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Strong Convergence of Two Hybrid Extragradient Methods for Solving Equilibrium and Fixed Point Problems*

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. In this paper we propose and we study two algorithmic methods for finding a common solution of an equilibrium problem and a fixed point problem in a Hilbert space. The strategy is to replace the proximal point iteration used in most papers by an extragradient procedure with or without an Armijo-backtracking linesearch. The strong convergence of the iterates generated by each method is obtained thanks to a shrinking projection method and under the assumptions that the fixed point mapping is a ξ-quasi-strict pseudo-contraction and the equilibrium function is monotone and Lipschitz-continuous for the pure extragradient method and pseudomonotone and weakly continuous for the extragradient method with linesearches. The particular case when the equilibrium problem is a variational inequality problem is considered in the last section.

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Key words. Equilibrium problem, fixed point problem, shrinking projection method, extragradient method, ξ-quasi-strict pseudo-contraction, Lipschitz continuity.

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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Let f be a function from $C \times C$ into $\mathbb{R}$ where $\mathbb{R}$ is the set of real numbers. The equilibrium problem associated with f consists in finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0 \quad \text{for every } y \in C.$$

The set of solutions of this problem is denoted by $E(f)$. When $f(x,y) = \langle F(x), y - x \rangle$ for every $x, y \in C$, where $F : C \to H$, the equilibrium problem reduces to a variational inequality problem that consists in finding a point $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0 \quad \text{for every } y \in C.$$

The class of equilibrium problems is very large. In particular, it includes the optimization problems, the variational inequality problems, and the Nash equilibria in noncooperative games (see [12], [3] and the references therein for more details).

In most papers the function f is assumed to satisfy the following conditions

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) $\limsup_{t \to 0^+} f(tz + (1-t)x, y) \leq f(x,y)$ for all $x, y, z \in C$;

(A4) $f(x, \cdot)$ is convex, subdifferentiable and lower semicontinuous for all $x \in C$.

Under these assumptions, for each $r > 0$ and $x \in H$, there exists a unique element $z \in C$ such that

$$f(z,y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \text{for every } y \in C \tag{1}$$

(see, for example, [8]).

The aim of this paper is to propose and study an algorithm for finding a point $x^* \in E(f)$ that is also a fixed point of a mapping S from C to C, i.e., such that $Sx^* = x^*$. The set of fixed points of S is denoted by $\mathrm{Fix}(S)$. The problem of finding such a point x^* has been studied in many recent papers (see [7, 15, 20, 21] and the references therein). Most of these algorithms are based on the inequality (1) for solving the underlying equilibrium problem, namely that the sequence $\{x_n\}$ is generated by $x_0 \in H$ and, for every $n \in \mathbb{N}$, by

$$x_{n+1} \in C \text{ such that } f(x_{n+1}, y) + \frac{1}{r_n} \langle y - x_{n+1}, x_{n+1} - x_n \rangle \geq 0 \text{ for every } y \in C,$$

where $\{r_n\} \subset (0, \infty)$ satisfies the condition $\liminf_{n \to \infty} r_n > 0$. Then the combination of this iteration with the fixed point iteration (corresponding to S) can be described as follows: $x_0 \in H$ and, for every $n \in \mathbb{N}$,

$$\begin{cases} z_n \in C \text{ such that } f(z_n, y) + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0 \text{ for every } y \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S z_n, \end{cases}$$

where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfies the condition $\liminf_{n\to\infty} r_n > 0$.

In [18], Tada and Takahashi have proven that if f satisfies conditions (A1)-(A4) and S is a nonexpansive mapping of C to H such that $\mathrm{Fix}(S) \cap E(f) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to $x^* \in \mathrm{Fix}(S) \cap E(f)$ where

$$x^* = \lim_{n\to\infty} P_{\mathrm{Fix}(S)\cap E(f)}(x_n).$$

Here P_K denotes the orthogonal projection mapping onto a nonempty closed convex subset K of H.

Let us recall that S is nonexpansive on C if

$$\|Sx - Sy\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

It is well known that if C is a nonempty bounded closed convex subset of H and if S is nonexpansive, then $\mathrm{Fix}(S)$ is a nonempty closed convex subset of C.

Recently, Ceng et al. [4] have proven that the sequence generated by the previous algorithm is weakly convergent under the same assumptions as in [18], except that the mapping S is supposed to be a ξ-strict pseudo-contraction and that the sequence $\{\alpha_n\}$ is supposed to belong to $[a, b]$ for some $a, b \in (\xi, 1)$. Let us recall ([9]) that S is a ξ-strict pseudo-contraction mapping if there exists $\xi \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \xi \, \|(I - S)x - (I - S)y\|^2 \quad \text{for all } x, y \in C,$$

where I denotes the identity mapping. It is easy to see that the class of strict pseudo-contraction mappings strictly includes the class of nonexpansive mappings.

In this paper we suppose that the mapping S is a ξ-quasi-strict pseudo-contraction in the sense that there exists $\xi \in [0, 1)$ such that

$$\|Sx - x^*\|^2 \leq \|x - x^*\|^2 + \xi\|x - Sx\|^2 \quad \text{for every } x \in C \text{ and } x^* \in \mathrm{Fix}(S).$$

We also suppose that the mapping $I - S$ is demiclosed at zero, i.e., that

$$x_n \rightharpoonup x \text{ (weakly)} \;\text{ and }\; Sx_n - x_n \to 0 \text{ (strongly)} \;\Rightarrow\; Sx = x.$$

Several techniques have been recently proposed to force the strong convergence of the sequence generated by the previous algorithm. One of them is the shrinking projection method [19, 9] where the sequence $\{x_n\}$ is defined as follows: $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$ and

$$\begin{cases} z_n \in C \text{ such that } f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \text{ for every } y \in C, \\ w_n = (1-\alpha_n)z_n + \alpha_n S z_n, \\ C_{n+1} = \{z \in C_n \mid \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0 \end{cases}$$

for every integer $n \geq 1$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfies the condition $\liminf_{n\to\infty} r_n > 0$.

In their paper, Jaiboon and Kumam [9] have proven that the sequence $\{x_n\}$ generated by the shrinking projection method converges strongly to the projection of x_0 onto $\text{Fix}(S) \cap E(f)$ provided that f satisfies the conditions (A1)-(A4), that the mapping S is a ξ-strict pseudo-contraction, and that $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (\xi, 1)$.

