Vietnam Journal of Mathematics **40**:2&3(2012) 355-369

Vietnam Journal of MATHEMATICS

Slopes of Multifunctions and Extensions of Metric Regularity

Huynh Van Ngai[1], **Alexander Y. Kruger**[2*] **and Michel Théra**[3]

[1] *Department of Mathematics, Quy Nhon University, 170 An Duong Vuong, Quy Nhon, Vietnam*

[2] *Centre for Informatics and Applied Optimization, School of Science, Information Technology and Engineering, University of Ballarat, P. O. B. 663, Ballarat, Vic, 3350, Australia*

[3] *Laboratoire XLIM, UMR-CNRS 6172, Université de Limoges, Limoges, France*

Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th-birthday

Received January 24, 2012
Revised April 24, 2012

Abstract. This article aims to demonstrate how the definitions of slopes can be extended to multi-valued mappings between metric spaces and applied for characterizing metric regularity. Several kinds of local and nonlocal slopes are defined and several metric regularity properties for set-valued mappings between metric spaces are investigated.

2000 Mathematics Subject Classification. 49J52, 49J53, 58C06, 47H04, 54C60.

Key words. Variational analysis, error bounds, slope, multifunction, metric regularity.

1. Introduction

This article aims to demonstrate how the definitions of slopes which have proved to be very useful tools for analyzing local properties of real-valued functions [1–

* The research was partially supported by the Australian Research Council, grant DP110102011.

3, 6, 10–13, 15–17] can be extended to multi-valued mappings between metric spaces and applied for characterizing metric regularity.

Several kinds of local and nonlocal slopes are defined in Section 2 following the scheme developed in [10] for real-valued functions and extended in [4, 5] to vector-valued functions. The idea is not quite new. Some elements of the definitions introduced in the current article are present implicitly in many publications [2, 3, 12, 13, 16, 15]. It seems the definitions can be useful and the time has come to formulate them explicitly.

In this article we investigate several metric regularity properties for set-valued mappings between metric spaces:

- conventional local *metric regularity* and *uniform metric regularity* for mappings depending on a parameter (Section 3);
- *metric regularity along a subspace* (Section 4);
- *metric multi-regularity* for mappings into product spaces (Section 5)

and formulate the corresponding necessary and sufficient regularity criteria in terms of slopes. For the definitions and characterizations of the mentioned above extensions of metric regularity we refer the reader to [8, 9].

Our basic notation is standard, see [14, 18]. Depending on the context, X and Y are either metric or normed spaces. Metrics in all spaces are denoted by the same symbol $d(\cdot,\cdot)$. $d(x,A) = \inf_{a\in A}\|x-a\|$ is the point-to-set distance from x to A. When dealing with product spaces we always assume that the product topology is given by the maximum type norm/distance. We also use the denotation $\alpha_+ = \max(\alpha, 0)$, where $\alpha \in \mathbb{R}$.

Recall that a set-valued mapping (multifunction) $F : X \rightrightarrows Y$ is a mapping which assigns to every $x \in X$ a subset (possibly empty) $F(x)$ of Y. As usual, we use the notation $\operatorname{gph} F := \{(x,y) \in X \times Y \,|\, y \in F(x)\}$ for the graph of F and $F^{-1} : Y \rightrightarrows X$ for the inverse of F. This inverse (which always exists) is defined by $F^{-1}(y) := \{x \in X \,|\, y \in F(x)\}$, $y \in Y$, and satisfies

$$(x,y) \in \operatorname{gph} F \quad \Leftrightarrow \quad (y,x) \in \operatorname{gph} F^{-1}.$$

2. Slopes

We start with considering an extended-real-valued function f on a metric space X. Recall that the local (strong) *slope* [7] of f at x ($|f(x)| < \infty$) is defined as

$$|\nabla f|(x) := \limsup_{u\to x,\, u\neq x} \frac{[f(x)-f(u)]_+}{d(u,x)}. \tag{1}$$

This quantity provides a convenient characterization of the local behaviour of f near x.

Given a $y \in \mathbb{R}$, we set

$$f_y(x) := \max\{f(x), y\}, \quad x \in X \tag{2}$$

and define the *nonlocal slope* of f at x relative to y:

$$|\nabla f|^{\diamond}_y(x) := \sup_{u \neq x} \frac{[f_y(x) - f_y(u)]_+}{d(u,x)}. \tag{3}$$

If $f(x) \le y$, then $f(x) \le f_y(u)$ and $f_y(x) \le f_y(u)$, and consequently $[f_y(x) - f_y(u)]_+ = [f(x) - f_y(u)]_+ = 0$. Hence, $[f_y(x) - f_y(u)]_+ = [f(x) - f_y(u)]_+$ for all x and u, and the subscript y in $f_y(x)$ in the last formula can be removed:

$$|\nabla f|^{\diamond}_y(x) = \sup_{u \neq x} \frac{[f(x) - f_y(u)]_+}{d(u,x)}. \tag{4}$$

As mentioned above, $|\nabla f|^{\diamond}_y(x) = 0$ if $f(x) \le y$. So only the case $f(x) > y$ can be of interest. Note that the supremum in the right-hand side of (3) (or (4)) can be restricted to a certain neighborhood of x since $[f_y(x) - f_y(u)]_+/d(u,x) \to 0$ as $d(u,x) \to \infty$.

