all $x \in \mathbb{R}^n$, we say that f is a lower semicontinuous function. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if $f((1-\mu)x + \mu y) \leq (1-\mu)f(x) + \mu f(y)$ for all $\mu \in [0,1]$, $x, y \in \mathbb{R}^n$. Moreover, we say that f is concave if $-f$ is convex. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function. The subdifferential of g at $a \in \operatorname{dom} g$ is given by

$$\partial g(a) = \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x-a \rangle \quad \text{for all } x \in \mathbb{R}^n\}.$$

As usual, for any proper convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its conjugate function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) = \sup_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - f(x)\}$ for all $x^* \in \mathbb{R}^n$. Clearly, f^* is a proper lower semicontinuous convex function and $\lambda \operatorname{epi} f^* = \operatorname{epi}(\lambda f)^*$ for any $\lambda > 0$. For details, see [15, 18].

Consider an uncertain multiobjective optimization problem:

$$\text{(UMP)} \qquad \begin{array}{ll} \text{minimize} & (f_1(x,u_1), \ldots, f_l(x,u_l)) \\ \text{subject to} & g_j(x,v_j) \leqq 0, \ j = 1, \ldots, m, \end{array}$$

where $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $i = 1, \ldots, l$ and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $j = 1, \ldots, m$ are continuous functions and u_i, v_j are uncertain parameters, and $u_i \in \mathcal{U}_i$, $v_j \in \mathcal{V}_j$ for some convex compact sets $\mathcal{U}_i$ in $\mathbb{R}^p$ and $\mathcal{V}_j$ in $\mathbb{R}^q$.

Recently, to find robust solutions which are less sensitive to small pertubations in variables, Deb and Gupta [7, 8] defined two kind robust solutions for multiobjective optimization problems; the emphasis of their robust multiobjective approaches is to find a robust frontier, instead of the Pareto frontier in the problems. In this paper, using worst-case approaches for the multiobjective optimization problems, we will define three kind robust solutions for the problems, which are different from the ones of Deb and Gupta [7, 8].

When $l = 1$, (UMP) becomes an uncertain optimization problem (UP), which has been intensively studied in [1–6] and [10]. In this case, to consider the worst-cases of objective functions and constraint sets, robust optimization, which has emerged as a powerful deterministic approach for studying (single-objective) optimization problem under uncertainty [2, 4–6], associates with (UP) its robust counterpart [1, 10]:

$$\text{(RP)} \qquad \min \left\{ \max_{u_1 \in \mathcal{U}_1} f_1(x,u_1) : g_j(x,v_j) \leqq 0 \quad \forall v_j \in \mathcal{V}_j, \ j = 1, \ldots, m \right\}.$$

In this paper, we treat the robust approach for (UMP), which is the worst-case approach for (UMP). Now we associate with the uncertain multiobjective optimization problem (UMP) its robust counterpart:

$$\text{(RMP)} \qquad \text{minimize} \quad \left(\max_{u_1\in\mathcal{U}_1} f_1(x,u_1),\ldots,\max_{u_l\in\mathcal{U}_l} f_l(x,u_l)\right)$$
$$\text{subject to} \quad \max_{v_j\in\mathcal{V}_j} g_j(x,v_j) \leqq 0,\ j=1,\ldots,m.$$

A vector $x \in \mathbb{R}^n$ is a robust feasible solution of (UMP) if $\max_{v_j\in\mathcal{V}_j} g_j(x,v_j) \leqq 0,\ j=1,\ldots,m$.

Let F be the set of all robust feasible solutions of (UMP).

By using (RMP), we define solution conceptions for (UMP) as follows. A robust feasible solution $\bar{x}$ of (UMP) is a robust efficient solution of (UMP) if there does not exist a robust feasible solution x of (UMP) such that

$$\max_{u_i\in\mathcal{U}_i} f_i(x,u_i) \leqq \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i), \quad i=1,\ldots,l,$$
$$\max_{u_j\in\mathcal{U}_j} f_j(x,u_j) < \max_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j) \quad \text{for some } \ j.$$

A robust feasible solution $\bar{x}$ of (UMP) is a weakly robust efficient solution of (UMP) if there does not exist a robust feasible solution x of (UMP) such that

$$\max_{u_i\in\mathcal{U}_i} f_i(x,u_i) < \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i), \quad i=1,\ldots,l.$$

A robust feasible solution $\bar{x}$ of (UMP) is a properly robust efficient solution of (UMP) if it is an efficient robust solution of (UMP) and if there is a number $M > 0$ such that for all i and $x \in F$ satisfying $\max_{u_i\in\mathcal{U}_i} f_i(x,u_i) < \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)$, there exists an index j such that $\max_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j) < \max_{u_j\in\mathcal{U}_j} f_j(x,u_j)$ and moreover

$$\frac{\max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i) - \max_{u_i\in\mathcal{U}_i} f_i(x,u_i)}{\max_{u_j\in\mathcal{U}_j} f_j(x,u_j) - \max_{u_j\in\mathcal{U}_j} f_j(\bar{x},u_j)} \leqq M.$$

In this paper, we give scalarizing methods for properly robust efficient solutions and weakly robust efficient solutions of (UMP), and establish necessary optimality theorems for weakly and properly robust efficient solutions for (UMP).