Let us also mention here that there exist two other procedures to force the strong convergence of the sequence $\{x_n\}$ to a point $x^* \in \text{Fix}(S) \cap E(f)$: The hybrid projection method (see, for example, [2, 9, 13]) and a method based on the viscosity principle (see, for example, [1, 11, 20]).

When the problem is to find a common solution to a variational inequality problem and a fixed point problem, many papers combine an extragradient iteration with a fixed point iteration (see, for example, [5, 6]). Recently the extragradient method has been extended to equilibrium problems [22], and in [1, 2] to the problem of finding a common solution $x^* \in \text{Fix}(S) \cap E(f)$. In other words, given $x_n \in C$, the proximal step computing z_n is replaced by the following two steps

$$y_n = \arg\min_{y\in C} \left\{ \lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}, \tag{2}$$

$$z_n = \arg\min_{y\in C} \left\{ \lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \right\}, \tag{3}$$

where $\lambda_n \in (0, \infty)$. In addition, the authors of [2] combine the extragradient method with the hybrid projection method to force the strong convergence of the sequence $\{x_n\}$. More precisely, after computing y_n and z_n as in (2) and (3), they calculate t_n and x_{n+1} as follows:

$$\begin{aligned} & t_n = \alpha_n z_n + (1-\alpha_n)\textstyle\sum_{i=1}^p \lambda_{n,i} S_i z_n, \\ & x_{n+1} = P_{P_n \cap Q_n}(x_0) \text{ with} \\ & \begin{cases} P_n = \{z \in C \mid \|t_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C \mid \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \end{cases} \end{aligned}$$

where the mappings S_i, $i = 1, \ldots, p$, are strict pseudo-contractions.

Under conditions on $\{\lambda_n\}$, $\{\lambda_{n,i}\}$ and $\{\alpha_n\}$, the authors of [2] prove that the sequence $\{x_n\}$ converges strongly to the projection of x_0 onto the set $\cap_{i=1}^p \text{Fix}(S_i) \cap E(f)$ provided that f satisfies the following Lipschitz-type condition

(A5) $\exists\, c_1 > 0, \exists\, c_2 > 0$ such that for every $x, y, z \in C$

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|y-x\|^2 - c_2\|z-y\|^2.$$

This assumption (A5) was introduced by Mastroeni [10] to prove the convergence of the Auxiliary Principle Method for equilibrium problems. When $f(x,y) = \langle F(x), y-x\rangle$ for every $x, y \in C$, it is easy to see that this condition holds when the map $F : C \to H$ is Lipschitz continuous.

However, the Lipschitz-type condition (A5) on f is rather strong. Furthermore, this condition is not necessary for proving the convergence of the method combining the proximal point iteration and the shrinking projection method. So our aim in this paper is to incorporate a linesearch iteration into the extragradient shrinking projection method to obtain the strong convergence of the sequence $\{x_n\}$ to some $x^* \in \mathrm{Fix}(S) \cap E(f)$ without imposing the Lipschitz-type condition (A5). Such a procedure is well known in the variational inequality framework for avoiding the introduction of a Lipschitz property. Recently, this strategy has been extended in [22] to the equilibrium problems and in [17] to the quasi-equilibrium problems.

The paper is organized as follows: In Section 2, some preliminary results are recalled. In Section 3, two algorithms are presented for finding a common solution to an equilibrium problem and a fixed point problem. First a linesearch procedure is introduced into the extragradient iteration and two different projections are considered giving rise to two algorithms. Then the resulting methods are combined with the shrinking projection method. This combination allows us to prove the strong convergence of the sequences generated by the two algorithms without imposing a Lipschitz-type condition on f. Finally, in Section 4, a particular case of variational inequality problems is examined.

2. Preliminaries

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. We denote the weak convergence and the strong convergence of a sequence $\{x_n\}$ to x in H by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for every $y \in C$. We know that P_C is a nonexpansive mapping from H onto C and that the two inequalities

$$\langle x - P_C x, P_C x - y\rangle \geq 0 \text{ and } \|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$$

are satisfied for every $x \in H$ and $y \in C$. The following lemmas are useful to establish our main results.

Lemma 2.1. [9] *For any $t \in [0,1]$ and for any $x, y \in H$, the following inequality holds:*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2.$$

Lemma 2.2. [9] *Let K be a nonempty closed convex subset of H. Let $u \in H$ and let $\{x_n\}$ be a sequence in H. If any weak limit point of $\{x_n\}$ belongs to K and if $\|x_n - u\| \leq \|u - P_K u\|$ for all $n \in \mathbb{N}$, then $x_n \to P_K u$.*

Lemma 2.3. [9] *Let K be a nonempty closed convex subset of H and let $S : K \to K$ be a mapping.*

1) *If S is a ξ-quasi-strict pseudo-contraction, then the fixed point set Fix(S) of S is closed and convex;*
2) *If S is a ξ-strict pseudo-contraction, then the mapping S is a ξ-quasi-strict pseudo-contraction and the mapping $I - S$ is demiclosed at zero.*

3. The shrinking projection method with linesearches

The extragradient algorithm considered in [2] has been proven to be strongly convergent under the Lipschitz-type condition (A5). This condition depends on two positive parameters c_1 and c_2 and, in some cases, they are unknown or difficult to approximate. In this section, we modify the second step of the extragradient iteration, i.e., the computation of z_n (see (3)), by considering a linesearch and two different projections onto a hyperplane and the set C. Doing so, we obtain two algorithms and we can prove the strong convergence of the generated sequences without assuming the Lipschitz-type condition (A5). However, in compensation, we have to slightly reinforce assumptions (A3) and (A4). More precisely, in this section, we suppose that the function f satisfies the following conditions

(B1) $f(x, x) = 0$ for all $x \in C$;

(B2) f is pseudomonotone on $E(f)$, i.e., $f(y, x^*) \leq 0$ for every $x^* \in E(f)$;

(B3) f is jointly weakly continuous on the product $\Delta \times \Delta$ where Δ is an open convex set containing C in the sense that if $x, y \in \Delta$ and if $\{x_n\}$ and $\{y_n\}$ are two sequences in Δ converging weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$;

(B4) $f(x, \cdot)$ is convex and subdifferentiable for all $x \in C$.

Under these assumptions, we have the following result (see, for example, [22]).

Proposition 3.1. *Assume that $f : C \times C \to \mathbb{R}$ satisfies* (B1)–(B4). *Then the set $E(f)$ of solutions to the equilibrium problem is closed and convex.*

We also suppose that the mapping S satisfies the conditions

(C1) S is a ξ-quasi-strict pseudo-contraction mapping for some $\xi \in [0, 1)$;

(C2) $I - S$ is demiclosed at zero.

