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Topological Minimax Theorems: Old and New

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Dedicated to Professor Phan Quoc Khanh on the occasion of his 65th birthday

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Abstract. We review and extend the main topological minimax theorems based on connectedness that have been developed over the years since the pioneering paper of Wu (1959). It is shown in particular that the topological minimax theorems of Geraghty and Lin (1984) are essentially a rediscovery of much earlier results of Tuy (1974), while the latter can be derived from a minimax theorem recently developed for functions involving a real variable. Several new topological minimax theorems are presented, including a general theorem containing both König's (1992) and Tuy's (1974) results and a minimax theorem for increasing-decreasing functions analogous to Sion's classical theorem.

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1. Introduction

Let C, D be subsets of two Hausdorff topological spaces X, Y, respectively, $F(x, y)$ a real function defined on $X \times Y$. A problem that dates back to the work of Von Neumann [6] and has important applications in various fields is to study conditions under which the following minimax equality for $F(x, y)$ on $C \times D$ holds:

$$\inf_{x \in C} \sup_{y \in D} F(x, y) = \sup_{y \in D} \inf_{x \in C} F(x, y). \qquad (1)$$

Throughout the years since the paper [6] numerous minimax theorems have been published for the case when X, Y are vector topological spaces. Among these the most important and best known is Sion's Theorem [16].

Subsequently, considering the problem in an abstract topological setting, Wu Wen tsun [26] developed purely topological conditions sufficient to guarantee the minimax equality (1). These conditions, based on the concept of strong connectedness of the function $F(x, y)$, were meant to generalize algebraic conditions related to convexity used in traditional minimax theory. In fact, Wu's topological minimax theorem contained Von Neumann's theorem as a special case. However, due to several restrictive assumptions (separability of the space Y, continuity of the function $F(x, y)$) it did not include some important results at the time, such as Nikaido's Theorem [7] and Sion's Theorem. This limitation has motivated a long refining process on topological minimax theorems, beginning with the work of Tuy [17] and culminating in 1992 with König's papers [4], [5].

In [17] the strong connectedness concept of Wu was replaced by a more general concept, called α-connectedness (see Section 2 below) which involves a special kind of arcwise connectedness for certain sets. Using this concept two theorems were established [17] that extended Wu's theorem as well as Sion's theorem and several other previously known minimax results. One should also mention a paper by M. A. Geraghty and B.-L. Lin [3], where a topological minimax theorem was presented in very abstract terms. However, close scrutiny has revealed that, except for the notation, these results are practically identical to those in [17].

The most significant contribution subsequent to Tuy's paper was made in 1992 by König [4], [5] who was able to prove two general minimax theorems with α-connectedness replaced by the general topological concept of connectedness. König's results are certainly very attractive. However, a question raised and left open in these results concerns the continuity assumptions on $F(x, y)$ which are stronger than those of Tuy in [17]. Specifically, in König's main theorem, the function $F(x, y)$ has to be lower semi-continuous jointly in (x, y), whereas only separate lower semi-continuity in each variable is required in Tuy's theorem.

The aim of the present paper is to clarify the somewhat ambiguous relationship between some among the best known results on topological minimax theorems based on connectedness and on the basis of this analysis to further extend both results in [17] and [4].

In Sections 2 and 3 we briefly review the earlier results of Tuy [17] and compare them with those of Geraghty and Lin [3]. In Sections 4 and 5 König's results [4] are discussed together with some further advances in topological minimax theory. In particular, it is shown how a recently developed minimax theorem for the special case $Y = \mathbb{R}$ can be used to recover earlier theorems based on α-connectedness and obtain a new general topological minimax theorem that simultaneously extends the earlier results in [17] and those of König in [4]. Also, in Section 6 a new minimax theorem for monotonic functions is derived that parallels the strongest version of Sion's theorem.