It is easy to see from definitions (1) and (3) that, when $y < f(x)$, the two slopes are related by the inequality:

$$|\nabla f|(x) \le |\nabla f|^{\diamond}_y(x).$$

At the same time, the nonlocal slope (3) is an important ingredient in the definition (1) of the local one: for any $y < f(x)$, it holds that

$$|\nabla f|(x) = \lim_{\varepsilon \downarrow 0} |\nabla f_{B_\varepsilon(x)}|^{\diamond}_y(x),$$

where $f_{B_\varepsilon(x)}$ is the restriction of f to $B_\varepsilon(x)$.

The following relations hold true:

$$|\nabla f|(x) = \limsup_{u \to x,\, u \neq x} \frac{[f(x) - \operatorname{cl} f(u)]_+}{d(u,x)}, \quad |\nabla f|^{\diamond}_y(x) = \sup_{u \neq x} \frac{[f(x) - \operatorname{cl} f_y(u)]_+}{d(u,x)}, \tag{5}$$

where $\operatorname{cl} f$ is the lower semicontinuous envelope of f (defined by $\operatorname{cl} f(x) = \liminf_{u \to x} f(u)$).

In the special case $y = 0$, we will omit y in the denotation of the nonlocal slope. Thus

$$|\nabla f|^{\diamond}(x) := \sup_{u \neq x} \frac{[f(x) - f_+(u)]_+}{d(u,x)}, \tag{6}$$

where the function f_+ is defined by $f_+(x) = [f(x)]_+$. We will refer to (6) simply as the *nonlocal slope* of f at x.

If f takes only nonnegative values, then (6) takes a simpler form:

$$|\nabla f|^{\diamond}(x) := \sup_{u \neq x} \frac{[f(x) - f(u)]_+}{d(u,x)} \tag{7}$$

and coincides with the global slope defined in [16].

Let $\bar{x} \in X$ and $\bar{y} = f(\bar{x})$, $|\bar{y}| < \infty$. Using (1) and (3), we define respectively the *strict outer* and *uniform strict slopes* [10, 11] of f at $\bar{x}$:

$$\overline{|\nabla f|}^{>}(\bar{x}) := \liminf_{x \to \bar{x},\ f(x) \downarrow f(\bar{x})} |\nabla f|(x), \tag{8}$$

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x \to \bar{x},\ f(x) \downarrow f(\bar{x})} |\nabla f|^{\diamond}_{\bar{y}}(x). \tag{9}$$

The word "strict" reflects the fact that slopes at nearby points contribute to definitions (8) and (9) making them analogues of the strict derivative. The word "outer" is used to emphasize that only points outside the set $S_{\bar{y}}(f) := \{x \in X | f(x) \le \bar{y}\}$ are taken into account. The word "uniform" emphasizes the non-local character of $|\nabla f|^{\diamond}_{\bar{y}}(x)$ involved in definition (9).

Taking into account (5), we have the relations:

$$\overline{|\nabla f|}^{>}(\bar{x}) := \liminf_{x \to \bar{x},\ \mathrm{cl}\, f(x) \downarrow f(\bar{x})} |\nabla (\mathrm{cl}\, f)|(x),$$

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x \to \bar{x},\ \mathrm{cl}\, f(x) \downarrow f(\bar{x})} |\nabla (\mathrm{cl}\, f)|^{\diamond}_{\bar{y}}(x).$$

Consider now a multifunction $F : X \rightrightarrows Y$ between metric spaces. We are going to define slopes of F using basically the same scheme as described above. To this end, an appropriate scalarization function is needed to replace (2). Given a $y \in Y$, we set

$$f_y(x) := d(y, F(x)), \quad x \in X. \tag{10}$$

Next we apply (1) and (7) to function (10) to define respectively the local and nonlocal slopes of F at x relative to y:

$$|\nabla F|_y(x) := |\nabla f_y|(x) = \limsup_{u \to x,\ u \ne x} \frac{[f_y(x) - f_y(u)]_+}{d(u,x)}, \tag{11}$$

$$|\nabla F|^{\diamond}_y(x) := |\nabla f_y|^{\diamond}(x) = \sup_{u \ne x} \frac{[f_y(x) - f_y(u)]_+}{d(u,x)}. \tag{12}$$

The following representations are straightforward:

$$|\nabla F|_y(x) = \limsup_{\substack{u \to x,\ u \ne x \\ v \in F(u)}} \frac{[f_y(x) - d(y,v)]_+}{d(u,x)},$$

$$|\nabla F|^{\diamond}_y(x) = \sup_{\substack{u \ne x \\ v \in F(u)}} \frac{[f_y(x) - d(y,v)]_+}{d(u,x)},$$

as well as the inequality:

$$|\nabla F|_y(x) \le |\nabla F|^{\diamond}_y(x).$$

Given a point $(\bar{x},\bar{y}) \in \operatorname{gph} F$, we now define the *strict outer* and *uniform strict slopes* of F at $(\bar{x},\bar{y})$

$$\overline{|\nabla F|}^{>}(\bar{x},\bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\ f_y(x)\downarrow 0} |\nabla F|_y(x), \tag{13}$$

$$\overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\ f_y(x)\downarrow 0} |\nabla F|_y^{\diamond}(x). \tag{14}$$

It is easy to check that quantities (13) and (14) do not change if function (10) is replaced in definitions (11), (12), (13), and (14) by its lower semicontinuous envelope. Note also the obvious inequality

$$\overline{|\nabla F|}^{>}(\bar{x},\bar{y}) \le \overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y}).$$