As a consequence of Lemma 2.3 and Proposition 3.1, the set Fix$(S) \cap E(f)$ is closed and convex when f satisfies (B1)–(B4) and S is a ξ-quasi-strict pseudo-contraction mapping. In that case, the orthogonal projection onto Fix$(S) \cap E(f)$ is well defined when Fix$(S) \cap E(f)$ is nonempty.

Now our first algorithm can be expressed as follows.

ALGORITHM 1

Step 0. Choose $\alpha \in (0,2)$, $\gamma \in (0,1)$ and the sequences

$$\{\alpha_n\} \subset [0,1),\ \{\beta_n\} \subset (0,1) \text{ and } \{\lambda_n\} \subset (0,1].$$

Step 1. Let $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1}x_0$. Set $n = 1$.

Step 2. Solve the strongly convex program

$$\min_{y\in C}\left\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}$$

to obtain the unique optimal solution y_n.

Step 3. If $y_n = x_n$, then set $z_n = x_n$. Otherwise

Step 3.1 Find m the smallest nonnegative integer such that

$$\begin{cases} z_{n,m} = (1-\gamma^m)x_n + \gamma^m y_n, \\ f(z_{n,m}, x_n) - f(z_{n,m}, y_n) \geq \frac{\alpha}{2\lambda_n}\|x_n - y_n\|^2. \end{cases}$$

Step 3.2 Set $\gamma_n = \gamma^m$, $z_n = z_{n,m}$ and go to Step 4.

Step 4. Select $g_n \in \partial_2 f(z_n, x_n) = \partial[f(z_n,\cdot)](x_n)$
and compute $w_n = P_C(x_n - \sigma_n g_n)$
where $\sigma_n = \frac{f(z_n,x_n)}{\|g_n\|^2}$ if $y_n \neq x_n$ and $\sigma_n = 0$ otherwise.

Step 5. Compute $t_n = \alpha_n x_n + (1-\alpha_n)\left[\beta_n w_n + (1-\beta_n)Sw_n\right]$.

If $y_n = x_n$ and $t_n = x_n$, then STOP: $x_n \in \mathrm{Fix}(S) \cap E(f)$.
Otherwise go to Step 6.

Step 6. Compute $x_{n+1} = P_{C_{n+1}}x_0$, where $C_{n+1} = \{z \in C_n \mid \|t_n - z\| \leq \|x_n - z\|\}$.

Step 7. Set $n := n + 1$ and go to Step 2.

The next lemma will be used in the proof of our strong convergence theorem.

Lemma 3.2. *Let $f : \Delta \times \Delta \to \mathbb{R}$ be a function satisfying conditions* (B3)–(B4). *Let $\bar{x}, \bar{z} \in \Delta$ and let $\{x_n\}$ and $\{z_n\}$ be two sequences in Δ converging weakly to $\bar{x}, \bar{z}$, respectively. Then, for any $\varepsilon > 0$, there exist $\eta > 0$ and $n_\varepsilon \in \mathbb{N}$ such that*

$$\partial_2 f(z_n, x_n) \subset \partial_2 f(\bar{z}, \bar{x}) + \frac{\varepsilon}{\eta}B$$

for every $n \geq n_\varepsilon$, where B denotes the closed unit ball in H.

This lemma is an infinite-dimensional version of Theorem 24.5 in [14] with a similar proof.

First we prove that Algorithm 1 is well defined. In that purpose we need to establish the following lemma.

Lemma 3.3. *For every integer $n \geq 1$ and every $y \in C$, the following inequality holds*

$$\langle x_n - y_n, y - y_n \rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, y_n).$$

Proof. Let n be an integer ≥ 1 and let $y \in C$. By definition of y_n, there exists $s_n \in \partial_2 f(x_n, y_n)$ such that

$$0 \in \lambda_n s_n + y_n - x_n + N_C(y_n)$$

where $N_C(y_n)$ denotes the normal cone to C at y_n.

Using successively the definition of the normal cone to C at y_n and the subdifferential of the convex function $f(x_n, \cdot)$ at y_n, we can write the following two inequalities

$$\langle -\lambda_n s_n - y_n + x_n, y - y_n \rangle \leq 0,$$

$$f(x_n, y) \geq f(x_n, y_n) + \langle s_n, y - y_n \rangle$$

for every $y \in C$. Then the desired inequality is easily obtained by combining these two inequalities. ■

Proposition 3.4. *Suppose that $y_n \neq x_n$ for some integer $n \geq 1$. Then*

1) *The linesearch corresponding to x_n and y_n (Step 3.1) is well defined;*
2) *$f(z_n, x_n) > 0$;*
3) *$0 \notin \partial_2 f(z_n, x_n)$.*

Proof. We only prove the first statement. The proof of the other ones can be found in [22], Lemma 4.5. Let n be an integer ≥ 1 and suppose, to get a contradiction, that the following inequality holds for every integer $m \geq 0$

$$f(z_{n,m}, x_n) - f(z_{n,m}, y_n) < \frac{\alpha}{2\lambda_n} \|x_n - y_n\|^2, \tag{4}$$

where $z_{n,m} = (1 - \gamma^m)x_n + \gamma^m y_n$ and $\gamma \in (0, 1)$.

Then $\{z_{n,m}\}_m$ converges strongly to x_n as $m \to \infty$ and thus also weakly. Since $f(\cdot, x)$ is weakly continuous on an open set $\Delta \supset C$ for every $x \in \Delta$, it follows that $f(z_{n,m}, x_n) \to f(x_n, x_n) = 0$ and that $f(z_{n,m}, y_n) \to f(x_n, y_n)$ as $m \to \infty$. So, taking the limit on m in (4) yields

$$-f(x_n, y_n) \leq \frac{\alpha}{2\lambda_n} \|y_n - x_n\|^2. \tag{5}$$

Now from Lemma 3.3 with $y = x_n$, we can write that

$$\|y_n - x_n\|^2 \leq -\lambda_n f(x_n, y_n).$$

Combining this inequality with (5), we obtain that $(1-\frac{\alpha}{2})\|y_n - x_n\|^2 \leq 0$. Since $\alpha \in (0,2)$, we deduce that $y_n = x_n$, which gives rise to a contradiction because we have supposed that $y_n \neq x_n$. Consequently the linesearch is well defined. ■

As a consequence of this proposition we have that σ_n is well defined and positive when $y_n \neq x_n$ (see Step 4 of Algorithm 1). In the next proposition we justify the stopping criterion.