2. Minimax theorems based on α-connectedness

For $\alpha \in \mathbb{R}$ and $y \in D$ define $C_\alpha(y) := \{x \in C | \ F(x,y) \le \alpha\}$. Following [17] the function $F(x,y)$ is said to be *α-connected* on $C \times D$ if

(H) for any finite set $H \subset D$ the set $C_H := \cap_{y \in H} C_\alpha(y)$ is either empty or connected;

(T) for any pair $a, b \in D$ there exists a continuous mapping $u : [0,1] \to D$ such that

$$u(0) = a, u(1) = b, \ C_\alpha(u(t)) \subset C_\alpha(u(\theta)) \cup C_\alpha(u(\theta')) \ \forall t \in [\theta, \theta'] \subset [0,1]. \tag{2}$$

Setting

$$\gamma := \inf_{x \in C} \sup_{y \in D} F(x,y); \quad \eta := \sup_{y \in D} \inf_{x \in C} F(x,y),$$

the following proposition was established by Tuy in 1974.

Theorem 2.1. (Tuy [17]) *Assume the set C is compact while the function $F(x,y)$ is lsc (lower semi-continuous) in x. The minimax equality (1) holds if, in addition, either of the following conditions is satisfied:*

(i) *$F(x,y)$ is usc in y and there exists a monotone decreasing sequence $\alpha_k \searrow \eta$ such that $F(x,y)$ is α_k-connected for every k;*

(ii) *$F(x,y)$ is lsc in y and there exists a noncreasing sequence $\alpha_k \to \eta$ such that $F(x,y)$ is α_k-connected for every k.*

In contrast to the proofs of earlier published minimax theorems ([6], [7], [26], [16], ...), the proof of Theorem 2.1 does not make use of any classical tool like: separation theorem for convex sets, fixed point theorem, Sperner Lemma, Helly theorem on intersection of convex sets, etc. It is simply based on the following Lemmas 2.2, 2.3 which in turn are established by purely set-theoretical arguments.

Lemma 2.2. *Let C be compact and $F(x,y)$ be lsc in x and usc in y. If $F(x,y)$ is α-connected for some $\alpha > \eta$, then for every pair $a, b \in C$*

$$C_\alpha(a) \cap C_\alpha(b) \neq \emptyset. \tag{3}$$

Lemma 2.3. *Let C be compact and $F(x,y)$ be lsc in each of the variables x, y. If $F(x,y)$ is α-connected for some $\alpha \ge \eta$ then (3) holds for every pair $a, b \in D$.*

Corollary 2.4. *Let C be compact and $F(x,y)$ be lsc in x and usc in y. If $F(x,y)$ is α-connected for every $\alpha > \eta$ then (1) holds.*

Corollary 2.5. *Let C be compact and $F(x,y)$ be lsc in each of the variables x, y. If $F(x,y)$ is α-connected for every $\alpha \ge \eta$ then (1) holds.*

An immediate consequence of Theorem 2.1 is the following sharpened version of Sion's minimax theorem:

Let $F(x,y)$ be quasiconvex in x and quasiconcave in y. If C is compact while $F(x,y)$ is lsc in x and either usc or lsc in y, then (1) holds. If D is compact while $F(x,y)$ is usc in y and either lsc or usc in x, then (1) holds.

3. Geraghty-Lin's minimax theorem

A decade after the appearance of Tuy's paper [17], Geraghty and Lin [3] published in 1984 a minimax theorem that for many years later was referred to by some authors as a new original result different from Tuy's one (see e.g. [15], [4]). Due to the overly abstract form in which this theorem was presented, it has been difficult to relate it to earlier published results. Nevertheless, upon the substitution

$$\mathcal{F} \leftarrow D,\ f \leftarrow y, f(x) \leftarrow -F(x,y),\ X \leftarrow C,\ S \leftarrow u,\ A_f^\alpha \leftarrow C_\alpha(y)$$

the main results in [3] can be described as follows.

Let $C, D, F(x,y)$ be as specified in the Introduction. The set D is said to be *submaximum* if for any $a, b \in D$ there exists a continuous mapping $u : [0,1] \to D$ such that $u(0) = a, u(1) = b$ and

$$F(x,u(t)) \geq \min\{F(x,u(\theta)), F(x,u(\theta'))\} \quad \forall x \in C, \tag{4}$$

for all $t \in [\theta, \theta'] \subset [0,1]$.