2. Scalarizing methods for robust efficient solutions

Now we give scalarizing methods for finding properly robust efficient solutions and weakly robust efficient solutions of (UMP).

Theorem 2.1. *Suppose that $f_i(\cdot,u_i),\ i=1,\ldots,l$ and $g_j(\cdot,v_j),\ j=1,\ldots,m$, are convex. Assume that $f_i(x,\cdot)$ are concave on $\mathcal{U}_i,\ i=1,\ldots,l$. Then the following holds: $\bar{x}\in F$ is a properly robust efficient solution of* (UMP) *if and only if there exist $\bar{\lambda}_i > 0$ and $\bar{u}_i \in \mathcal{U}_i,\ i=1,\ldots,l$ such that for any $x \in F$,*

$$\sum_{i=1}^{l} \bar{\lambda}_i f_i(x,\bar{u}_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x},\bar{u}_i)$$

$$\textit{and} \;\; f_i(\bar{x}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \;\; i = 1, \ldots, l.$$

Proof. Notice that $\max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i),\ i = 1, \ldots, l$, are convex. By Theorem 2 in [12], $\bar{x} \in F$ is a properly efficient robust solution of (UMP) if and only if there exist $\bar{\lambda}_i > 0,\ i = 1, \ldots, l$ such that for any $x \in F$,

$$\sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i),$$

that is,

$$\inf_{x \in F} \max_{(u_1, \ldots, u_l) \in \Pi_{i=1}^{l} \mathcal{U}_i} \sum_{i=1}^{l} \bar{\lambda}_i f_i(x, u_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

By min-max theorem [15, Corollary 37.3.2],

$$\max_{(u_1, \ldots, u_l) \in \Pi_{i=1}^{l} \mathcal{U}_i} \inf_{x \in F} \sum_{i=1}^{l} \bar{\lambda}_i f_i(x, u_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

So, there exist $\bar{u}_i \in \mathcal{U}_i,\ i = 1, \ldots, l$ such that for any $x \in F$,

$$\sum_{i=1}^{l} \bar{\lambda}_i f_i(x, \bar{u}_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

Thus, for any $x \in F$,

$$\sum_{i=1}^{l} \bar{\lambda}_i f_i(x, \bar{u}_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x}, \bar{u}_i)$$

$$\text{and} \;\; \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

Since $\bar{\lambda}_i > 0$ and $f_i(\bar{x}, \bar{u}_i) \leqq \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i),\ i = 1, \ldots, l$, we have

$$f_i(\bar{x}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \;\; i = 1, \ldots, l.$$

Hence the conclusion holds. ■

By Theorem 2.6 in [13], we can get the following theorem:

Theorem 2.2. $\bar{x} \in F$ *is a properly robust efficient solution of* (UMP) *if and only if there exist* $M > 0$ *and* $\lambda_i > 0,\ i = 1, \ldots, l$ *such that*

$$\min_{x \in F} \hat{f}(x) = \hat{f}(\bar{x}) = 0,$$

where

$$\begin{aligned}\hat{f}(x) = &\sum_{i=1}^{l} \lambda_i \left[\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\right] \\ &+ M\left(\sum_{i=1}^{l} \lambda_i\right) \max_{i \in \{1, \cdots, l\}} \left[\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\right].\end{aligned}$$

Theorem 2.3. *Suppose that $f_i(\cdot, u_i),\ i = 1, \ldots, l$ and $g_j(\cdot, v_j),\ j = 1, \ldots, m$, are convex. Assume that $f_i(x, \cdot)$ are concave on $\mathcal{U}_i,\ i = 1, \ldots, l$. Then the following holds: $\bar{x} \in F$ is a weakly robust efficient solution of* (UMP) *if and only if there exist $\bar{\lambda}_i \geqq 0,\ i = 1, \ldots, l$, not all zero, $\bar{u}_i \in \mathcal{U}_i,\ i = 1, \ldots, l$ such that for any $x \in F$,*

$$\sum_{i=1}^{l} \bar{\lambda}_i f_i(x, \bar{u}_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x}, \bar{u}_i)$$