Proposition 3.5. *If $y_n = x_n$, then $x_n \in E(f)$. If $y_n = x_n$ and $t_n = x_n$, then $w_n = x_n$ and $x_n \in \mathrm{Fix}(S) \cap E(f)$.*

Proof. When $y_n = x_n$, it follows from Lemma 3.3 that

$$0 \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, x_n) = \lambda_n f(x_n, y)$$

holds for every $y \in C$. But this means that $x_n \in E(f)$.

On the other hand, when $y_n = x_n$ and $t_n = x_n$, we have that $w_n = x_n$ because $\sigma_n = 0$ and $x_n \in C$. Hence

$$x_n = \alpha_n x_n + (1-\alpha_n)\,[\beta_n x_n + (1-\beta_n) S x_n].$$

Since, by assumption, $1-\alpha_n > 0$ and $1-\beta_n > 0$, it follows that $x_n = \beta_n x_n + (1-\beta_n) S x_n$ and that $x_n = S x_n$. So $x_n \in \mathrm{Fix}(S)$. ■

Remark 3.6. Step 4 of Algorithm 1 can also be expressed as

$$\begin{cases} \text{Select } g_n \in \partial_2 f(z_n, x_n) \text{ and compute } w_n = P_C u_n, \\ \text{where } u_n = P_{H_n} x_n \text{ and } H_n = \{x \in H \mid \langle g_n, x_n - x\rangle \geq f(z_n, x_n)\}. \end{cases}$$

Indeed, let w_n be the vector computed in Step 4 of Algorithm 1. When $y_n \neq x_n$, it is easy to see that $x_n - \sigma_n g_n = P_{H_n} x_n$ and thus that $w_n = P_C u_n$ with $u_n = P_{H_n} x_n$. On the other hand when $y_n = x_n$, we have that $\sigma_n = 0$ and $w_n = P_C x_n$. Since $f(z_n, x_n) = f(x_n, x_n) = 0$, we have that $x_n \in H_n$. So $u_n = x_n$ and $w_n = P_C u_n$.

Now we suppose that the STOP never occurs in Algorithm 1 and we establish the strong convergence of the infinite sequence $\{x_n\}$ generated by the algorithm to the projection of x_0 onto $\mathrm{Fix}(S) \cap E(f)$.

Theorem 3.7. *Let C be a nonempty closed convex subset of H. Let f be a function from $\Delta \times \Delta$ into $\mathbb{R}$ satisfying conditions* (B1)–(B4) *and let S be a mapping from C to C satisfying conditions* (C1)–(C2) *such that* $\mathrm{Fix}(S) \cap E(f) \neq \emptyset$. *Let $\alpha \in (0,2)$, $\gamma \in (0,1)$ and suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions*

$$\{\alpha_n\} \subset [0,c] \text{ for some } c < 1, \{\beta_n\} \subset [d,b] \text{ for some } 0 \leq \xi < d \leq b < 1,$$

$$\text{and } \{\lambda_n\} \subset [\lambda, 1] \text{ for some } 0 < \lambda \leq 1.$$

Then the sequence $\{x_n\}$ *generated by Algorithm 1 converges strongly to the projection of* x_0 *onto* $\mathrm{Fix}(S) \cap E(f)$.

Proof. Let $\{x_n\}$ be the infinite sequence generated by Algorithm 1. The proof of the strong convergence of this sequence is done in several steps.

Step 1. The sequence $\{x_n\}$ is well defined.

We prove by induction that each subset C_n is closed convex and contains the nonempty set $\mathrm{Fix}(S) \cap E(f)$. This property is obvious for $n = 1$. So we suppose that C_n is a closed convex subset of H that contains $\mathrm{Fix}(S) \cap E(f)$ and we prove that this property remains true for $n+1$. Since C_{n+1} can be rewritten as

$$C_{n+1} = \{z \in C_n \mid 2\langle x_n - t_n, z\rangle \leq \|x_n\|^2 - \|t_n\|^2\}$$

we have immediately that C_{n+1} is closed and convex.

Next let $x^* \in \mathrm{Fix}(S) \cap E(f)$. Using the convexity of $\|\cdot\|^2$ and Lemma 2.1, we obtain successively

$$\begin{aligned}
\|t_n - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)[\beta_n w_n + (1-\beta_n)Sw_n - x^*]\|^2 \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|\beta_n w_n + (1-\beta_n)Sw_n - x^*\|^2 \\
&= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[\beta_n\|w_n - x^*\|^2 + \\
&\quad + (1-\beta_n)\|Sw_n - x^*\|^2 - \beta_n(1-\beta_n)\|w_n - Sw_n\|^2] \\
&\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)[\beta_n\|w_n - x^*\|^2 + (1-\beta_n)\|w_n - x^*\|^2 + \\
&\quad + (1-\beta_n)\xi\|w_n - Sw_n\|^2 - \beta_n(1-\beta_n)\|w_n - Sw_n\|^2] \\
&= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|w_n - x^*\|^2 - \\
&\quad - (1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|w_n - Sw_n\|^2, \qquad (6)
\end{aligned}$$

where we have used the ξ-quasi-strict pseudo-contraction property of the mapping S to get the last inequality.

On the other hand, using successively the definition of the subgradient g_n, the pseudomonotonicity of f on $E(f)$, and the definition of σ_n, we can write that

$$\langle g_n, x_n - x^*\rangle \geq f(z_n, x_n) - f(z_n, x^*) \geq f(z_n, x_n) = \sigma_n\|g_n\|^2.$$

(Let us point out that this property also holds when $y_n = x_n$ because in that case $z_n = x_n$ and $\sigma_n = 0$). Hence we obtain for every integer $n \geq 1$ (P_C being nonexpansive) that

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|P_C(x_n - \sigma_n g_n) - P_C x^*\|^2 \\
&\leq \|x_n - \sigma_n g_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 - 2\sigma_n\langle g_n, x_n - x^*\rangle + \sigma_n^2\|g_n\|^2 \\
&\leq \|x_n - x^*\|^2 - \sigma_n^2\|g_n\|^2, \qquad (7)
\end{aligned}$$

where $\sigma_n = 0$ when $y_n = x_n$.