Theorem 3.1. ([3], Theorem 2) *Let C be compact and $F(x,y)$ be lsc in x and continuous in y. If D is a submaximum set such that for any $a^1, \dots, a^n$ in D and any $\alpha \in \mathbb{R}$, the set $\{x \in C|\ F(x,a^i) \leq \alpha, i = 1,\dots,n\}$ is either empty or connected, then the minimax equality holds for $F(x,y)$ on $C \times D$.*

It is not hard to see that this theorem is contained in Theorem 2.1. Indeed, condition (4) amounts to requiring that for every $t \in [\theta, \theta'] \subset [0,1]$:

$$F(x,u(t)) \leq \alpha \Rightarrow \min\{F(x,u(\theta')), F(x,u(\theta))\} \leq \alpha$$

and this in turn amounts to requiring (2) for every $\alpha \in \mathbb{R}$. So the assumption in Theorem 3.1 that

> *D is a submaximum set such that for any $a^1, \dots, a^n$ in D and any $\alpha \in \mathbb{R}$, the set $\{x \in C|\ F(x,a^i) \leq \alpha, i = 1,\dots,n\}$ is either empty or connected*

is equivalent to requiring that $F(x,y)$ be α-connected for every $\alpha \in \mathbb{R}$. Therefore, Theorem 3.1 is contained in Theorem 2.1; it is actually weaker since it assumes continuity of the function $y \mapsto F(x,y)$ for fixed $x \in C$, whereas Theorem 2.1 only assumes either upper semi-continuity or lower semi-continuity of this function.

Furthermore, the proof given in [3] for Theorem 3.1 does not differ much from that of Tuy for Theorem 2.1 in [17]. In fact, the proof in [3] is based on the following proposition.

Lemma 3.2. [3, Lemma 1] *Let C be compact, $F(x,y)$ be lsc in x for fixed y, while D is a submaximum set and for some $\alpha \in \mathbb{R}$ the set $C_\alpha(y) := \{x \in C |\ F(x,y) \le \alpha\}$ is either empty or connected for every $y \in D$. Then for any $a, b \in D$ such that* $\max\{F(x,a), F(x,b)\} > \alpha \quad \forall x \in C$, *there exists $y \in D$ such that $F(x,y) > \alpha\ \forall x \in C$.*

Clearly the conclusion of this proposition amounts to saying that for any $a, b \in D$ if $C_\alpha(a) \cap C_\alpha(b) = \emptyset$ then $\alpha < \eta := \sup_{y\in D} \inf_{x\in C} F(x,y)$; or alternatively, if $\alpha \ge \eta$ then $C_\alpha(a) \cap C_\alpha(b) \ne \emptyset$ for every pair $a, b \in D$. Therefore, this proposition can be equivalently formulated as: if C is compact, $F(x,y)$ is lsc in x while D is a submaximum set and for some $\alpha \ge \eta$ the set $C_\alpha(y)$ is either empty or connected for every $y \in D$, then $C_\alpha(a) \cap C_\alpha(b) \ne \emptyset$ for every pair $a, b \in D$. This is exactly Lemma 2.3. Furthermore, the proof of Lemma 3.2 in [3] proceeds much along the same line as that of Lemma 2.3 in [17], but it is even more involved, while using a stronger assumption (continuity of $F(x,y)$ in y instead of lower semi-continuity as in Lemma 2.3).

Remark 3.3. The results in [3] are thus all contained in those of Tuy [17] published many years earlier . This fact has been overlooked by many researchers (see e.g. [15], [4]), although prior to [3], the results in [17] had been described with due references e.g. in [4], [8], [25].

4. König's theorems

Letting $\theta = 0, \theta' = 1$ condition (2) implies that for every $x \in C$:

$$[F(x,a) > \alpha, F(x,b) > \alpha] \ \Rightarrow\ F(x,u(t)) > \alpha\ \forall t \in [0,1].$$

In other words, for every $x \in C$ the set $D_\alpha(x) := \{y \in D |\ F(x,y) > \alpha\}$ is arcwise connected (indeed, any two points $a, b \in D_\alpha(x)$ are connected by the continuous curve $u(t), t \in [0,1]$ entirely lying in it). Then, given any nonempty set $K \subset C$, if $a, b \in D,\ F(x,a) > \alpha, F(x,b) > \alpha\ \forall x \in K$ the mapping $u : [0,1] \to D$ satisfies $F(x,u(t)) > \alpha\ \forall x \in K\ \forall t \in [0,1]$, so that the set $D^K := \cap_{x\in K}\{y \in D |\ F(x,y) > \alpha\}$ is arcwise connected. The question then arises as to whether condition (T) in the concept of α-connectedness used in Theorem 2.1 could be weakened by requiring only connectedness of the set D^K for every nonempty set $K \subset C$.

