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Optimality Conditions and Duality for Nondifferentiable Multiobjective Semi-infinite Programming

Shashi Kant Mishra and Monika Jaiswal

Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi, 221 005, India

Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. This paper deals with a nondifferentiable multiobjective semi-infinite programming problem. We establish sufficient optimality conditions for the nondifferentiable multiobjective semi-infinite programming problem under generalized convexity assumptions. We formulate Mond-Weir type dual for the problem and establish weak, strong and strict converse duality theorems relating to the problem and dual problem using generalized invexity.

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1. Introduction

The field of multi-objective programming, also known as vector programming, has grown remarkably since the 1980s. This follows from the fact that optimization problems with several objectives have a wide range of applications in economics, optimal control, decision theory, game theory and several other applied sciences and engineering disciplines.

Convexity plays a vital role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems see, for example, Mangasarian [9], Bazaraa et al. [1] and Mishra et al. [13]. To relax convexity assumptions imposed on the functions in theorems on sufficient optimality and duality, various generalized convexity notions have been proposed. Generalized convexity is the core of many important subjects in the context of various research fields such as mathematics, economics, management science, engineering and other applied sciences, see Cambini and Martein [2]. For many problems encountered in economics and engineering the notion of convexity does no longer suffice. Hence, it is necessary to extend the notion of convexity to the notions of pseudo-convexity, quasi-convexity and to many other important classes of generalized convex functions.

In many cases, generalized convex functions preserve some of the valuable properties of convex functions. One of the important generalizations of convex functions is invex functions, a notion originally introduced for differentiable functions by Hanson [5] and named by Craven [3]. Hanson [5] noted that the usual convexity (or pseudo-convexity or quasi-convexity) requirements, appearing in the sufficient Kuhn-Tucker conditions for a mathematical programming problem, can be further weakened.

In this paper, we consider the following nondifferentiable multiobjective semi-infinite programming problem:

$$\text{(NDSIP)} \qquad \begin{aligned} \text{Minimize } & \left(f_1(x) + (x^T B_1 x)^{1/2}, \ldots, f_p(x) + (x^T B_p x)^{1/2}\right) \\ \text{subject to } & g(x,u) \leq 0 \quad \forall u \in U, \\ & x \in X_0, \end{aligned}$$

where $X_0 \subset R^n$ is a nonempty set and $U \subset R^m$ is an infinite countable set. Let $f = (f_1, \ldots, f_p) : X_0 \to R^p, g : X_0 \times U \to R$ are differentiable functions and $B_1, \ldots, B_p$ are positive semi-definite (symmetric) matrices. We define I as the index set such that $\left\{u^i\right\}_{i \in I} = U$ and for a feasible point x^*, let $I(x^*) = \left\{i : g(x^*, u^i) = 0, u^i \in U\right\}$.

Such types of problems are referred to as nondifferentiable programming problems because the square root of a positive semidefinite quadratic form appearing in the objective function is nondifferentiable. Many authors have studied nondifferentiable programming problem containing the square root of a positive semidefinite quadratic form. We refer Zhang and Mond [18] and the references therein.

Recently, many scholars have been interested in semi-infinite programming problems, as this model naturally arises in an abundant number of applications in different fields of mathematics, economics and engineering, see for example [4, 6, 8, 17]. Semi-infinite programming deals with extremal problems that involve infinitely many constraints in a finite dimensional space. Shapiro [16] has given results on Lagrangian duality for convex semi-infinite programming problems. Kostyukova and Tchemisova [7] have proved sufficient optimality conditions for convex semi-infinite programming.

The article is organized as follows: in Section 2, definitions and preliminaries have been given. Section 3 contains necessary and sufficient optimality conditions for (NDSIP). In Section 4, we formulate Mond-Weir type dual (NDSID) for (NDSIP) and establish weak, strong and strict-converse duality theorems relating to (NDSIP) and (NDSID) under generalized invexity and regularity conditions.

2. Definitions and preliminaries

The following definitions are taken from Mishra and Giorgi [11].