Combining (6) and (7), we deduce that for every integer $n \geq 1$

$$\|t_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1-\alpha_n)\sigma_n^2 \|g_n\|^2 - (1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|w_n - Sw_n\|^2. \tag{8}$$

Since $1 - \alpha_n > 0$, $1 - \beta_n > 0$, and $\beta_n - \xi > 0$, we have in particular that

$$\|t_n - x^*\| \leq \|x_n - x^*\|,$$

i.e. , $x^* \in C_{n+1}$. But this means that $\mathrm{Fix}(S) \cap E(f) \subset C_{n+1}$. In particular, the subsets C_n are all nonempty closed and convex and thus the sequence $\{x_n\}$ is well defined.

Step 2. The sequence $\{x_n\}$ is bounded, $\|x_{n+1} - x_n\| \to 0$ and $\|t_n - x_n\| \to 0$.

Let n be an integer ≥ 1. Since $x_n = P_{C_n} x_0$, it follows from the definition of the projection that

$$\langle x_0 - x_n, x_n - y \rangle \geq 0 \quad \text{for every } y \in C_n. \tag{9}$$

Since $\mathrm{Fix}(S) \cap E(f) \subset C_n$ (see Step 1 of this proof), we can write

$$\langle x_0 - x_n, x_n - x^* \rangle \geq 0 \quad \text{for every } x^* \in \mathrm{Fix}(S) \cap E(f).$$

So $0 \leq \langle x_0 - x_n, x_n - x_0 + x_0 - x^* \rangle \leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \, \|x_0 - x^*\|$, which can be rewritten as

$$\|x_n - x_0\| \leq \|x^* - x_0\| \quad \text{for every } x^* \in \mathrm{Fix}(S) \cap E(f). \tag{10}$$

The integer n being arbitrary, this means that the sequence $\{x_n\}$ is bounded. Next to prove that $\|x_{n+1} - x_n\| \to 0$, we observe that $\|x_n - x_0\| \to a \geq 0$. Indeed, since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, it follows from (9) that

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0, \tag{11}$$

i.e., that $\langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1} \rangle \geq 0$ and thus that $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. Hence, the sequence $\{x_n\}$ being bounded, the sequence $\{\|x_n - x_0\|\}$ is convergent to some $a \geq 0$. Consequently we can deduce, using (11), that

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_{n+1} - x_0\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1} \rangle + \|x_{n+1} - x_0\|^2 \\
&= -\|x_n - x_0\|^2 + 2\langle x_n - x_0, x_n - x_{n+1} \rangle + \|x_{n+1} - x_0\|^2 \\
&\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}$$

Since the right-hand side of the previous inequality tends to zero, we obtain that $\|x_n - x_{n+1}\| \to 0$.

On the other hand, since $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have by the definition of C_{n+1} that

$$\|t_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|.$$

Hence $\|t_n - x_n\| \le \|t_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le 2\|x_{n+1} - x_n\|$, and $\|t_n - x_n\| \to 0$ because $\|x_{n+1} - x_n\| \to 0$.

Step 3. $\sigma_n\|g_n\| \to 0$ and $\|w_n - Sw_n\| \to 0$.

Since $\|x_n - x^*\|^2 - \|t_n - x^*\|^2 \le \|x_n - t_n\| \, [\|x_n - x^*\| + \|t_n - x^*\|]$, it follows from (8) that the next two inequalities hold

$$(1-\alpha_n)\sigma_n^2 \, \|g_n\|^2 \le \|x_n - t_n\| \, [\|x_n - x^*\| + \|t_n - x^*\|] \quad \text{and}$$
$$(1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|w_n - Sw_n\|^2 \le \|x_n - t_n\| \, [\|x_n - x^*\| + \|t_n - x^*\|] \, .$$

Since $1-\alpha_n \ge 1-c > 0$, $1-\beta_n \ge 1-b > 0$, $\beta_n - \xi \ge d - \xi > 0$, $\|x_n - t_n\| \to 0$, and since the sequences $\{x_n\}$ and $\{t_n\}$ are bounded, we easily deduce from the first inequality that $\sigma_n\|g_n\| \to 0$ and from the second inequality that $\|w_n - Sw_n\| \to 0$.

Step 4. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ converging weakly to some $\bar{x} \in C$. Then the sequences $\{y_{n_i}\}$, $\{z_{n_i}\}$ and $\{g_{n_i}\}$ are bounded. Furthermore $f(z_{n_i}, x_{n_i}) \to 0$.

The first statement is immediate because the sequence $\{x_n\}$ is bounded and the subset C is convex and closed and thus weakly closed. Now to show that the sequence $\{y_{n_i}\}$ is bounded, it suffices to prove that there exists $M > 0$ such that $\|x_{n_i} - y_{n_i}\| \le M$ for i large enough. Without loss of generality we can suppose that $x_{n_i} \ne y_{n_i}$ for all i and we set

$$A(y) = \lambda_{n_i} f(x_{n_i}, y) + \frac{1}{2}\|y - x_{n_i}\|^2$$

for every $y \in C$. Since $f(x_{n_i}, \cdot)$ is convex, it is easy to see that $A(y)$ is a strongly convex function on C with modulus $\nu = 1$. So, for all $y_1, y_2 \in C$, $s(y_1) \in \partial A(y_1)$ and $s(y_2) \in \partial A(y_2)$, we have the inequality

$$\langle s(y_1) - s(y_2), y_1 - y_2 \rangle \ge \|y_1 - y_2\|^2.$$

Taking $y_1 = x_{n_i}$ and $y_2 = y_{n_i}$ and noting that $0 \in \partial A(y_{n_i}) + N_C(y_{n_i})$ by the definition of y_{n_i}, we obtain that there exists $s(y_{n_i}) \in \partial A(y_{n_i})$ such that $-s(y_{n_i}) \in N_C(y_{n_i})$, i.e.,

$$\langle -s(y_{n_i}), y - y_{n_i} \rangle \le 0 \quad \text{for every } y \in C. \tag{12}$$

So, for every $s(x_{n_i}) \in \partial A(x_{n_i})$, we have

$$\begin{aligned} \langle s(x_{n_i}), x_{n_i} - y_{n_i} \rangle &\ge \langle s(y_{n_i}), x_{n_i} - y_{n_i} \rangle + \|x_{n_i} - y_{n_i}\|^2 \\ &\ge \|x_{n_i} - y_{n_i}\|^2, \end{aligned}$$

where we have used (12) with $y = x_{n_i}$. Consequently, we obtain that

$$\|x_{n_i} - y_{n_i}\| \le \|s(x_{n_i})\| \quad \text{for every } s(x_{n_i}) \in \partial A(x_{n_i}). \tag{13}$$

On the other hand, since $x_{n_i} \rightharpoonup \bar{x}$, it follows from Lemma 3.2 that for any $\varepsilon > 0$, there exist $\eta > 0$ and $i_0 \in \mathbb{N}$ such that

$$\partial_2 f(x_{n_i}, x_{n_i}) \subset \partial_2 f(\bar{x}, \bar{x}) + \frac{\varepsilon}{\eta} B \tag{14}$$

for all $i \geq i_0$, where B denotes the closed unit ball of H.