$$\textit{and} \quad \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

Proof. Notice that $\max_{u_i \in \mathcal{U}_i} f_i(\cdot, u_i),\ i = 1, \ldots, l$, are convex. By Gordan's alternative theorem in [14], $\bar{x} \in F$ is a weakly efficient robust solution of (UMP) if and only if there exist $\bar{\lambda}_i \geqq 0,\ i = 1, \ldots, l$, not all zero, such that for any $x \in \mathbb{R}^n$,

$$\sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(x, u_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i),$$

that is,

$$\inf_{x \in \mathbb{R}^n} \max_{(u_1, \ldots, u_l) \in \Pi_{i=1}^{l} \mathcal{U}_i} \sum_{i=1}^{l} \bar{\lambda}_i f_i(x, u_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

By min-max theorem [15, Corollary 37.3.2],

$$\max_{(u_1, \ldots, u_l) \in \Pi_{i=1}^{l} \mathcal{U}_i} \inf_{x \in \mathbb{R}^n} \sum_{i=1}^{l} \bar{\lambda}_i f_i(x, u_i) \geqq \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

Thus the conclusion holds. ■

3. Robust optimality conditions

We say that the Slater type strict feasibility condition for (UMP) is satisfied if there exists x_0 in $\mathbb{R}^n$ such that for any $j \in J := \{1, \ldots, m\}$ and any $v_j \in \mathcal{V}_j$,

$$g_j(x_0, v_j) < 0.$$

The following lemma, which is the robust version of an alternative theorem, can be obtained from Proposition 2.3 and Theorem 2.4 in [10].

Lemma 3.1 (Robust theorem of the alternative). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $g_i : \mathbb{R}^n \times \mathbb{R}^q$, $i = 1, \ldots, m$ be continuous functions such that $g_i(\cdot, v_i)$ is a convex function for each $v_i \in \mathbb{R}^q$. Let $\mathcal{V}_i$ be a nonempty convex subset of $\mathbb{R}^q$, $i = 1, \ldots, m$. Let $F := \{x \in \mathbb{R}^n \mid g_i(x, v_i) \leqq 0 \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m\} \neq \emptyset$. Suppose that for each $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is a concave function.*
Then exactly one of the following two statements holds:

(i) $(\exists x \in \mathbb{R}^n) \ f(x) < 0, \quad g_i(x, v_i) \leqq 0 \ \forall v_i \in \mathcal{V}_i, \quad i = 1, \ldots, m,$

(ii) $(0,0) \in \operatorname{epi} f^* + \operatorname{cl}\left(\bigcup_{v_i \in \mathcal{V}_i, \lambda_i \geqq 0} \operatorname{epi}\left(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i) \right)^* \right).$

By Theorem 2.1, Lemma 3.1, and Proposition 2.1 in [9], we can prove the following necessary and sufficient optimality theorem for properly efficient solutions for (UMP).

Theorem 3.2. *Suppose that $f_i(\cdot, u_i)$, $i = 1, \ldots, l$, and $g_j(\cdot, v_j)$, $j = 1, \ldots, m$, are convex. Assume that the Slater-type strict feasibility condition for* (UMP) *is satisfied and $f_i(x, \cdot)$ are concave on $\mathcal{U}_i$, $i = 1, \ldots, l$. Then the following holds: $\bar{x} \in F$ is a properly robust efficient solution of* (UMP) *if and only if there exist $\bar{\lambda}_i > 0$ and $\bar{u}_i \in \mathcal{U}_i$, $i = 1, \ldots, l$, $\bar{\mu}_j \geqq 0$ and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \ldots, m$ such that for any $x \in F$,*

$$0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial_1 f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{m} \bar{\mu}_j \partial_1 g_j(\bar{x}, \bar{v}_j),$$

$$\mu_j g_j(\bar{x}, \bar{v}_j) = 0, \quad j = 1, \ldots, m,$$

$$\text{and} \qquad f_i(\bar{x}, \bar{u}_i) = \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i), \quad i = 1, \ldots, l.$$

By Theorem 2.3, Lemma 3.1, and Proposition 2.1 in [9], we can prove the following necessary and sufficient optimality theorem for weakly efficient solutions for (UMP).