Definition 2.1. A feasible point $\overline{x}$ is said to be an efficient solution for (NDSIP) if and only if there does not exist another feasible point x such that $f_j(x) + \left(x^T B_j x\right)^{1/2} \leq f_j(\overline{x}) + \left(\overline{x}^T B_j \overline{x}\right)^{1/2}$ for $j = 1, \dots, p$ and $f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(\overline{x}) + \left(\overline{x}^T B_k \overline{x}\right)^{1/2}$ for at least one $k \in \{1, \dots, p\}$.

Definition 2.2. Let X_0 be an open subset of R^n. A vector function $f : X_0 \to R^p$ is said to be invex at $\overline{x} \in X_0$ with respect to η, for a function $\eta : X_0 \times X_0 \to R^n$, if for each $x \in X_0$ and for $j = 1, \dots, p$,

$$f_j(x) - f_j(\overline{x}) \geq \nabla f_j(\overline{x}) \eta(x, \overline{x}).$$

Definition 2.3. Let X_0 be an open subset of R^n. A vector function $f : X_0 \to R^p$ is said to be pseudo-invex at $\overline{x} \in X_0$ with respect to η, for a function $\eta : X_0 \times X_0 \to R^n$, if for each $x \in X_0$ and for $j = 1, \dots, p$,

$$\nabla f_j(\overline{x}) \eta(x, \overline{x}) \geq 0 \Rightarrow f_j(x) \geq f_j(\overline{x}).$$

Definition 2.4. Let X_0 be an open subset of R^n. A vector function $f : X_0 \to R^p$ is said to be strictly pseudo-invex at $\overline{x} \in X_0$ with respect to η, for a function $\eta : X_0 \times X_0 \to R^n$, if for each $x \in X_0, x \neq \overline{x}$ and for $j = 1, \dots, p$,

$$\nabla f_j(\overline{x}) \eta(x, \overline{x}) \geq 0 \Rightarrow f_j(x) > f_j(\overline{x}).$$

Definition 2.5. Let X_0 be an open subset of R^n. A vector function $f : X_0 \to R^p$ is said to be quasi-invex at $\overline{x} \in X_0$ with respect to η, for a function $\eta : X_0 \times X_0 \to R^n$, if for each $x \in X_0$ and for $j = 1, \dots, p$,

$$f_j(x) \leq f_j(\overline{x}) \Rightarrow \nabla f_j(\overline{x}) \eta(x, \overline{x}) \leq 0.$$

Definition 2.6. [10] A function g is said to satisfy the generalized Slater's constraint qualification at $x^* \in X_0$ if g is invex at x^* and there exists $\overline{x} \in X_0$ such that $g\left(\overline{x}, u^i\right) < 0, i \in I\left(x^*\right)$.

In the subsequent analysis, we shall frequently use the following generalized Schwarz inequality, see Riesz and Sz-Nagy [15, pp. 262]:

$$x^T B z \leq \left(x^T B x\right)^{1/2} \left(z^T B z\right)^{1/2} \quad \forall x, z \in R^n, \tag{1}$$

where B is an $n \times n$ positive semi-definite (symmetric) matrix.

3. Optimality conditions

We now recall a Kuhn-Tucker type necessary condition.

Lemma 3.1 (Kuhn-Tucker type necessary conditions). (Mishra et al. [12]) *Let x^* be an efficient solution for* (NDSIP)*. Then $I(x^*)$ is non-empty and there exist $\tau \in R^p, \lambda_i \geq 0$ for $i \in I(x^*)$ with $\lambda_i \neq 0$ for finitely many i and $z_j \in R^n$ such that*

$$\sum_{j=1}^{p} \tau_j \left[\nabla f_j(x^*) + B_j z_j\right] + \sum_{i \in I(x^*)} \lambda_i \nabla g(x^*, u^i) = 0, \tag{2}$$

$$\lambda_i g(x^*, u^i) = 0, \quad i \in I(x^*), \tag{3}$$

$$z_j^T B_j z_j \leq 1, \quad j = 1, 2, \ldots, p, \tag{4}$$

$$\left(x^{*T} B_j x^*\right)^{1/2} = x^{*T} B_j z_j, \quad j = 1, 2, \ldots, p, \tag{5}$$