Since $s(x_{n_i}) \in \partial A(x_{n_i}) = \lambda_{n_i} \partial_2 f(x_{n_i}, x_{n_i})$ for all i and $\partial_2 f(\bar{x}, \bar{x})$ is bounded, and as $0 < \lambda \leq \lambda_{n_i} \leq 1$ for all i, the inclusion (14) implies that the right-hand side of (13) is bounded. So there exists $M > 0$ such that $\|x_{n_i} - y_{n_i}\| \leq M$ for all $i \geq i_0$ and the sequence $\{y_{n_i}\}$ is bounded.

The sequence $\{z_{n_i}\}$, being a convex combination of x_{n_i} and y_{n_i}, is also bounded, and there exists a subsequence of $\{z_{n_i}\}$, again denoted by $\{z_{n_i}\}$, that converges to $\bar{z} \in C$. Then it follows from Lemma 3.2 that for any $\varepsilon > 0$, there exist $\eta > 0$ and i_0 such that

$$\partial_2 f(z_{n_i}, x_{n_i}) \subset \partial_2 f(\bar{z}, \bar{x})) + \frac{\varepsilon}{\eta} B$$

for all $i \geq i_0$. Since $g_{n_i} \in \partial_2 f(z_{n_i}, x_{n_i})$ for all i and as B and $\partial_2 f(\bar{z}, \bar{x})$ are bounded, we deduce that $\{g_{n_i}\}$ is bounded. But then $f(z_{n_i}, x_{n_i}) = \sigma_{n_i} \|g_{n_i}\| \cdot \|g_{n_i}\| \to 0$. Indeed, the previous equality comes from the definition of σ_{n_i} when $y_{n_i} \neq x_{n_i}$ and from $\sigma_{n_i} = 0$ and $f(z_{n_i}, x_{n_i}) = 0$ (because $z_{n_i} = x_{n_i}$) when $y_{n_i} = x_{n_i}$. Finally, the convergence to zero follows directly from the boundedness of $\{g_{n_i}\}$ and from $\sigma_{n_i} \|g_{n_i}\| \to 0$ (see Step 3 of this proof).

Step 5. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ converging weakly to some $\bar{x} \in C$. Then $\|x_{n_i} - y_{n_i}\| \to 0$ and $\bar{x}$ belongs to $E(f)$.

Let $x_{n_i} \rightharpoonup \bar{x}$. If $y_{n_i} = x_{n_i}$ for an infinite number of indices n_i, then it follows from Lemma 3.3 that, for such $n_i's$,

$$0 \leq \lambda_{n_i} f(x_{n_i}, y) \quad \forall y \in C. \tag{15}$$

Since $\lambda_{n_i} \in [\lambda, 1]$ with $\lambda > 0$ and as $f(\cdot, y)$ is weakly continuous on the open set $\Delta \supset C$, we obtain, after taking the limit in (15) as $i \to \infty$, that $0 \leq f(\bar{x}, y)$ for every $y \in C$, i.e., $\bar{x} \in E(f)$.

Now we suppose that $y_{n_i} \neq x_{n_i}$ for i large enough and let i be such an index. Since the function $f(z_{n_i}, \cdot)$ is convex, we can write

$$\begin{aligned} \rho_{n_i} f(z_{n_i}, y_{n_i}) + (1 - \rho_{n_i}) f(z_{n_i}, x_{n_i}) &\geq f\left(z_{n_i}, \rho_{n_i} y_{n_i} + (1 - \rho_{n_i}) x_{n_i}\right) \\ &= f(z_{n_i}, z_{n_i}) = 0. \end{aligned}$$

Hence $\rho_{n_i} [f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i})] \leq f(z_{n_i}, x_{n_i})$. Now, by the definition of line-search, we have

$$\frac{\alpha}{2\lambda_{n_i}} \|y_{n_i} - x_{n_i}\|^2 \leq f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i}).$$

Multiplying both sides of the previous inequality by ρ_{n_i}, we obtain

$$\frac{\rho_{n_i} \alpha}{2\lambda_{n_i}} \|y_{n_i} - x_{n_i}\|^2 \leq \rho_{n_i} [f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i})]$$

$$\leq f(z_{n_i}, x_{n_i}) \to 0. \tag{16}$$

Since $0 < \lambda \leq \lambda_{n_i} \leq 1$ for every i, it follows from (16) that

$$\|x_{n_i} - y_{n_i}\| \to 0 \quad \text{and} \quad y_{n_i} \rightharpoonup \bar{x}.$$

Indeed, in the case where $\limsup \rho_{n_i} > 0$, there exist $\bar{\rho} > 0$ and a subsequence of $\{\rho_{n_i}\}$ denoted again by $\{\rho_{n_i}\}$ such that $\rho_{n_i} \to \bar{\rho}$. Then, by (16), $\|x_{n_i} - y_{n_i}\| \to 0$ as $i \to \infty$. In the case where $\rho_{n_i} \to 0$, let $\{m_i\}$ be the sequence of the smallest positive integers such that, for every i,

$$f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i}) \geq \frac{\alpha}{2\lambda_{n_i}} \|x_{n_i} - y_{n_i}\|^2,$$

where $z_{n_i} = (1 - \gamma^{m_i})x_{n_i} + \gamma^{m_i} y_{n_i}$. Since $\rho_{n_i} = \gamma^{m_i} \to 0$, it follows that $m_i > 1$ for i sufficiently large, and consequently that

$$f(\bar{z}_{n_i}, x_{n_i}) - f(\bar{z}_{n_i}, y_{n_i}) < \frac{\alpha}{2\lambda_{n_i}} \|x_{n_i} - y_{n_i}\|^2 \tag{17}$$

where $\bar{z}_{n_i} = (1 - \gamma^{m_i - 1})x_{n_i} + \gamma^{m_i - 1} y_{n_i}$. On the other hand, using Lemma 3.3 with $n = n_i$ and $y = x_{n_i}$, we can write

$$\|x_{n_i} - y_{n_i}\|^2 \leq -\lambda_{n_i} f(x_{n_i}, y_{n_i}). \tag{18}$$

Combining (17) with (18), we obtain

$$f(\bar{z}_{n_i}, x_{n_i}) - f(\bar{z}_{n_i}, y_{n_i}) < -\frac{\alpha}{2} f(x_{n_i}, y_{n_i}). \tag{19}$$