Theorem 3.3. *Suppose that $f_i(\cdot, u_i)$, $i = 1, \ldots, l$, and $g_j(\cdot, v_j)$, $j = 1, \ldots, m$, are convex. Assume that the Slater-type strict feasibility condition for* (UMP) *is satisfied and $f_i(x, \cdot)$ are concave on $\mathcal{U}_i$ $i = 1, \ldots, l$. Then the following holds: $\bar{x} \in F$ is a weakly robust efficient solution of* (UMP) *if and only if there exist $\bar{\lambda}_i \geqq 0$, $i = 1, \ldots, l$, not all zero, $\bar{u}_i \in \mathcal{U}_i$, $i = 1, \ldots, l$, $\bar{\mu}_j \geqq 0$ and $\bar{v}_j \in \mathcal{V}_j$, $j = 1, \ldots, m$ such that for any $x \in F$,*

$$0 \in \sum_{i=1}^{l} \bar{\lambda}_i \partial_1 f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{m} \bar{\mu}_j \partial_1 g_j(\bar{x}, \bar{v}_j),$$

$$\mu_j g_j(\bar{x}, \bar{v}_j) = 0, \quad j = 1, \ldots, m,$$

$$\text{and} \qquad \sum_{i=1}^{l} \bar{\lambda}_i f_i(\bar{x}, \bar{u}_i) = \sum_{i=1}^{l} \bar{\lambda}_i \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i).$$

Lemma 3.4. [16, page 352]*Let Θ be a nonempty, compact topological space and let $h : \mathbb{R}^n \times \Theta \to \mathbb{R}$ be such that $h(\cdot, \theta)$ is differentiable for every $\theta \in \Theta$ and $\nabla_1 h(x, \theta)$ is continuous on $\mathbb{R}^n \times \Theta$. Let $\phi(x) = \sup_{\theta \in \Theta} h(x, \theta)$. Denote $\bar{\Theta}(x) := \arg\max_{\theta \in \Theta} h(x, \theta)$. Then the function $\phi(x)$ is locally Lipschitz continuous, directionally differentiable and*

$$\phi'(x; d) = \sup_{\theta \in \bar{\Theta}(x)} \nabla_1 h(x, \theta)^T d,$$

where $\phi'(x; d) = \lim_{t \to 0+} \frac{\phi(x+td) - \phi(x)}{t}$.

Lemma 3.5. *Let $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$, be continuous and directionally differentiable and $\bar{x} \in \mathbb{R}^n$. Suppose that $g_1(\bar{x}) = \cdots = g_m(\bar{x})$ and let $h(x) = \max\{g_1(x), \ldots, g_m(x)\}$. Then $h'(\bar{x}; d) = \max\{g_1'(\bar{x}; d), \ldots, g_m'(\bar{x}; d)\}$ for any $d \in \mathbb{R}^n$.*

Proof. Let $d \in \mathbb{R}^n$. Then, we have

$$\begin{aligned}
h'(\bar{x}; d) &= \lim_{t \to 0+} \frac{h(\bar{x} + td) - h(\bar{x})}{t} \\
&= \lim_{t \to 0+} \frac{\max\limits_{i \in \{1,\ldots,m\}} \{g_i(\bar{x} + td)\} - \max\limits_{i \in \{1,\ldots,m\}} \{g_i(\bar{x})\}}{t} \\
&= \lim_{t \to 0+} \frac{\max\limits_{i \in \{1,\ldots,m\}} \{g_i(\bar{x} + td) - g_i(\bar{x})\}}{t} \\
&= \lim_{t \to 0+} \max_{i \in \{1,\ldots,m\}} \left\{ \frac{g_i(\bar{x} + td) - g_i(\bar{x})}{t} \right\} \\
&= \max_{i \in \{1,\ldots,m\}} \left\{ \lim_{t \to 0+} \frac{g_i(\bar{x} + td) - g_i(\bar{x})}{t} \right\} \\
&= \max_{i \in \{1,\ldots,m\}} \{g_i'(\bar{x}; d)\}.
\end{aligned}$$

■

Consider the function

$$F(x) = \sup_{y \in Y} f(x, y).$$

We assume that $f : \mathbb{R}^n \times Y \to \bar{\mathbb{R}}$ satisfies the following conditions:

(i) $f(\cdot, y)$ is convex for all $y \in Y$;

(ii) $f(x, \cdot)$ is upper semicontinuous for all x in a certain neighborhood of a point x_0;

(iii) The set $Y \subset \mathbb{R}^m$ is compact.

We denote by $\hat{Y}(x)$ the set of $y \in Y$ at which $f(x,y) = F(x)$.