$$\tau > 0, \ \sum_{j=1}^{p} \tau_j = 1.$$

Theorem 3.2 (Sufficient optimality conditions). *Let x^* be feasible for* (NDSIP)*. Let $I(x^*)$ be non-empty and there exist $\tau \in R^p, \tau > 0$ and scalars $\lambda_i \geq 0, i \in I(x^*)$ with $\lambda_i \neq 0$ for finitely many i and $z_j \in R^n$ such that equations* (2)–(5) *are satisfied. If $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ and $\lambda_i g(\cdot, u^i), i \in I(x^*)$ are invex with respect to the same η, then x^* is an efficient solution for* (NDSIP)*.*

Proof. Suppose to the contrary that x^* is not an efficient solution for (NDSIP). Then there exists a feasible point x for (NDSIP) such that

$$f_j(x) + \left(x^T B_j x\right)^{1/2} \leq f_j(x^*) + \left(x^{*T} B_j x^*\right)^{1/2} \quad \text{for} \quad j = 1, \ldots, p$$

and

$$f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(x^*) + \left(x^{*T} B_k x^*\right)^{1/2} \quad \text{for at least one } k \in \{1, \ldots, p\}.$$

Thus, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \left(x^T B_j x\right)^{1/2}\right] < \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + \left(x^{*T} B_j x^*\right)^{1/2}\right]. \tag{6}$$

Since $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ and $\lambda_i g(\cdot, u^i), i \in I(x^*)$ are invex with respect to the same η, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + x^T B_j z_j\right] - \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + x^{*T} B_j z_j\right]$$

$$\geq \sum_{j=1}^{p} \tau_j \left[\nabla f_j(x^*) + B_j z_j\right] \eta(x, x^*),$$

and

$$\sum_{i \in I(x^*)} \lambda_i g(x, u^i) - \sum_{i \in I(x^*)} \lambda_i g(x^*, u^i) \geq \sum_{i \in I(x^*)} \lambda_i \nabla g(x^*, u^i) \eta(x, x^*).$$

Adding the above two inequalities and using equations (2) and (3), we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + x^T B_j z_j\right] - \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + x^{*T} B_j z_j\right] + \sum_{i \in I(x^*)} \lambda_i g(x, u^i) \geq 0,$$

which implies

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + x^T B_j z_j\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + x^{*T} B_j z_j\right],$$

as $g(x, u^i) \leq 0, i \in I(x^*)$.
Using equations (4), (5) and the generalized Schwarz inequality (1), the above inequality becomes

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \left(x^T B_j x\right)^{1/2}\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + \left(x^{*T} B_j x^*\right)^{1/2}\right].$$

This contradicts (6). Thus, x^* is an efficient solution for (NDSIP). ■

We can also prove sufficient optimality conditions for (NDSIP) by means of further relaxations on invexity requirements which follows as:

Theorem 3.3 (Sufficient optimality conditions). *Let x^* be feasible for* (NDSIP)*. Let $I(x^*)$ be non-empty and there exist $\tau \in R^p, \tau > 0$ and scalars $\lambda_i \geq 0, i \in I(x^*)$ with $\lambda_i \neq 0$ for finitely many i and $z_j \in R^n$ such that equations* (2)–(5) *are satisfied. If $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ is pseudo-invex and $\lambda_i g(\cdot, u^i), i \in I(x^*)$ are quasi-invex with respect to the same η, then x^* is an efficient solution for* (NDSIP)*.*

Proof. Suppose to the contrary that x^* is not an efficient solution for (NDSIP). Then there exists a feasible point x for (NDSIP) such that (6) holds.
As x is feasible for (NDSIP) and since $\lambda_i g(x^*, u^i) = 0, i \in I(x^*)$, hence

$$\sum_{i \in I(x^*)} \lambda_i g(x, u^i) \leq \sum_{i \in I(x^*)} \lambda_i g(x^*, u^i).$$