After long efforts a positive answer was obtained in 1992 by König [4] (see also [5]), at the expense, however, of somewhat stronger continuity assumptions on $F(x,y)$.

Specifically, the following theorem was proved by König:

Theorem 4.1. (König [4], [5]) *Assume C is compact, D is connected, while $F(x,y)$ is either lsc in x, usc in y or it is lsc jointly in (x,y). Let $\eta := \sup_{y\in D}\inf_{x\in C} F(x,y)$, and $\Lambda \subset (\eta,+\infty)$ be a nonempty set such that* $\inf \Lambda = \eta$. *Then the minimax equality* (1) *holds, provided for every $\alpha \in \Lambda$ the following assumptions are satisfied:*

(H) *For every nonempty finite set $H \subset D$ the set $C_H := \cap_{y\in H}\{x \in C|\ F(x,y) \le \alpha\}$ is connected;*

(K) *For every nonempty set $K \subset C$ the set $D^K := \cap_{x\in K}\{y \in D|\ F(x,y) > \alpha\}$ is connected.*

Since (T) implies (K) as we just saw, König's Theorem 1.2 in [4] is indeed an extension of Tuy's Theorem 1 in [17] (i.e., Theorem 2.1 with assumption (i)). However, as was mentioned in [4], Theorem 1.3 in [4] *is not really an extension* of Tuy's Theorem 1' in [17] (i.e., Theorem 2.1 with assumption (ii)), because it assumes joint lower semi-continuity of $F(x,y)$ in x,y and not simply separate lower semi-continuity in each variable x,y as was done in Tuy's Theorem 1'. Furthermore, as was shown in [18], [22], in contrast to König's Theorem 4.1, Tuy's Theorem 2.1 still remains true when D is convex while lower semi-continuity of the function $y \mapsto F(x,y)$ is required only on every line segment of D.

Note that $D^K = D$ if $K = \emptyset$, so the connectedness of D would follow from condition (K) with the word 'nonempty' removed. Also, as mentioned in [5], the connectedness of both C,D would be implied if conditions (H) and (K) were assumed for every $\alpha \in \mathbb{R}$.

5. Further advances

In recent years, concentrating on the special case $Y = \mathbb{R}$ a number of topological minimax theorems have been developed ([2], [9]-[13], [14], [24] and the references therein) that cannot be derived from earlier known minimax theorems where Y is a more general topological space. Despite the restrictiveness of the assumption $Y = \mathbb{R}$ these new minimax theorems have proved to be more useful in several questions of functional analysis and optimization theory where traditional conditions related to convexity and/or continuity are not met. Furthermore, as we are going to show, this new line of research on topological minimax theorems with $Y = \mathbb{R}$ may lead to new ideas and suggest new ways for recovering, strengthening and extending many classical results.

Theorem 5.1. *Let C be a nonempty compact subset of a Hausdorff topological space X, D be an interval of $\mathbb{R}$ and $F(x,y) : C \times D \to \mathbb{R}$ a function on $C \times D$, lsc in x for every fixed $y \in D$. Assume there exists a real number $\alpha^* > \eta := \sup_{y\in D}\inf_{x\in C} F(x,y)$ such that for every $\alpha \in (\eta,\alpha^*)$:*