Example 2.1. Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x) = (x_1 + x_2, x_1 - x_2)$, where $x = (x_1, x_2)$. If $y = (y_1, y_2)$, then

$$f_y(x) = \|y_1 - (x_1 + x_2), y_2 - (x_1 - x_2)\|.$$

Let $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ be such that $f_y(x) > 0$. Denote

$$z_1 := \frac{y_1 + y_2}{2} - x_1 \quad \text{and} \quad z_2 := \frac{y_1 - y_2}{2} - x_2.$$

Then

$$z_1 + z_2 = y_1 - (x_1 + x_2), \quad z_1 - z_2 = y_2 - (x_1 - x_2),$$

and $\|z_1, z_2\| \neq 0$. Indeed, if we assume that $z_1 = z_2 = 0$, then $x_1 + x_2 = y_1$ and $x_1 - x_2 = y_2$ which contradicts the assumption that $f_y(x) > 0$. Take $u_1 = x_1 + tz_1$, $u_2 = x_2 + tz_2$ for $t > 0$, and $u = (u_1, u_2)$. Then

$$\begin{aligned} f_y(u) &= \|y_1 - (x_1 + x_2) - t(z_1 + z_2), y_2 - (x_1 - x_2) - t(z_1 - z_2)\| \\ &= (1 - t)\|z_1 + z_2, z_1 - z_2\| \end{aligned}$$

and

$$\frac{f(x) - f(u)}{d(u,x)} = \frac{\|z_1 + z_2, z_1 - z_2\|}{\|z_1, z_2\|} \ge \gamma > 0,$$

where the positive constant γ depends only on the norm on $\mathbb{R}^2$. For instance, if $\mathbb{R}^2$ is equipped with the maximum type norm, then denoting $\alpha := |z_1|/|z_2|$ if $|z_1| \le |z_2|$ or $\alpha := |z_2|/|z_1|$ otherwise, one has

$$\frac{f(x) - f(u)}{d(u,x)} = \max\{1 + \alpha, 1 - \alpha\} \ge 1$$

and we can take $\gamma = 1$.

By (11) and (12), it follows that $|\nabla F|_y(x) \ge \gamma$ and $|\nabla F|_y^{\diamond}(x) \ge \gamma$. Since x and y are arbitrary, it also follows from (13) and (14) that $\overline{|\nabla F|}^{>}(0,0) \ge \gamma$ and $\overline{|\nabla F|}^{\diamond}(0,0) \ge \gamma$.

3. Metric regularity

Recall (see e.g. [14, 18]) that a multifunction $F : X \rightrightarrows Y$ between metric spaces is said to be *metrically regular* near $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exist a $\tau > 0$ and neighborhoods U and V of $\bar{x}$ and $\bar{y}$ respectively such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \forall x \in U,\ \forall y \in V. \tag{15}$$

The following (possibly infinite) constant is convenient for characterizing the metric regularity property:

$$r[F](\bar{x}, \bar{y}) := \liminf_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ (x,y)\notin \operatorname{gph} F}} \frac{d(y, F(x))}{d(x, F^{-1}(y))}. \tag{16}$$

It is easy to check that F is metrically regular near $(\bar{x}, \bar{y})$ if and only if $r[F](\bar{x}, \bar{y}) > 0$. Moreover, when positive, constant (16) provides a quantitative characterization of this property. It coincides with the reciprocal of the infimum of all positive τ such that (15) holds for some U and V (metric regularity modulus). Constant (16) is also known as the *rate* or *modulus of surjection* or *covering* (see [12, 14]).

The next theorem provides an equivalent characterization of the metric regularity property in terms of slopes (13) and (14). It follows from [16, Theorem 5] where a slightly more general statement is established and formulated without the explicit use of constants (13), (14) and (16).

Theorem 3.1. *Let X and Y be a complete metric space and a metric space, respectively, $F : X \rightrightarrows Y$ be a closed multifunction, and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

$$r[F](\bar{x}, \bar{y}) = \overline{|\nabla F|}^{\diamond}(\bar{x}, \bar{y}) \geq \overline{|\nabla F|}^{>}(\bar{x}, \bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 3.2. *Let X and Y be a complete metric space and a metric space, respectively, $F : X \rightrightarrows Y$ be a closed multifunction, and $(\bar{x}, \bar{y}) \in \operatorname{gph} F$. Consider the following conditions:*

(i) *F is metrically regular near $(\bar{x}, \bar{y})$;*
(ii) $\overline{|\nabla F|}^{\diamond}(\bar{x}, \bar{y}) > 0$*;*
(iii) $\overline{|\nabla F|}^{>}(\bar{x}, \bar{y}) > 0$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i)*.*

Moreover, the following assertions are true:

(a) *if* (15) *holds with some $\tau > 0$, U and V, then $\tau^{-1} \leq \overline{|\nabla F|}^{\diamond}(\bar{x}, \bar{y})$;*
(b) *if $0 < \tau^{-1} < \overline{|\nabla F|}^{\diamond}(\bar{x}, \bar{y})$, then* (15) *holds with some U and V.*

If, additionally, Y is a normed linear space, then $\overline{|\nabla F|}^{\diamond}(\bar{x},\bar{y})$ in (a) *and* (b) *above can be replaced by $\overline{|\nabla F|}^{>}(\bar{x},\bar{y})$.*

Example 3.3. Considering the linear continuous mapping $F:\mathbb{R}^2\to\mathbb{R}^2$ from Example 2.1 given by $F(x)=(x_1+x_2,\, x_1-x_2)$, where $x=(x_1,x_2)$, we see that it is surjective and consequently metrically regular near $(0,0)$. This conclusion also follows from Corollary 3.2 thanks to the estimates for the strict slopes of F established in Example 2.1.