Lemma 3.6. [17]*Assume conditions* (i)–(iii). *Then*

$$\partial F(x_0) \supset \operatorname{co}\left(\bigcup_{y \in \hat{Y}(x_0)} \partial_1 f(x_0, y) \right).$$

If, in addition, the function $f(\cdot, y)$ is continuous at x_0 for all $y \in Y$, then

$$\partial F(x_0) = \operatorname{co}\left(\bigcup_{y \in \hat{Y}(x_0)} \partial_1 f(x_0, y) \right).$$

Let $\bar{x} \in F$ and let us decompose $J := \{1, \ldots, m\}$ into two index sets $J = J_1(\bar{x}) \cup J_2(\bar{x})$, where $J_1(\bar{x}) = \{j \in J \mid \exists v_j \in \mathcal{V}_j \text{ s.t. } g_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = J \setminus J_1(\bar{x})$. Since $\bar{x} \in F$, $J_1(\bar{x}) = \{j \in J \mid \max_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = \{j \in J \mid \max_{v_j \in \mathcal{V}_j} g_j(\bar{x}, v_j) < 0\}$. Let $\mathcal{V}_j^0 = \{v_j \in \mathcal{V}_j \mid g_j(\bar{x}, v_j) = 0\}$ for $j \in J_1(\bar{x})$.

Assume that $f_i : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R},\ i = 1, \ldots, l$, and $g_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R},\ j = 1, \ldots, m$ are continuously differentiable in the first variable.

Now we define an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) for (UMP) as follows: there exists $d \in \mathbb{R}^n$ such that for any $j \in J_1(\bar{x})$ and any $v_j \in \mathcal{V}_j^0$,

$$\nabla_1 g_j(\bar{x}, v_j)^T d < 0.$$

Now we establish necessary optimality theorems for weakly and properly robust efficient solutions for (UMP). For the proof of the following theorem, we follow the proof approach for Theorem 3.1 (robust KKT necessary optimality conditions) in [11].

Theorem 3.7. *Let $\bar{x} \in F$ be a weakly robust efficient solution of* (UMP). *Suppose that $f_i(\bar{x}, \cdot)$ are concave on $\mathcal{U}_i,\ i = 1, \ldots, l$ and that $g_j(\bar{x}, \cdot)$ are concave on $\mathcal{V}_j,\ j = 1, \ldots, m$. Then there exist $\lambda_i \geqq 0,\ i = 1, \ldots, l,\ \mu_j \geqq 0,\ j = 1, \ldots, m$, not all zero, and $\bar{u}_i \in \mathcal{U}_i\ i = 1, \ldots, l,\ \bar{v}_j \in \mathcal{V}_j,\ j = 1, \ldots, m$ such that*

$$\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x}, \bar{u}_i) + \sum_{j=1}^{m} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0 \qquad (1)$$

$$\text{and} \qquad \mu_j g_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \ldots, m. \qquad (2)$$

Moreover, if we further assume that the Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) *holds, then there exist $\lambda_i \geqq 0,\ i = 1, \ldots, l$, not all zero, and $\bar{u}_i \in \mathcal{U}_i,\ i = 1, \ldots, l,\ \mu_j \geq 0$ and $\bar{v}_j \in \mathcal{V}_j,\ j = 1, \ldots, m$ such that* (1) *and* (2) *hold.*

Proof. We assume that $\max_{v_j \in \mathcal{V}_j} g_i(\bar{x}, v_j) < 0,\ j = 1, \ldots, m$, that is, $J_1(\bar{x}) = \phi$.

Then $\bar{x} \in \text{int}F$, where $\text{int}F$ is the interior of F. Let $\psi_i(x) = \max_{u_i \in \mathcal{U}_i} f_i(x, u_i)$ and $\mathcal{U}_i^0 = \{u_i \in \mathcal{U}_i \mid f_i(\bar{x}, u_i) = \psi_i(\bar{x})\},\ i = 1, \ldots, l$. Then $\mathcal{U}_i^0$ is convex and compact. By Lemma 3.4, for any $d \in \mathbb{R}^n$,

$$\psi_i'(\bar{x}; d) = \max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}; u_i)^T d, \quad i = 1, \ldots, l.$$

Suppose to the contrary that there exists $\bar{d} \in \mathbb{R}^n$ such that

$$\max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}; u_i)^T \bar{d} < 0, \quad i = 1, \ldots, l.$$

Then there exists $\delta > 0$ such that for any $t \in (0, \delta)$,

$$\psi_i(\bar{x} + t\bar{d}) < \psi_i(\bar{x}).$$

Since $\bar{x} \in \text{int}F$, this contradicts the weakly robust efficiency of $\bar{x}$. Thus there does not exist $d \in \mathbb{R}^n$ such that

$$\max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}, u_i)^T d < 0,\ i = 1, \ldots, l.$$

Since $d \mapsto \max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}; u_i)^T d$ is convex, by Gordan alternative theorem in [14], there exist $\lambda_i \geqq 0,\ i = 1, \ldots, l$, not all zero, such that for any $d \in \mathbb{R}^n$

$$\sum_{i=1}^{l} \lambda_i \max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}; u_i)^T d \geqq 0.$$