T1. *For every $x \in C$ the set $\{y \in D|\ F(x,y) > \alpha\}$ is connected;*

T2. *At least one of the following conditions* T2a, T2b, T2c *holds:*

a) *for every* $y \in D$ *the set* $C_\alpha(y) := \{x \in C|\ F(x,y) \leq \alpha\}$ *is connected while for every* $x \in C$ *the function* $F(x,\cdot)$ *is either usc or lsc;*

b) *for each segment* $\Delta \subset D$ *there exists a closed set-valued map* φ *from* Δ *to* C *with nonempty closed and connected values* $\varphi(y)$ *satisfying*

$$\varphi(y) \subset C_\alpha(y) \qquad \forall y \in \Delta;$$

c) *for every* $x \in C$ *the function* $F(x,\cdot)$ *is continuous and for every* $y \in D$ *any local minimizer of* $F(\cdot,y)$ *on* C *is a global one.*

Then we have the minimax equality

$$\min_{x\in C}\sup_{y\in D} F(x,y) = \sup_{y\in D}\inf_{x\in C} F(x,y). \tag{5}$$

Proof. This theorem could be deduced, though not trivially, from recent results in [24] (Theorem 1, Corollary 2 and remark following Corollary 2 in [24]). We give here a direct proof.

Consider an arbitrary $\alpha \in (\eta,\alpha^*)$ and an arbitrary segment $\Delta = [a,b] \subset D$. By assumption the function $F(x,y)$ is lsc in x for every fixed y, so the sets $C_\alpha(y) := \{x \in C|\ F(x,y) \leq \alpha\}, y \in D$, are closed. To simplify the notation we drop the subscript α and write $C(y)$ for $C_\alpha(y)$. The first step is to show that

$$C(a) \cap C(b) \neq \emptyset. \tag{6}$$

By (T1), for every $x \in C$ if $F(x,a) > \alpha, F(x,b) > \alpha$ then $F(x,y) > \alpha\ \forall y \in \Delta := [a,b]$. This implies

$$C(y) \subset C(a) \cup C(b) \quad \forall y \in \Delta. \tag{7}$$

I. Case (T2a) is satisfied.

Arguing by contradiction suppose that

$$C(a) \cap C(b) = \emptyset.$$

Since by (T2a) $C(y)$ is connected, for every $y \in \Delta$, $C(y) \subset C(a)$ or $C(y) \subset C(b)$ but not both. Setting

$$M_a := \{y \in \Delta|\ C(y) \subset C(a)\}, \quad M_b := \{y \in \Delta|\ C(y) \subset C(b)\},$$

we then have

$$\Delta = M_a \cup M_b, \quad M_a \cap M_b = \emptyset.$$

Consider the case when $F(x,y)$ is usc in y and let $\bar{y} \in M_a$, so that $C(\bar{y}) \subset C(a)$. Since $\inf_{x\in C} F(x,\bar{y}) \leq \eta < \alpha$, there exists $\bar{x} \in C$ satisfying $F(\bar{x},\bar{y}) < \alpha$. Since $F(\bar{x},\cdot)$ is usc there exists a neighborhood V of $\bar{y}$ in Δ such that $F(\bar{x},y) < \alpha\ \forall y \in V$, i.e., $\bar{x} \in C(\bar{y}) \cap C(y)\ \forall y \in V$. So $C(a) \cap C(y) \neq \emptyset$, i.e., $C(y) \subset$

$C(a)\ \forall y \in V$, hence $V \subset M_a$. Thus, the set M_a is open. Similarly, M_b is open, and so the segment Δ is the union of two disjoint open sets, which cannot be. Hence (6) must hold.

Turning to the case when $F(x,y)$ is lsc in y, let $\bar{y} \in M_a$, so that $F(x,\bar{y}) > \alpha\ \forall x \in C(b)$. Then for every $x \in C(b)$ since $F(x,\bar{y}) > \alpha$, there exists a neighborhood $V(x) = [y_1(x), y_2(x)]$ of $\bar{y}$ satisfying $F(x,y) > \alpha\ \forall y \in V(x)$. Next, for each $i = 1,2$ since $F(., y_i)$ is lsc, there exists a neighborhood $W_i(x)$ of x such that $F(x', y_i(x)) > \alpha\ \forall x' \in W_i(x)$. Letting $W(x) = W_1(x) \cap W_2(x)$ we thus have $F(x', y_i(x)) > \alpha,\ i = 1,2,\ \ \forall x' \in W(x)$, hence by (T1) $F(x', y) > \alpha\ \forall x' \in W(x),\ \forall y \in V(x)$. Using the compactness of $C(b)$ (as a closed subset of the compact set C) one can find a finite set $Q \subset C(b)$ such that $C(b) \subset \cup_{x \in C(b)} W(x)$. Let $V = \cap_{x\in Q} V(x)$. If $x \in C(b)$ then $x \in W(x')$ for some $x' \in Q$, hence $F(x,y) > \alpha\ \forall y \in V$, so $C(b) \cap C(y) = \emptyset\ \forall y \in V$, hence $C(y) \subset C(a)\ \forall y \in V$, proving that M_a is open. Similarly, M_b is open, and just as in the previous case, this conflicts with the connectedness of the segment Δ, and hence proves (6).