The statement of Theorem 3.1 can be extended to the case of set-valued mappings depending on a parameter.

Consider a multifunction $F: P\times X\rightrightarrows Y$, where X and Y are metric spaces and P is a topological space. Denote $F_p=F(p,\cdot):X\rightrightarrows Y$. Let $(\bar{p},\bar{x},\bar{y})\in\operatorname{gph}F$.

We say that F is *uniformly metrically regular* (see e.g. [8]) near $(\bar{p},\bar{x},\bar{y})$ with respect to (x,y) if there exist a $\tau>0$ and neighborhoods U, V and W of $\bar{x}$, $\bar{y}$ and $\bar{p}$, respectively, such that

$$d(x,F_p^{-1}(y))\le \tau d(y,F(p,x))\quad \forall x\in U,\ \forall y\in V, \forall p\in W. \tag{17}$$

This property can be equivalently characterized using the following analogue of (16):

$$r_{\bar{p}}[F](\bar{x},\bar{y}):=\liminf_{\substack{(p,x,y)\to(\bar{p},\bar{x},\bar{y})\\ (p,x,y)\notin\operatorname{gph}F}}\frac{d(y,F(p,x))}{d(x,F_p^{-1}(y))}. \tag{18}$$

F is uniformly metrically regular near $(\bar{p},\bar{x},\bar{y})$ with respect to (x,y) if and only if $r_{\bar{p}}[F](\bar{x},\bar{y})>0$.

To formulate uniform metric regularity criteria in terms of slopes, some modifications of definitions (10)–(14) are required:

$$f_{y,p}(x):=d(y,F(p,x)),\quad x\in X,$$

$$|\nabla F|_{y,p}(x):=|\nabla f_{y,p}|(x)=\limsup_{u\to x,\,u\ne x}\frac{[f_{y,p}(x)-f_{y,p}(u)]_+}{d(u,x)}, \tag{19}$$

$$|\nabla F|^{\diamond}_{y,p}(x):=|\nabla f_{y,p}|^{\diamond}(x)=\sup_{u\ne x}\frac{[f_{y,p}(x)-f_{y,p}(u)]_+}{d(u,x)}, \tag{20}$$

$$\overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x},\bar{y}):=\liminf_{(p,x,y)\to(\bar{p},\bar{x},\bar{y}),\ f_{y,p}(x)\downarrow 0}|\nabla F|_{y,p}(x), \tag{21}$$

$$\overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x},\bar{y}):=\liminf_{(p,x,y)\to(\bar{p},\bar{x},\bar{y}),\ f_{y,p}(x)\downarrow 0}|\nabla F|^{\diamond}_{y,p}(x). \tag{22}$$

The required characterization of the uniform metric regularity property in terms of slopes (21) and (22) is similar to the one provided by Theorem 3.1 and follows from [16, Theorem 8], the latter one being formulated without slopes (21) and (22) and regularity constant (18).

Theorem 3.4. *Let X, Y and P be a complete metric space, a metric space and a topological space respectively, $F : P \times X \rightrightarrows Y$ be a closed multifunction, and $(\bar{p}, \bar{x}, \bar{y}) \in \operatorname{gph} F$. Then*

$$r_{\bar{p}}[F](\bar{x}, \bar{y}) = \overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y}) \geq \overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x}, \bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 3.5. *Let X, Y and P be a complete metric space, a metric space and a topological space respectively, $F : P \times X \rightrightarrows Y$ be a closed multifunction, and $(\bar{p}, \bar{x}, \bar{y}) \in \operatorname{gph} F$. Consider the following conditions:*

(i) *F is uniformly metrically regular near $(\bar{p}, \bar{x}, \bar{y})$;*
(ii) $\overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y}) > 0$*;*
(iii) $\overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x}, \bar{y}) > 0$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Moreover, the following assertions are true:

(a) *if* (17) *holds with some $\tau > 0$, U, V and W, then $\tau^{-1} \leq \overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y})$;*
(b) *if $0 < \tau^{-1} < \overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y})$, then* (17) *holds with some U, V and W.*

If, additionally, Y is a normed linear space, then $\overline{|\nabla F|}^{\diamond}_{\bar{p}}(\bar{x}, \bar{y})$ in (a) *and* (b) *above can be replaced by $\overline{|\nabla F|}^{>}_{\bar{p}}(\bar{x}, \bar{y})$.*

4. Metric regularity along a subspace

Consider a multifunction $F : X \rightrightarrows Y$ from a normed linear space to a metric space. Let H be a (closed) subspace of X. F is called *metrically regular along H* [9] near $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if there exist a $\tau > 0$ and neighborhoods U and V of $\bar{x}$ and $\bar{y}$, respectively, such that

$$\inf_{h \in H} \{\|h\| \big| x + h \in F^{-1}(y)\} \leq \tau d(y, F(x)) \quad \forall x \in U,\ y \in V. \tag{23}$$

Obviously, if $H = X$, then this property coincides with the conventional metric regularity of F near $(\bar{x}, \bar{y})$.