Thus, from quasi-invexity of $\lambda_i g(\cdot, u^i), i \in I(x^*)$, we get

$$\sum_{i\in I(x^*)} \lambda_i \nabla g(x^*, u^i)\eta(x, x^*) \leq 0.$$

Therefore, from (2), we have

$$\sum_{j=1}^{p} \tau_j \left[\nabla f_j(x^*) + B_j z_j\right] \eta(x, x^*) \geq 0.$$

Thus, from pseudo-invexity of $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + x^T B_j z_j\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(x^*) + x^{*T} B_j z_j\right].$$

The rest of the proof is similar to that of Theorem 3.2. ■

4. Duality

In relation to (NDSIP) we associate the following dual nondifferentiable multi-objective maximization problem (Mond et al. [14]):

(NDSID) Maximize $\left(f_1(y) + (y^T B_1 z_1), \ldots, f_p(y) + (y^T B_p z_p)\right)$

subject to
$$\sum_{j=1}^{p} \tau_j \left[\nabla f_j(y) + B_j z_j\right] + \sum_{i\in I} \lambda_i \nabla g(y, u^i) = 0, \quad (7)$$
$$z_j^T B_j z_j \leq 1, \quad j = 1, \ldots, p, \quad (8)$$
$$\sum_{i\in I} \lambda_i g(y, u^i) \geq 0,$$
$$\tau > 0, \sum_{j=1}^{p} \tau_j = 1, \lambda = (\lambda_i)_{i\in I} \geq 0 \text{ and } \lambda_i \neq 0 \quad (9)$$

for finitely many $i \in I$ and $z_j \in R^n$.

Theorem 4.1 (Weak duality). *Let x be feasible for* (NDSIP) *and $(y, \tau, \lambda, z_1, \ldots, z_p)$ be feasible for* (NDSID). *Let $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ and $\lambda_i g(\cdot, u^i), i \in I$ be invex with respect to the same η. Then the following cannot hold:*

$$f_j(x) + \left(x^T B_j x\right)^{1/2} \leq f_j(y) + y^T B_j z_j \quad \forall j = 1, \ldots, p$$

and

$$f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(y) + y^T B_k z_k \quad \text{for at least one}\, k \in \{1, \ldots, p\}.$$

Proof. Suppose contrary to the result of the theorem that

$$f_j(x) + \left(x^T B_j x\right)^{1/2} \leq f_j(y) + y^T B_j z_j \quad \forall j = 1, \ldots, p$$

and

$$f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(y) + y^T B_k z_k \quad \text{for at least one}\, k \in \{1, \ldots, p\}\,.$$

Therefore,

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \left(x^T B_j x\right)^{1/2}\right] < \sum_{j=1}^{p} \tau_j \left[f_j(y) + y^T B_j z_j\right]. \tag{10}$$

As x is feasible for (NDSIP) and $(y, \tau, \lambda, z_1, \ldots, z_p)$ is feasible for (NDSID), we have

$$\sum_{i \in I} \lambda_i g(x, u^i) \leq \sum_{i \in I} \lambda_i g(y, u^i).$$

That is,

$$\sum_{i \in I} \lambda_i g(x, u^i) - \sum_{i \in I} \lambda_i g(y, u^i) \leq 0. \tag{11}$$

From the definition of invexity, we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + x^T B_j z_j\right] - \sum_{j=1}^{p} \tau_j \left[f_j(y) + y^T B_j z_j\right] \geq \sum_{j=1}^{p} \tau_j \left[\nabla f_j(y) + B_j z_j\right] \eta(x, y),$$

and

$$\sum_{i \in I} \lambda_i g(x, u^i) - \sum_{i \in I} \lambda_i g(y, u^i) \geq \sum_{i \in I} \lambda_i \nabla g(y, u^i) \eta(x, y).$$

Adding the above two inequalities and using equations (7) and (11), we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + x^T B_j z_j\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(y) + y^T B_j z_j\right]. \tag{12}$$