Taking the limit in (19) as $i \to \infty$, using the weak continuity of f, and recalling that $x_{n_i} \rightharpoonup \bar{x}$, $y_{n_i} \rightharpoonup \bar{y}$, and $\gamma^{m_i} \to 0$, we have that $\bar{z}_{n_i} \rightharpoonup \bar{x}$ and

$$-f(\bar{x}, \bar{y}) \leq -\frac{\alpha}{2} f(\bar{x}, \bar{y}),$$

i.e., $f(\bar{x}, \bar{y}) \geq 0$ because $\alpha \in (0, 2)$.
Then it follows from (18) that $\|x_{n_i} - y_{n_i}\| \to 0$ and thus, since $x_{n_i} \rightharpoonup \bar{x}$, that $y_{n_i} \rightharpoonup \bar{x}$. Taking the limit in the inequality given by Lemma 3.3 with $n = n_i$, we obtain that $0 \leq \bar{\lambda} f(\bar{x}, y)$ for every $y \in C$, i.e., $\bar{x} \in E(f)$.

Step 6. Any weak limit point of $\{x_n\}$ belongs to $\mathrm{Fix}(S)$.

Let $x_{n_i} \rightharpoonup \bar{x}$. Since $\|w_n - x_n\| \leq \|x_n - \sigma_n g_n - x_n\| = \sigma_n \|g_n\| \to 0$, we have immediately that $\|w_n - x_n\| \to 0$ and thus that $w_{n_i} \rightharpoonup \bar{x}$. As the operator $I - S$ is demiclosed at zero and $\|Sw_{n_i} - w_{n_i}\| \to 0$, we can conclude that $S\bar{x} - \bar{x} = 0$, i.e., $\bar{x} \in \mathrm{Fix}(S)$.

Step 7. $\{x_n\}$ converges strongly to the projection of x_0 onto $\mathrm{Fix}(S) \cap E(f)$.

Since any weak limit point of $\{x_n\}$ belongs to $\mathrm{Fix}(S) \cap E(f)$ and since, from (10),

$$\|x_n - x_0\| \leq \|x_0 - x^*\| \quad \text{for every } x^* \in \mathrm{Fix}(S) \cap E(f)$$

we can use Lemma 2.2 with $K = \mathrm{Fix}(S) \cap E(f)$ and $u = x_0$ to obtain that the sequence $\{x_n\}$ converges strongly to the projection of x_0 onto $\mathrm{Fix}(S) \cap E(f)$. ■

In Algorithm 1 two projections are needed to get w_n (see Remark 4.5). First x_n is projected onto the halfspace

$$H_n = \{x \in H \mid \langle g_n, x_n - x\rangle \geq f(z_n, x_n)\}$$

to get $x_n - \sigma_n g_n$ and after that, this point is projected onto C to get w_n. In the next algorithm, denoted by Algorithm 2, only one projection of x_n is performed onto the intersection $C \cap H_n$ to get w_n. In other words, Algorithm 2 is the same as Algorithm 1 except that Step 4 is replaced by

Step 4bis. Select $g_n \in \partial_2 f(z_n, x_n)$ and compute $w_n = P_{C\cap H_n}(x_n)$,
where $H_n = \{x \in H \mid \langle g_n, x_n - x\rangle \geq f(z_n, x_n)\}$.

In order to prove the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 2, it is interesting to rewrite Step 4bis under an equivalent form. It is the aim of the next proposition whose proof is similar to the one given in [16].

Proposition 3.8. *Let $w_n = P_{C\cap H_n}(x_n)$. Then $w_n = P_{C\cap H_n}(u_n)$ where $u_n = P_{H_n}(x_n)$.*

Thanks to this proposition, the only difference between the two algorithms is in the computation of w_n. For Algorithm 1 we have $w_n = P_C(u_n)$ while for Algorithm 2 we have $w_n = P_{C\cap H_n}(u_n)$. Since $\mathrm{Fix}(S) \cap E(f) \subset C \cap H_n \subset C$ (see the proof of Theorem 3.9), we have that the point w_n computed in Algorithm 2 is closer to the solution set than the point w_n computed in Algorithm 1. As in Theorem 3.7 we suppose that the STOP never occurs in Algorithm 2 and thus that the sequence $\{x_n\}$ generated by this algorithm is infinite.

Theorem 3.9. *Let C be a nonempty closed convex subset of H. Let f be a function from $\Delta \times \Delta$ into $\mathbb{R}$ satisfying conditions* (B1)–(B4) *and let S be a mapping from C to C satisfying conditions* (C1)–(C2) *such that $Fix(S) \cap E(f) \neq \emptyset$. Let $\alpha \in (0, 2)$, $\gamma \in (0, 1)$ and suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions:*

$$\{\alpha_n\} \subset [0, c] \text{ for some } c < 1, \{\beta_n\} \subset [d, b] \text{ for some } 0 \leq \xi < d \leq b < 1,$$
$$\text{and } \{\lambda_n\} \subset [\lambda, 1] \text{ for some } 0 < \lambda \leq 1.$$

Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to the projection of x_0 onto $\mathrm{Fix}(S) \cap E(f)$.

Proof. Let $\{x_n\}$ be the infinite sequence generated by Algorithm 2 and let $x^* \in \mathrm{Fix}(S) \cap E(f)$. First we observe that x^* and z_n belong to $C \cap H_n$ for every integer $n \geq 1$. Indeed by definition of g_n, we have

$$f(z_n, y) \geq f(z_n, x_n) + \langle g_n, y - x_n \rangle \quad \forall y \in C.$$

Taking $y = x^*$ and remembering that $f(z_n, x^*) \leq 0$, we deduce that $\langle g_n, x_n - x^* \rangle \geq f(z_n, x_n)$, i.e., $x^* \in H_n$. Similarly taking $y = z_n$ and remembering that $f(z_n, z_n) = 0$, we obtain that $\langle g_n, x_n - z_n \rangle \geq f(z_n, x_n)$, i.e., $z_n \in H_n$.