In the definition of metric regularity along H, it is convenient to use the point-to-set *distance along* H defined for $x \in X$ and $M \subset X$ as

$$d_H(x, M) := \inf_{h \in H} \{\|h\| \big| x + h \in M\} = d(0, (M - x) \cap H).$$

Of course, it is not a real distance on X. For instance, $d_H(x_1, x_2) = \infty$ if $x_1 - x_2 \notin H$. In general, $d_H(x, M) \geq d(x, M)$, and the equality holds when $H = X$.

The above property can be equivalently characterized using the following constant:

$$r_H[F](\bar{x},\bar{y}) := \liminf_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\notin \operatorname{gph} F}} \frac{d(y,F(x))}{d_H(x,F^{-1}(y))}. \tag{24}$$

F is metrically regular along H near $(\bar{x},\bar{y})$ if and only if $r_H[F](\bar{x},\bar{y}) > 0$.

Evidently, $r_H[F](\bar{x},\bar{y}) \le r[F](\bar{x},\bar{y})$, and metric regularity of F along some subspace implies its conventional the metric regularity.

The metric regularity along a subspace can be treated in the framework of the previously considered property of parametric metric regularity.

Given a multifunction $F : X \rightrightarrows Y$, define another multifunction $\Phi : X \times H \rightrightarrows Y$ by the formula

$$\Phi(x,h) := F(x+h), \quad x \in X,\ h \in H. \tag{25}$$

Then, for this multifunction, X can be viewed as a parameter space and the above parametric definitions can be reformulated for this particular case, the point $\bar{h} = 0$ being of special interest. The next proposition (cf. [9, Proposition 4.1 (iii)]) shows that the uniform metric regularity of Φ near $(\bar{x},0,\bar{y})$ is exactly the metric regularity of F near $(\bar{x},\bar{y})$ along H.

Proposition 4.1. *Let the mapping* $\Phi : X \times H \rightrightarrows Y$ *be defined by* (25). *Then* $r_H[F](\bar{x},\bar{y}) = r_{\bar{x}}[\Phi](0,\bar{y})$.

Proof. Taking into account (25) and the obvious relations

$$F(x) = \Phi(x,0),\quad \Phi_x^{-1}(y) = (F^{-1}(y)-x)\cap H,\quad d(h,\Phi_x^{-1}(y)) = d_H(x+h,F^{-1}(y)),$$

we have

$$\begin{aligned} r_{\bar{x}}[\Phi](0,\bar{y}) = &\liminf_{\substack{(x,h,y)\to(\bar{x},0,\bar{y})\\(x+h,y)\notin \operatorname{gph} F}} \frac{d(y,F(x+h))}{d_h(x+h,F^{-1}(y))} \\ &\le \liminf_{\substack{(x,y)\to(\bar{x},\bar{y})\\(x,y)\notin \operatorname{gph} F}} \frac{d(y,F(x))}{d_h(x,F^{-1}(y))} = r_H[F](\bar{x},\bar{y}). \end{aligned}$$

On the other hand,

$$\begin{aligned} r_{\bar{x}}[\Phi](0,\bar{y}) = &\lim_{\delta\downarrow 0} \inf_{\substack{(x,h,y)\in B_\delta(\bar{x},0,\bar{y})\\(x+h,y)\notin \operatorname{gph} F}} \frac{d(y,F(x+h))}{d_h(x+h,F^{-1}(y))} \\ &= \lim_{\delta\downarrow 0} \inf_{\substack{x\in B_{2\delta}(\bar{x}),\, y\in B_\delta(\bar{y})\\(x,y)\notin \operatorname{gph} F}} \frac{d(y,F(x))}{d_h(x,F^{-1}(y))} \end{aligned}$$

$$\geq \lim_{\delta\downarrow 0} \inf_{\substack{(x,y)\in B_{2\delta}(\bar{x},\bar{y})\\ (x,y)\notin \operatorname{gph} F}} \frac{d(y,F(x))}{d_h(x,F^{-1}(y))} = r_H[F](\bar{x},\bar{y}).$$

■

Formulas (19)–(22) applied to multifunction (25) lead to the following definitions:

$$|\nabla F|_{y,H}(x) := \limsup_{u\to x,\, u\neq x,\, u-x\in H} \frac{[f_y(x)-f_y(u)]_+}{d(u,x)}, \tag{26}$$

$$|\nabla F|^{\diamond}_{y,H}(x) := \sup_{u\neq x,\, u-x\in H} \frac{[f_y(x)-f_y(u)]_+}{d(u,x)}, \tag{27}$$

$$\overline{|\nabla F|}^{>}_{H}(\bar{x},\bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\ f_y(x)\downarrow 0} |\nabla F|_{y,H}(x), \tag{28}$$

$$\overline{|\nabla F|}^{\diamond}_{H}(\bar{x},\bar{y}) := \liminf_{(x,y)\to(\bar{x},\bar{y}),\ f_y(x)\downarrow 0} |\nabla F|^{\diamond}_{y,H}(x), \tag{29}$$

where f_y is defined by (10).

The next theorem is a consequence of Theorem 3.4.