By the generalized Schwarz inequality we know that

$$x^T B_j z_j \leq \left(x^T B_j x\right)^{1/2} \left(z_j^T B_j z_j\right)^{1/2},$$

which on using (8) becomes

$$x^T B_j z_j \leq \left(x^T B_j x\right)^{1/2}. \tag{13}$$

Now using (9) and (13) in (12), we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \left(x^T B_j x\right)^{1/2}\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(y) + y^T B_j z_j\right],$$

which is a contradiction to (10). Hence the proof is complete. ■

Theorem 4.2 (Strong duality). *Let x be an efficient solution for* (NDSIP) *at which the generalized Slater's constraint qualification is satisfied. Let the invexity assumptions of the weak duality theorem be satisfied. Then there exists $(\tau, \lambda, z_1, \dots, z_p)$ such that $(x, \tau, \lambda, z_1, \dots, z_p)$ is an efficient solution for* (NDSID) *and the respective objective values are equal.*

Proof. Since x is an efficient solution for (NDSIP) and the generalized Slater's constraint qualification is satisfied at x, from the Kuhn-Tucker necessary conditions, there exists $(\tau, \lambda, z_1, \dots, z_p)$ such that $(x, \tau, \lambda, z_1, \dots, z_p)$ is feasible for (NDSID).
On the other hand by weak duality theorem, the following cannot hold for any feasible y for (NDSID):

$$f_j(x) + \left(x^T B_j x\right)^{1/2} \le f_j(y) + y^T B_j z_j \quad \forall j = 1, \dots, p$$

and

$$f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(y) + y^T B_k z_k \quad \text{for at least one}\, k \in \{1, \dots, p\}.$$

From the Kuhn-Tucker necessary conditions, we have

$$\left(x^T B_j x\right)^{1/2} = x^T B_j z_j, \quad j = 1, \dots, p. \tag{14}$$

Hence, the following cannot hold for any feasible y for (NDSID):

$$f_j(x) + x^T B_j z_j \le f_j(y) + y^T B_j z_j \quad \forall j = 1, \dots, p$$

and

$$f_k(x) + x^T B_k z_k < f_k(y) + y^T B_k z_k \quad \text{for at least one}\, k \in \{1, \dots, p\}.$$

Thus, $(x, \tau, \lambda, z_1, \dots, z_p)$ is an efficient solution for (NDSID).
Also using (14), we get that the objective values of (NDSIP) and (NDSID) are equal at x. ∎

Theorem 4.3 (Strict converse duality). *Let x^* be an efficient solution for* (NDSIP) *at which the generalized Slater's constraint qualification is satisfied. Assume that $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \dots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ and $\lambda_i g(\cdot, u^i), i \in I$ are invex with respect to the same η. If $(\overline{x}, \tau, \lambda, z_1, \dots, z_p)$ is an efficient solution for* (NDSID) *and $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \dots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ is strictly-invex at $\overline{x}$, then $\overline{x} = x^*$.*

Proof. We prove this by contradiction. Assume that $\overline{x} \neq x^*$. Then by strong duality theorem there exists $(\tau, \lambda, z_1, \dots, z_p)$ such that $(x^*, \tau, \lambda, z_1, \dots, z_p)$ is an efficient solution for (NDSID) and

$$f_j(x^*) + \left(x^{*T} B_j x^*\right)^{1/2} = f_j(\overline{x}) + \overline{x}^T B_j z_j, \quad j = 1, \dots, p,$$

which on using (14) becomes

$$f_j(x^*) + x^{*T} B_j z_j = f_j(\overline{x}) + \overline{x}^T B_j z_j, \quad j = 1, \ldots, p. \tag{15}$$

As x^* is an efficient solution for (NDSIP), $\lambda_i \geq 0$ and $(\overline{x}, \tau, \lambda, z_1, \ldots, z_p)$ is an efficient solution for (NDSID), we have

$$\sum_{i \in I} \lambda_i g(x^*, u^i) \leq \sum_{i \in I} \lambda_i g(\overline{x}, u^i).$$

That is,

$$\sum_{i \in I} \lambda_i g(x^*, u^i) - \sum_{i \in I} \lambda_i g(\overline{x}, u^i) \leq 0. \tag{16}$$