The proof of this strong convergence theorem is similar to the proof of Theorem 3.7 except the inequality (7) and the first part of Step 6, namely that $w_{n_i} \rightharpoonup \bar{x}$. Let us prove these facts.

First we observe that inequality (7) is also true for Algorithm 2 because $w_n = P_{C\cap H_n}(u_n) = P_{C\cap H_n}(x_n - \sigma_n g_n)$, $P_{C\cap H_n}$ is nonexpansive, and $x^* \in C \cap H_n$. On the other hand, let $x_{n_i} \rightharpoonup \bar{x}$. Since $z_{n_i} \in C \cap H_{n_i}$, $w_{n_i} = P_{C\cap H_{n_i}}(u_{n_i})$, and $u_{n_i} = P_{H_{n_i}}(x_{n_i})$, we have

$$\|w_{n_i} - z_{n_i}\| \leq \|u_{n_i} - z_{n_i}\| \leq \|x_{n_i} - z_{n_i}\|.$$

Hence $\|w_{n_i} - z_{n_i}\| \to 0$ because $\|x_{n_i} - z_{n_i}\| = \gamma_{n_i}\|x_{n_i} - y_{n_i}\| \to 0$ (see Step 5 of the proof of Theorem 3.7). Consequently, it follows from $x_{n_i} \rightharpoonup \bar{x}$ that $z_{n_i} \rightharpoonup \bar{x}$ and thus that $w_{n_i} \rightharpoonup \bar{x}$. ∎

4. The particular case of variational inequalities

When the equilibrium function f is defined, for every $x, y \in C$, by $f(x, y) = \langle F(x), y-x \rangle$, with $F : C \to H$, the equilibrium problem reduces to the variational inequality problem: Find $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geq 0 \quad \text{for every } y \in C.$$

The set of solutions of the variational inequality problem is denoted by VI(F, C). In that particular situation, Algorithm 1 becomes

ALGORITHM 1-VI

Step 0. Choose $\alpha \in (0, 2)$, $\gamma \in (0, 1)$ and the sequences

$$\{\alpha_n\} \subset [0, 1),\ \{\beta_n\} \subset (0, 1) \text{ and } \{\lambda_n\} \subset (0, 1].$$

Step 1. Let $x_0 \in H$, $C_1 = C$, $x_1 = P_{C_1} x_0$. Set $n = 1$.

Step 2. Compute $y_n = P_C(x_n - \lambda_n F(x_n))$.

Step 3. If $y_n = x_n$, then set $z_n = x_n$. Otherwise

Step 3.1 Find m the smallest nonnegative integer such that

$$\begin{cases} z_{n,m} = (1 - \gamma^m)x_n + \gamma^m y_n, \\ \langle F(z_{n,m}), x_n - y_n \rangle \geq \frac{\alpha}{2\lambda_n} \|x_n - y_n\|^2. \end{cases}$$

Step 3.2 Set $\gamma_n = \gamma^m$, $z_n = z_{n,m}$ and go to Step 4.

Step 4. Compute $w_n = P_C(x_n - \sigma_n F(z_n))$

where $\sigma_n = \frac{\langle F(z_n), x_n - z_n \rangle}{\|F(z_n)\|^2}$ if $y_n \neq x_n$ and $\sigma_n = 0$ otherwise.

Step 5. Compute $t_n = \alpha_n x_n + (1 - \alpha_n) [\beta_n w_n + (1 - \beta_n) S w_n]$.

If $y_n = x_n$ and $t_n = x_n$, then STOP: $x_n \in \mathrm{Fix}(S) \cap \mathrm{VI(F,C)}$.

Otherwise go to Step 6.

Step 6. Compute $x_{n+1} = P_{C_{n+1}} x_0$

where $C_{n+1} = \{z \in C_n \mid \|t_n - z\| \leq \|x_n - z\|\}$.

Step 7. Set $n := n + 1$ and go to Step 2.

In order to apply Theorem 3.7 to the variational inequality problem, let us observe that in this particular case, conditions (B1) and (B4) are always satisfied and that condition (B2) becomes: F is pseudomonotone on VI(F, C), i.e.,

$$\langle F(y), x^* - y \rangle \leq 0 \quad \text{for every } x^* \in \mathrm{VI(F,C)}.$$

Furthermore, if $F : \Delta \to H$ is such that, for any sequence $\{x_n\} \subset \Delta$,

$$x_n \rightharpoonup x \ \Rightarrow \ F(x_n) \to F(x), \tag{20}$$

then the corresponding function f is weakly continuous on $\Delta \times \Delta$.

Theorem 4.1. *Let C be a nonempty closed convex subset of H. Let $F : \Delta \to H$ be a mapping pseudomonotone on* VI(F, C) *and satisfying* (20)*. Let S be a mapping from C to C satisfying conditions* (C1)–(C2) *such that* $\mathrm{Fix}(S) \cap \mathrm{VI(F,C)} \neq \emptyset$*. Let $\alpha \in (0, 2)$, $\gamma \in (0, 1)$ and suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the conditions:*

$$\{\alpha_n\} \subset [0, c] \text{ for some } c < 1, \{\beta_n\} \subset [d, b] \text{ for some } 0 \leq \xi < d \leq b < 1,$$
$$\text{and } \{\lambda_n\} \subset [\lambda, 1] \text{ for some } 0 < \lambda \leq 1.$$

Then the sequence $\{x_n\}$ generated by Algorithm 1-VI converges strongly to the projection of x_0 onto $\mathrm{Fix}(S) \cap \mathrm{VI(F,C)}$.

Finally, to obtain the algorithm and the convergence theorem corresponding to Algorithm 2 but for the variational inequality problem, it suffices to replace in Step 4 the calculation of w_n by $w_n = P_{C \cap H_n}(x_n)$ where $H_n = \{x \in H \mid \langle F(z_n), z_n - x \rangle \geq 0\}$.

5. Conclusion

Two new iterative methods have been introduced for finding a common solution to an equilibrium problem and a fixed point problem in a Hilbert space. The proximal point iteration used in most papers has been replaced by an extragradient step with a linesearch procedure. The strong convergence of the iterates has been obtained thanks to a shrinking projection method. Another approach known as the viscosity method has been studied in the literature to obtain strong convergence theorems. It is mainly based on the proximal point method. Using this viscosity approach but with the extragradient procedure instead of the proximal point method could be an interesting variant to the method developed in this paper. This will be the subject of a forthcoming research paper.

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