Theorem 4.2. *Let X and Y be a Banach space and a metric space, respectively, $F : X \rightrightarrows Y$ be a closed multifunction, and $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Suppose H is a subspace of X. Then*

$$r_H[F](\bar{x},\bar{y}) = \overline{|\nabla F|}^{\diamond}_{H}(\bar{x},\bar{y}) \geq \overline{|\nabla F|}^{>}_{H}(\bar{x},\bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 4.3. *Let X and Y be a Banach space and a metric space, respectively, $F : X \rightrightarrows Y$ be a closed multifunction, and $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Suppose H is a subspace of X. Consider the following conditions:*

(i) *F is metrically regular along H near $(\bar{x},\bar{y})$;*
(ii) $\overline{|\nabla F|}^{\diamond}_{H}(\bar{x},\bar{y}) > 0$;
(iii) $\overline{|\nabla F|}^{>}_{H}(\bar{x},\bar{y}) > 0$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Moreover, the following assertions are true:

(a) *if* (23) *holds with some $\tau > 0$, U and V, then $\tau^{-1} \leq \overline{|\nabla F|}^{\diamond}_{H}(\bar{x},\bar{y})$;*
(b) *if $0 < \tau^{-1} < \overline{|\nabla F|}^{\diamond}_{H}(\bar{x},\bar{y})$, then* (23) *holds with some U and V.*

If, additionally, Y is a normed linear space, then $\overline{|\nabla F|}^{\diamond}_{H}(\bar{x},\bar{y})$ in (a) *and* (b) *above can be replaced by $\overline{|\nabla F|}^{>}_{H}(\bar{x},\bar{y})$.*

Example 4.4. Consider again the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F(x) = (x_1 + x_2,\, x_1 - x_2)$, where $x = (x_1, x_2)$. As established in Examples 2.1 and 3.3, it is metrically regular near $(0, 0)$. We are going to show that it is not metrically regular near $(0, 0)$ along the subspace $H = \mathbb{R} \times \{0\}$. For simplicity, we assume that $\mathbb{R}^2$ is equipped with the maximum type norm. Take $x = (0, \alpha)$ with $\alpha \neq 0$ and $y = (0, 0)$. Then $f_y(x) = \|-\alpha, \alpha\| = |\alpha|$ and, for any $h = (\beta, 0) \in H$, $f_y(x+h) = \|-(\alpha+\beta), \alpha-\beta\| = \max\{|\alpha+\beta|, |\alpha-\beta|\} \geq |\alpha|$. Hence, $\overline{|\nabla F|}{}^{\diamond}_{H}(0, 0) = \overline{|\nabla F|}{}^{>}_{H}(0, 0) = |\nabla F|^{\diamond}_{y,H}(x) = |\nabla F|_{y,H}(x) = 0$. The claimed assertion follows from Corollary 4.3.

5. Metric multi-regularity

Let $F : X \rightrightarrows Y$ be a mapping between a normed linear space X and the product of $n \geq 1$ metric spaces $Y = Y_1 \times Y_2 \times \ldots \times Y_n$. Throughout this section we assume that F can be represented as $F = (F_1, F_2, \ldots, F_n)$, where each F_i is a mapping from X into Y_i. This means that for any $x \in X$ its image $F(x)$ under F is the product of the images:

$$F(x) = F_1(x) \times F_2(x) \times \ldots \times F_n(x). \tag{30}$$

If F is single-valued, this assumption is fulfilled automatically.

Let $\bar{x} \in X$ and $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in F(\bar{x})$.

Besides considering metric regularity of F, one can also examine this property componentwise. The next proposition which strengthens [9, Proposition 5.2 (ii)] shows that the metric regularity of F implies metric regularity of all its components.

Proposition 5.1. *$r[F](\bar{x}, \bar{y}) \leq \min_{1\leq i\leq n} r[F_i](\bar{x}, \bar{y}_i)$.*

Proof. If $r[F](\bar{x}, \bar{y}) = 0$, the inequality holds true trivially. Let $r[F](\bar{x}, \bar{y}) > 0$. Take any neighborhoods U of $\bar{x}$ and $V = V_1 \times V_2 \times \ldots \times V_n$ of $\bar{y}$. By definition (16), taking a smaller U if necessary, we can ensure that $F(x) \cap V \neq \emptyset$ for all $x \in U$. Take any i, $1 \leq i \leq n$, any $x \in U$ and any $y_i \in V_i$. For all $j \neq i$ take some $y_j \in F_j(x) \cap V_j$ and compose $y = (y_1, y_2, \ldots, y_n)$. Then $y \in V$, $d(y, F(x)) = d(y_i, F_i(x))$ and $d(x, F^{-1}(y)) = d(x, F_i^{-1}(y_i))$. By the definition (16), $r[F](\bar{x}, \bar{y}) \leq r[F_i](\bar{x}, \bar{y}_i)$. Since this inequality is valid for any i, the assertion has been proved. ■

The inequality in Proposition 5.1 can be strict [9, Example 5.3].

There is another way of dealing with mappings into product spaces. The following local regularity property of F near $(\bar{x}, \bar{y})$, taking into account the behaviour of its components, can be of interest.

F is called *metrically multi-regular* [8] at $(\bar{x}, \bar{y})$ if there exist a $\tau > 0$ and neighborhoods U of $\bar{x}$ and V_i of $\bar{y}_i$, $i = 1, 2, \ldots, n$, such that

$$d(0, \bigcap_{i=1}^{n}(F_i^{-1}(y_i) - x_i)) \le \tau \max_{1\le i\le n} d(y_i, F_i(x_i))$$
$$\forall x_i \in U,\ \forall y_i \in V_i,\ i = 1, 2, \ldots, n. \quad (31)$$

Obviously, when $n = 1$, the above property coincides with the conventional one. When $n > 1$, this property is stronger than the metric regularity which corresponds to taking $x_i = \bar{x}$, $i = 1, 2, \ldots, n$, in the above definition.