Thus

$$\inf_{d \in \mathbb{R}^n} \max_{(u_1, \ldots, u_l) \in \Pi_{i=1}^l \mathcal{U}_i^0} \sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x}; u_i)^T d \geqq 0. \tag{3}$$

For fixed $d \in \mathbb{R}^n$, define $h(u_1, \ldots, u_l) = \sum_{i=1}^l \lambda_i \nabla_1 f_i(\bar{x}; u_i)^T d$ for each $(u_1, \ldots, u_l) \in \Pi_{i=1}^l \mathcal{U}_i^0$. Then, for any $(u_1, \ldots, u_l),\ (u_1', \ldots, u_l') \in \Pi_{i=1}^l \mathcal{U}_i^0$ and $\alpha \in (0, 1)$, we have

$$\begin{aligned}
& h(\alpha(u_1, \ldots, u_l) + (1-\alpha)(u_1', \ldots, u_l')) \\
= & \sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x}; \alpha u_i + (1-\alpha)u_i')^T d \\
= & \left[\nabla_1 \sum_{i=1}^{l} \lambda_i f_i(\bar{x}; \alpha u_i + (1-\alpha)u_i')\right]^T d \\
= & \lim_{t \to 0+} \frac{\sum_{i=1}^l \lambda_i f_i(\bar{x} + td; \alpha u_i + (1-\alpha)u_i') - \sum_{i=1}^l \lambda_i f_i(\bar{x}; \alpha u_i + (1-\alpha)u_i')}{t} \\
\geqq & \lim_{t \to 0+} \frac{\alpha \sum_{i=1}^l \lambda_i f_i(\bar{x} + td; u_i) + (1-\alpha)\sum_{i=1}^l \lambda_i f_i(\bar{x} + td; u_i') - \sum_{i=1}^l \lambda_i \psi_i(\bar{x})}{t}
\end{aligned}$$

$$= \alpha \lim_{t\to 0+} \frac{\sum_{i=1}^{l} \lambda_i f_i(\bar{x}+td;u_i) - \sum_{i=1}^{l} \lambda_i f_i(\bar{x};u_i)}{t}$$
$$+ (1-\alpha) \lim_{t\to 0+} \frac{\sum_{i=1}^{l} \lambda_i f_i(\bar{x}+td;u_i') - \sum_{i=1}^{l} \lambda_i f_i(\bar{x};u_i')}{t}$$
$$= \alpha \sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x};u_i)^T d + (1-\alpha) \sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x};u_i')^T d$$
$$= \alpha h(u_1,\ldots,u_l) + (1-\alpha) h(u_1',\ldots,u_l').$$

The above inequality is due to the concavity of $f_i(x,\cdot)$. Hence h is concave on $\Pi_{i=1}^{l}\mathcal{U}_i^0$. So, by min-max theorem [15, Corollary 37.3.2], from (3),

$$\max_{(u_1,\ldots,u_l)\in\Pi_{i=1}^{l}\mathcal{U}_i^0} \inf_{d\in\mathbb{R}^n} \sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x};u_i)^T d \geqq 0.$$

Thus there exists $(\bar{u}_1,\ldots,\bar{u}_l) \in \Pi_{i=1}^{l}\mathcal{U}_i^0$ such that $\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x};\bar{u}_i)^T d \geqq 0$ for any $d\in\mathbb{R}^n$, that is, $\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x};\bar{u}_i) = 0$.

We assume that $J_1(\bar{x}) \neq \emptyset$. Let $\Psi_j(x) = \max_{v_j\in\mathcal{V}_j} g_j(x,v_j),\ j=1,\ldots,l$. By Lemma 3.4, $\Psi_j(x)$ is locally Lipschitz, $j=1,\ldots,l$ and for $j\in J_1(\bar{x})$,

$$\Psi_j'(x;d) = \max_{v_j\in\mathcal{V}_j^0} \nabla_1 g_j(x;v_j)^T d,$$

where $\Psi_j'(x;d) = \lim_{t\to 0+} \frac{\Psi_j(x+td)-\Psi_j(x)}{t}$. Assume to the contrary that the following system has a solution d:

$$\max_{u_i\in\mathcal{U}_i^0} \nabla_1 f_i(\bar{x},u_i)^T d < 0, i=1,\ldots,l \text{ and } \max_{v_j\in\mathcal{V}_j^0} \nabla_1 g_j(\bar{x};v_j)^T d < 0,\ j\in J_1(\bar{x}),$$

where $\mathcal{U}_i^0 = \{u_i\in\mathcal{U}_i \mid f_i(\bar{x},u_i) = \max_{u_i\in\mathcal{U}_i} f_i(\bar{x},u_i)\}$. Thus, by Lemma 3.4, the following system has a solution d:

$$\psi_i'(\bar{x};d) < 0,\ i=1,\ldots,l \text{ and } \Psi_j'(\bar{x};d) < 0,\ j\in J_1(\bar{x}),$$

where $\psi_i(x) = \max_{u_i\in\mathcal{U}_i} f_i(x,u_i)$. Since $\Psi_j(x)$ is continuous, $j\in J_2(\bar{x})$, there exists $\delta>0$ such that for any $t\in(0,\delta)$,

$$\begin{aligned} &\psi_i(\bar{x}+t\bar{d}) < \psi_i(\bar{x}),\ i=1,\ldots,l,\\ &\Psi_j(\bar{x}+t\bar{d}) < \Psi_j(\bar{x}) = 0,\ j\in J_1(\bar{x}) \text{ and}\\ &\Psi_j(\bar{x}+t\bar{d}) < 0,\ j\in J_2(\bar{x}). \end{aligned}$$

This contradicts the weakly robust efficiency of $\bar{x}$. Thus the following system has no solution:

$$\max_{u_i\in\mathcal{U}_i^0} \nabla_1 f_i(\bar{x},u_i)^T d < 0, i=1,\ldots,l \text{ and } \max_{v_j\in\mathcal{V}_j^0} \nabla_1 g_j(\bar{x};v_j)^T d < 0,\ j\in I(\bar{x}).$$

Thus, by Gordan alternative theorem in [14], there exist $\lambda_i \geqq 0,\ i = 1,\dots,l,\ \mu_j \geqq 0,\ j \in J_1(\bar{x})$ not all zero, such that for all $d \in \mathbb{R}^n$,

$$\sum_{i=1}^{l} \lambda_i \max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}, u_i)^T d + \sum_{j \in J_1(\bar{x})} \mu_j \max_{v_j \in \mathcal{V}_j^0} \nabla_1 g_j(\bar{x}, v_j)^T d \geqq 0$$

that is,

$$\inf_{d \in \mathbb{R}^n} \max_{u_i \in \mathcal{U}_i^0 v_j \in \mathcal{V}_j^0} \left[\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x}, u_i)^T d + \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, v_j)^T d \right] \geqq 0.$$

Since $u_i \mapsto \nabla_1 f_i(\bar{x}; u_i),\ i = 1,\dots,l$ and $v_j \mapsto \nabla_1 g_j(\bar{x}; v_j)^T d,\ j \in J_1(\bar{x})$ are concave on $\mathcal{U}_i^0$ and $\mathcal{V}_j^0$, respectively, we have

$$\max_{\substack{u_i \in \mathcal{U}_i^0 \\ v_j \in \mathcal{V}_j^0}} \inf_{d \in \mathbb{R}^n} \left[\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x}, u_i)^T d + \sum_{j \in J_1(\bar{x})} \mu_j \nabla_1 g_j(\bar{x}, v_j)^T d \right] \geqq 0.$$

Thus there exist $\bar{u}_i \in \mathcal{U}_i^0,\ i = 1,\dots,l$ and $\bar{v}_j \in \mathcal{V}_j^0,\ j \in J_1(\bar{x})$ such that for any $d \in \mathbb{R}^n$,

$$\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x}, \bar{u}_i)^T d + \sum_{j \in J_1(x)} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j)^T d \geqq 0,$$

that is,

$$\sum_{i=1}^{l} \lambda_i \nabla_1 f_i(\bar{x}, \bar{u}_i) + \sum_{j \in J_1(x)} \mu_j \nabla_1 g_j(\bar{x}, \bar{v}_j) = 0.$$

Thus the conclusions of Theorem 3.7 hold. ■

Theorem 3.8. *Let $\bar{x} \in F$ be a properly robust efficient solution of* (UMP). *Assume that the Extended Mangasarian-Fromovitz constraint qualification* (EMFCQ) *holds. Then, there exist $\lambda_i^k > 0,\ \bar{u}_i^k \in \mathcal{U}_i\ \sum_{k=1}^{m_i} \lambda_i^k = 1,\ i = 1,\dots,l,\ k = 1,\dots,m_i$, and $\mu_j^k \geqq 0,\ \bar{v}_j^k \in \mathcal{V}_i,\ j = 1,\dots,m,\ k = 1,\dots,m_j$ such that*

$$\sum_{i=1}^{l} \sum_{k=1}^{m_i} \lambda_i^k \nabla_1 f_i(\bar{x}, \bar{u}_i^k) + \sum_{j=1}^{m} \sum_{k=1}^{m_j} \mu_j^k \nabla_1 g_j(\bar{x}, \bar{v}_j^k) = 0$$

and $\mu_j^k \nabla_1 g_j(\bar{x}, \bar{v}_j^k) = 0,\ j = 1,\dots,m$ and $k = 1,\dots,m_j$.