From the invexity of $\lambda_i g(\cdot, u^i), i \in I$, and strict-invexity of $\big(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\big)$ at $\overline{x}$, we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x^*) + x^{*T} B_j z_j\right] - \sum_{j=1}^{p} \tau_j \left[f_j(\overline{x}) + \overline{x}^T B_j z_j\right] > \sum_{j=1}^{p} \tau_j \left[\nabla f_j(\overline{x}) + B_j z_j\right] \eta(x^*, \overline{x})$$

and

$$\sum_{i \in I} \lambda_i g(x^*, u^i) - \sum_{i \in I} \lambda_i g(\overline{x}, u^i) \geq \sum_{i \in I} \lambda_i \nabla g(\overline{x}, u^i) \eta(x^*, \overline{x}).$$

Adding the above two inequalities and using equations (7) and (16), we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x^*) + x^{*T} B_j z_j\right] > \sum_{j=1}^{p} \tau_j \left[f_j(\overline{x}) + \overline{x}^T B_j z_j\right],$$

which contradicts (15).
Therefore, $x^* = \overline{x}$. ■

We can also prove the duality theorems for (NDSID) by means of further relaxations on invexity requirements which follows as:

Theorem 4.4 (Weak Duality). *Let x be feasible for* (NDSIP) *and $(y, \tau, \lambda, z_1, \ldots, z_p)$ be feasible for* (NDSID). *Let $\big(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\big)$ be pseudo-invex and $\lambda_i g(\cdot, u^i), i \in I$ be quasi-invex with respect to the same η. Then the following cannot hold:*

$$f_j(x) + \left(x^T B_j x\right)^{1/2} \leq f_j(y) + y^T B_j z_j \quad \forall j = 1, \ldots, p$$

and

$$f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(y) + y^T B_k z_k \quad \textit{for at least one } k \in \{1, \ldots, p\}.$$

Proof. Suppose contrary to the result of the theorem that

$$f_j(x) + \left(x^T B_j x\right)^{1/2} \leq f_j(y) + y^T B_j z_j \quad \forall j = 1, \ldots, p$$

and

$$f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(y) + y^T B_k z_k \quad \text{for at least one}\, k \in \{1, \ldots, p\}\,.$$

Therefore,

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \left(x^T B_j x\right)^{1/2}\right] < \sum_{j=1}^{p} \tau_j \left[f_j(y) + y^T B_j z_j\right]. \tag{17}$$

As x is feasible for (NDSIP) and $(y, \tau, \lambda, z_1, \ldots, z_p)$ is feasible for (NDSID), we have

$$\sum_{i \in I} \lambda_i g(x, u^i) \leq \sum_{i \in I} \lambda_i g(y, u^i).$$

Then by quasi-invexity of $\lambda_i g(\cdot, u^i), i \in I$, we get

$$\sum_{i \in I} \lambda_i \nabla g(y, u^i) \eta(x, y) \leq 0. \tag{18}$$

Now using equations (7) and (18), we have

$$\sum_{j=1}^{p} \tau_j \left[\nabla f_j(y) + B_j z_j\right] \eta(x, y) \geq 0.$$

Thus, from pseudo-invexity of $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$, we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + x^T B_j z_j\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(y) + y^T B_j z_j\right]. \tag{19}$$

By the generalized Schwarz inequality we know that

$$x^T B_j z_j \leq \left(x^T B_j x\right)^{1/} \left(z_j^T B_j z_j\right)^{1/2},$$

which on using (8) becomes

$$x^T B_j z_j \leq \left(x^T B_j x\right)^{1/2}. \tag{20}$$

Now from (19) and (20), we have

$$\sum_{j=1}^{p} \tau_j \left[f_j(x) + \left(x^T B_j x\right)^{1/2}\right] \geq \sum_{j=1}^{p} \tau_j \left[f_j(y) + y^T B_j z_j\right],$$

which contradicts (17). Hence the proof is complete. ■

Theorem 4.5 (Strong duality). *Let x be an efficient solution for* (NDSIP) *at which the generalized Slater's constraint qualification is satisfied. Let the pseudo-invexity and quasi-invexity assumptions of the weak duality theorem be satisfied. Then there exists $(\tau, \lambda, z_1, \ldots, z_p)$ such that $(x, \tau, \lambda, z_1, \ldots, z_p)$ is an efficient solution for* (NDSID) *and the respective objective values are equal.*