A multifunction $F : X \rightrightarrows Y$ of the type (30) can be used, for instance, to define a system of *generalized equations*:

$$0_{Y_i} \in F_i(x), \quad i = 1, 2, \ldots, n. \quad (32)$$

If $\bar{x}$ is a solution of (32), then the metric multi-regularity of F at $(\bar{x}, 0)$ means the existence of a joint "stabilizing" action satisfying an "error bound" type estimate when both the right-hand sides and variables of each of the generalized equations are perturbed independently.

The following constant corresponds to the above metric multi-regularity property:

$$\hat{r}[F](\bar{x}, \bar{y}) := \liminf_{\substack{(x_i, y_i)\to(\bar{x}, \bar{y}_i),\, i=1,2,\ldots,n \\ (y_1,\ldots,y_n)\notin F_1(x_1)\times\ldots\times F_n(x_n)}} \frac{\max\limits_{1\le i\le n} d(y_i, F_i(x_i))}{d(0, \bigcap\limits_{i=1}^{n}(F_i^{-1}(y_i) - x_i))}. \quad (33)$$

Its relationship with (16) is straightforward:

$$\hat{r}[F](\bar{x}, \bar{y}) \le r[F](\bar{x}, \bar{y}),$$

where the equality holds if $n = 1$.

The metric multi-regularity property can be treated in the framework of metric regularity along a subspace examined above. Indeed, let $Z = X^n$ and $z = (x_1, x_2, \ldots, x_n) \in Z$. One can consider the multifunction $\Phi : Z \rightrightarrows Y$ defined by

$$\Phi(z) = F_1(x_1) \times F_2(x_2) \times \ldots \times F_n(x_n). \quad (34)$$

Note that each "component" of Φ in the above formula depends on its own argument.

In the space Z, one can consider the diagonal subspace

$$H = \{(x_1, x_2, \ldots, x_n) \in X^n \,|\, x_1 = x_2 = \ldots = x_n\}. \quad (35)$$

Evidently, $\Phi(z) = F(x)$ if $z = (x, x, \ldots, x) \in H$, and $(\bar{z}, \bar{y}) \in \operatorname{gph} \Phi$, where $\bar{z} = (\bar{x}, \bar{x}, \ldots, \bar{x})$.

The next proposition shows that the metric regularity of Φ near $(\bar{z}, \bar{y})$ along H is exactly the metric multi-regularity of F near $(\bar{x}, \bar{y})$ (cf. [9, Proposition 5.5 (iv)]).

Proposition 5.2. *Let the multifunction $\Phi : Z \rightrightarrows Y$ and the subspace H of Z be defined by* (34) *and* (35)*, respectively. Then $\hat{r}[F](\bar{x},\bar{y}) = r_H[\Phi](\bar{z},\bar{y})$.*

Proof. It follows immediately from definition (34) that, for any $z = (x_1, x_2, \ldots, x_n) \in Z$ and $y = (y_1, y_2, \ldots, y_n) \in Y$, one has

$$d(y, \Phi(z)) = \max_{1\le i\le n} d(y_i, F_i(x_i)),$$
$$\Phi^{-1}(y) = F_1^{-1}(y_1) \times F_2^{-1}(y_2) \times \ldots \times F_n^{-1}(y_n),$$
$$d_H(z, \Phi^{-1}(y)) = d(0, \bigcap_{i=1}^{n} (F_i^{-1}(y_i) - x_i)).$$

The assertion follows by comparing definitions (24) and (33). ■

Formulas (26)–(29) applied to multifunction (34) and subspace (35) lead to the following definitions, where $\hat{y} = (y_1, y_2, \ldots, y_n) \in Y$:

$$f_y^i(x) := d(y, F_i(x)), \quad x \in X,\ y \in Y_i,$$
$$f_{\hat{y}}(x_1, \ldots, x_n) := \max_{1\le i\le n} f_{y_i}^i(x_i),$$
$$|\nabla F|_{\hat{y}}(x_1, \ldots, x_n) := \limsup_{0\ne u\to 0_X} \frac{[f_{\hat{y}}(x_1, \ldots, x_n) - f_{\hat{y}}(x_1 + u, \ldots, x_n + u)]_+}{\|u\|},$$
$$|\nabla F|^{\diamond}_{\hat{y}}(x_1, \ldots, x_n) := \sup_{u\ne 0_X} \frac{[f_{\hat{y}}(x_1, \ldots, x_n) - f_{\hat{y}}(x_1 + u, \ldots, x_n + u)]_+}{\|u\|},$$
$$\widehat{\overline{|\nabla F|}}{}^{>}(\bar{x},\bar{y}) := \liminf_{\substack{(x_i,y_i)\to(\bar{x},\bar{y}),\, i=1,2,\ldots,n\\ f_{\hat{y}}(x_1,\ldots,x_n)\downarrow 0}} |\nabla F|_{\hat{y}}(x_1, \ldots, x_n),$$
$$\widehat{\overline{|\nabla F|}}{}^{\diamond}(\bar{x},\bar{y}) := \liminf_{\substack{(x_i,y_i)\to(\bar{x},\bar{y}),\, i=1,2,\ldots,n\\ f_{\hat{y}}(x_1,\ldots,x_n)\downarrow 0}} |\nabla F|^{\diamond}_{\hat{y}}(x_1, \ldots, x_n).$$

Application of Theorem 4.2 to the setting of metric multi-regularity yields the following statement.