Proof. Let $\bar{x} \in F$ be a properly robust efficient solution of (UMP). Then by Theorem 2.2, there exist $M > 0$ and $\lambda_i > 0,\ i = 1,\dots,l$ such that

$$\min_{x \in F} \hat{f}(x) = \hat{f}(\bar{x}) = 0,$$

where

$$\hat{f}(x) = \sum_{i=1}^{l} \lambda_i \left[\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\right] + M\left(\sum_{i=1}^{l} \lambda_i\right) \max_{i \in \{1,\ldots,l\}} \left[\max_{u_i \in \mathcal{U}_i} f_i(x, u_i) - \max_{u_i \in \mathcal{U}_i} f_i(\bar{x}, u_i)\right].$$

By Lemmas 3.4 and 3.5, for any $d \in \mathbb{R}^n$,

$$\begin{aligned}\hat{f}'(\bar{x}; d) &= \sum_{i=1}^{l} \lambda_i \max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}, u_i)^T d + M\left(\sum_{i=1}^{l} \lambda_i\right) \max_{i \in \{1,\ldots,l\}} \max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}, u_i)^T d \\ &= \sum_{i=1}^{l} \lambda_i \max_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}, u_i)^T d + M\left(\sum_{i=1}^{l} \lambda_i\right) \sigma_{\mathrm{co}(A)}(d),\end{aligned}$$

where $A = \cup_{i=1}^{l} \{\nabla_1 f_i(\bar{x}, u_i) \mid u_i \in \mathcal{U}_i^0\}$ and $\sigma_{\mathrm{co}(A)}(d) = \sup_{z \in \mathrm{co}(A)} z^T d$. Hence A is compact and hence $\mathrm{co}(A)$ is compact. Now we assume that $J_1(\bar{x}) \neq \emptyset$. Then the following system has no solution:

$$\hat{f}'(\bar{x}; d) < 0 \quad \text{and} \quad \max_{v_j \in \mathcal{V}_j^0} \nabla_1 g_j(\bar{x}; v_j)^T d < 0, \ j \in J_1(\bar{x}).$$

Since $d \mapsto \hat{f}'(\bar{x}; d)$ and $d \mapsto \max_{v_j \in \mathcal{V}_j^0} \nabla_1 g_j(\bar{x}; v_j)^T d$ are convex, by Gordan alternative theorem in [14], there exist $\tilde{\lambda}_0 \geqq 0, \ \tilde{\lambda}_j \geqq 0, \ j \in J_1(\bar{x})$ such that for any $d \in \mathbb{R}^n$,

$$\tilde{\lambda}_0 \hat{f}'(\bar{x}; d) + \sum_{j \in J_1(\bar{x})} \tilde{\lambda}_j \max_{v_j \in \mathcal{V}_j^0} \nabla_1 g_j(\bar{x}; v_j)^T d \geqq 0.$$

From (EMFCQ), $\tilde{\lambda}_0$ can not be 0. Thus we may assume that $\tilde{\lambda}_0 = 1$. Thus $0 \in \partial \hat{f}(\bar{x}) + \sum_{j \in J_1(\bar{x})} \tilde{\lambda}_j \mathrm{co}\{\cup_{v_j \in \mathcal{V}_j^0} \nabla_1 g_j(\bar{x}; v_j)\}$. On the other hand, by Lemma 3.6,

$$\partial \hat{f}(\bar{x}) = \sum_{i=1}^{l} \lambda_i \mathrm{co}\{\cup_{u_i \in \mathcal{U}_i^0} \nabla_1 f_i(\bar{x}, u_i)\} + M\left(\sum_{i=1}^{l} \lambda_i\right) \mathrm{co}(A).$$

Thus there exist $\lambda_i^k > 0, \ \bar{u}_i^k \in \mathcal{U}_i, \ \sum_{k=1}^{m_i} \lambda_i^k = 1, \ i = 1, \ldots, l, \ k = 1, \ldots, m_i$ and $\mu_j^k \geqq 0, \ \bar{v}_j^k \in \mathcal{V}_i, \ j = 1, \ldots, m, \ k = 1, \ldots, m_j$ such that

$$\sum_{i=1}^{l} \sum_{k=1}^{m_i} \lambda_i^k \nabla_1 f_i(\bar{x}, \bar{u}_i^k) + \sum_{j=1}^{m} \sum_{k=1}^{m_j} \mu_j^k \nabla_1 g_j(\bar{x}, \bar{v}_j^k) = 0$$

and $\mu_j^k \nabla_1 g_j(\bar{x}, \bar{v}_j^k) = 0, \ j = 1, \ldots, m$ and $k = 1, \ldots, m_j$. ■

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