Proof. Since x is an efficient solution for (NDSIP) and the generalized Slater's constraint qualification is satisfied at x, from the Kuhn-Tucker necessary conditions, there exists $(\tau, \lambda, z_1, \ldots, z_p)$ such that $(x, \tau, \lambda, z_1, \ldots, z_p)$ is feasible for (NDSID).
On the other hand by weak duality theorem, the following cannot hold for any feasible y for (NDSID):

$$f_j(x) + \left(x^T B_j x\right)^{1/2} \leq f_j(y) + y^T B_j z_j \quad \forall j = 1, \ldots, p$$

and

$$f_k(x) + \left(x^T B_k x\right)^{1/2} < f_k(y) + y^T B_k z_k \quad \text{for at least one } k \in \{1, \ldots, p\} .$$

From the Kuhn-Tucker necessary conditions, we have

$$\left(x^T B_j x\right)^{1/2} = x^T B_j z_j, \quad j = 1, \ldots, p. \tag{21}$$

Hence, the following cannot hold for any feasible y for (NDSID):

$$f_j(x) + x^T B_j z_j \leq f_j(y) + y^T B_j z_j \quad \forall j = 1, \ldots, p$$

and

$$f_k(x) + x^T B_k z_k < f_k(y) + y^T B_k z_k \quad \text{for at least one } k \in \{1, \ldots, p\} .$$

Thus, $(x, \tau, \lambda, z_1, \ldots, z_p)$ is an efficient solution for (NDSID).
Also using (21), we get that the objective values of (NDSIP) and (NDSID) are equal at x. ■

Theorem 4.6 (Strict converse duality). *Let x^* be an efficient solution for* (NDSIP) *at which the generalized Slater's constraint qualification is satisfied. Assume that $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ is pseudo-invex and $\lambda_i g(\cdot, u^i), i \in I$ are quasi-invex with respect to the same η. If $(\overline{x}, \tau, \lambda, z_1, \ldots, z_p)$ is an efficient solution for* (NDSID) *and $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ is strictly pseudo-invex at $\overline{x}$, then $\overline{x} = x^*$.*

Proof. We prove this by contradiction. Assume that $\overline{x} \neq x^*$. Then by strong duality theorem there exists $(\tau, \lambda, z_1, \ldots, z_p)$ such that $(x^*, \tau, \lambda, z_1, \ldots, z_p)$ is an efficient solution for (NDSID) and

$$f_j(x^*) + \left(x^{*T} B_j x^*\right)^{1/2} = f_j(\overline{x}) + \overline{x}^T B_j z_j, \quad j = 1, \ldots, p,$$

which on using (21) becomes

$$f_j(x^*) + x^{*T} B_j z_j = f_j(\overline{x}) + \overline{x}^T B_j z_j, \quad j = 1, \ldots, p. \tag{22}$$

As x^* is an efficient solution for (NDSIP), $\lambda_i \geq 0$, we have

$$\sum_{i \in I} \lambda_i g(x^*, u^i) \leq \sum_{i \in I} \lambda_i g(\overline{x}, u^i).$$

Then from quasi-invexity of $\lambda_i g(\cdot, u^i), i \in I$, we get

$$\sum_{i \in I} \lambda_i \nabla g(\overline{x}, u^i) \eta(x^*, \overline{x}) \leq 0. \tag{23}$$

Now using equations (7) and (23), we have

$$\sum_{j=1}^{p} \tau_j \left[\nabla f_j(\overline{x}) + B_j z_j\right] \eta(x^*, \overline{x}) \geq 0.$$

Thus, from strict pseudo-invexity of $\left(\tau_1(f_1 + \cdot^T B_1 z_1), \ldots, \tau_p(f_p + \cdot^T B_p z_p)\right)$ at $\overline{x}$, we get

$$\sum_{j=1}^{p} \tau_j \left[f_j(x^*) + x^{*T} B_j z_j\right] > \sum_{j=1}^{p} \tau_j \left[f_j(\overline{x}) + \overline{x}^T B_j z_j\right],$$

which contradicts (22).
Therefore, $x^* = \overline{x}$. ■

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