Theorem 5.3. *Let X be a Banach space and $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ be the product of $n \ge 1$ metric spaces. Suppose that $F : X \rightrightarrows Y$ is a closed multifunction which can be represented as $F = (F_1, F_2, \ldots, F_n)$ where $F_i : X \rightrightarrows Y_i$, $i = 1, 2, \ldots, n$, and $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in F(\bar{x})$. Then*

$$\hat{r}[F](\bar{x},\bar{y}) = \widehat{\overline{|\nabla F|}}{}^{\diamond}(\bar{x},\bar{y}) \ge \widehat{\overline{|\nabla F|}}{}^{>}(\bar{x},\bar{y}).$$

If, additionally, Y is a normed linear space, then the last inequality holds as equality.

Corollary 5.4. *Let X be a Banach space and $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ be the product of $n \ge 1$ metric spaces. Suppose that $F : X \rightrightarrows Y$ is a closed multifunction which* *can be represented as* $F = (F_1, F_2, \ldots, F_n)$ *where* $F_i : X \rightrightarrows Y_i$, $i = 1, 2, \ldots, n$, *and* $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_n) \in F(\bar{x})$. *Consider the following conditions:*

(i) F *is metrically multi-regular near* $(\bar{x}, \bar{y})$*;*

(ii) $\widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y}) > 0$*;*

(iii) $\widehat{\overline{|\nabla F|}}^{>}(\bar{x}, \bar{y}) > 0$.

Then (iii) $\Rightarrow$ (ii) $\Leftrightarrow$ (i).

Moreover, the following assertions are true:

(a) *if* (31) *holds with some* $\tau > 0$, U, $V_1,\ldots,V_n$, *then* $\tau^{-1} \le \widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y})$*;*

(b) *if* $0 < \tau^{-1} < \widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y})$, *then* (31) *holds with some* U, $V_1,\ldots,V_n$.

If, additionally, $Y_1, \ldots, Y_n$ *are normed linear spaces, then* $\widehat{\overline{|\nabla F|}}^{\diamond}(\bar{x}, \bar{y})$ *in* (a) *and* (b) *above can be replaced by* $\widehat{\overline{|\nabla F|}}^{>}(\bar{x}, \bar{y})$.

Acknowledgements. The authors wish to acknowledge the very thorough refereeing of the paper and express their gratitude to the two anonymous referees for valuable and helpful comments and suggestions.

References

1. D. Azé, A survey on error bounds for lower semicontinuous functions, in *Proceedings of 2003 MODE-SMAI Conference* (2003), ESAIM Proc., Vol. 13, EDP Sci., Les Ulis, pp. 1–17.
2. D. Azé and J.-N. Corvellec, Characterizations of error bounds for lower semicontinuous functions on metric spaces, *ESAIM, Control Optim. Calc. Var.* **10** (2004), 409–425.
3. D. Azé and J.-N. Corvellec, On some variational properties of metric spaces, *J. Fixed Point Theory Appl.* **5** (2009), 185–200.
4. E. M. Bednarczuk and A. Y. Kruger, Error bounds for vector-valued functions: necessary and sufficient conditions, *Nonlinear Anal.* **75** (2012), 1124–1140.
5. E. M. Bednarczuk and A. Y. Kruger, Error bounds for vector-valued functions on metric spaces, *Vietnam J. Math.* **40** (2012).
6. J.-N. Corvellec and V. V. Motreanu, Nonlinear error bounds for lower semicontinuous functions on metric spaces, *Math. Program., Ser. A* **114** (2008), 291–319.
7. E. De Giorgi, A. Marino and M. Tosques, Problems of evolution in metric spaces and maximal decreasing curve, *Atti Accad. Naz. Lincei, Rend. Cl. Sci. Fis. Mat. Nat.* (8) **68** (1980), 180–187.
8. A. V. Dmitruk and A. Y. Kruger, Metric regularity and systems of generalized equations, *J. Math. Anal. Appl.* **342** (2008), 864–873.
9. A. V. Dmitruk and A. Y. Kruger, Extensions of metric regularity, *Optimization* **58** (2009), 561–584.
10. M. J. Fabian, R. Henrion, A. Y. Kruger and J. V. Outrata, Error bounds: necessary and sufficient conditions, *Set-Valued Var. Anal.* **18** (2010), 121–149.
11. M. J. Fabian, R. Henrion, A. Y. Kruger and J. V. Outrata, About error bounds in metric spaces, in: *Proc. Int. Conf. Operations Research (OR 2011), Zurich, August 30–September 2, 2011*, Springer, 2012. To appear.

12. A. D. Ioffe, Metric regularity and subdifferential calculus, *Russ. Math. Surv.* **55** (2000), 501–558.
13. A. D. Ioffe, Regularity on a fixed set, *SIAM J. Optim.* **21** (2011), 1345–1370.
14. B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation I: Basic Theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 330, Springer-Verlag, Berlin, 2006.
15. H. V. Ngai and M. Théra, Error bounds for systems of lower semicontinuous functions in Asplund spaces, *Math. Program., Ser. B* **116** (2009), 397–427.
16. H. V. Ngai and M. Théra, Directional metric regularity via error bounds, To be published.
17. J.-P. Penot, Error bounds, calmness and their applications in nonsmooth analysis, in: *Nonlinear Analysis and Optimization II: Optimization*, A. Leizarowitz, B. S. Mordukhovich, I. Shafrir and A. J. Zaslavski (eds.), Contemp. Math., Vol. 514, AMS, Providence, RI, 2010, pp. 225–248.
18. R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 317, Springer-Verlag, Berlin, 1998.