II. Case (T2b) is satisfied.

Since $\varphi(y) \subset C(y)\ \forall y \in \Delta$, we have from (7):

$$\varphi(y) \subset C(a) \cup C(b).$$

If (6) does not hold, i.e., if $C(a) \cap C(b) = \emptyset$ then, setting

$$M_a = \{y \in \Delta |\ \varphi(y) \subset C(a)\}, \quad M_b = \{y \in \Delta |\ \varphi(y) \subset C(b)\}$$

we have $\Delta = M_a \cup M_b,\ M_a \cap M_b = \emptyset$. Now, since the set-valued map φ is closed, if $y^k \to \bar{y}, \varphi(y^k) \subset C(a)$ then it is easily seen that $\varphi(\bar{y}) \subset C(a)$. Indeed, let $x^k \in \varphi(y^k) \subset C(y^k) \subset C$. Since C is compact, by passing to subsequences if necessary we can suppose that $x^k \to \bar{x} \in C, y^k \to \bar{y}$, hence $\bar{x} \in \varphi(\bar{y}) \subset C(a)$. Therefore the set M_a is closed. Similarly, M_b is closed. So the segment Δ is the union of two disjoint closed sets, which is impossible. Hence (6) must be true.

III. Case (T2c) is satisfied.

For every set $G \subset \Delta$ define $C(G) = \cap_{y\in G} C(y)$. Using the lower semi-continuity of $F(x,\cdot)$ it is easily seen that the set $\{y \in \Delta |\ C_\alpha([a,y]) \neq \emptyset\}$ is closed. Indeed, let $y_k \nearrow \bar{y},\ C_\alpha([a,y_k]) \neq \emptyset$. Since $C_\alpha([a,y_h]) \subset C_\alpha([a,y_k])\ \forall h > k$, while C is compact, $C_\alpha([a,y_k]), k = 1,2,\dots,$ form a nested sequence of nonempty closed subsets of a compact set, so there exists $\bar{x} \in \cap_{k=1}^{+\infty} C_\alpha([a,y_k]) \neq \emptyset$. For every $z \in [a,\bar{y})$ we have $z \in [a,y_k]$ for some k, hence $F(\bar{x},z) \leq \alpha$. Letting $z \to \bar{y}$ yields $F(\bar{x},\bar{y}) \leq 0$ by lower semi-continuity of $F(\bar{x},\cdot)$, hence $\bar{x} \in C_\alpha([a,\bar{y}])$, proving the closedness of the set $\{y \in \Delta |\ C_\alpha([a,y]) \neq \emptyset\}$. It then follows that if $s := \sup\{y \in \Delta |\ C_\alpha([a,y]) \neq \emptyset\}$ then $C_\alpha([a,s]) \neq \emptyset$ and proving (6) now reduces to showing that $s = b$. Suppose by contradiction that $s < b$, and let $t = \inf\{y \in \Delta |\ C([y,b]) \neq \emptyset$. Then just as $C([a,s])$ is closed and nonempty, so is $C([t,b])$ and necessarily $s < t$. So we can take an $x^0 \in C([a,s] \subset W := C \setminus C([t,b])$. We claim that $F(x,s) \geq \alpha$ for every $x \in W$. Indeed, if $F(x,s) < \alpha$ then, by upper semi-continuity of $F(x,\cdot)$ there exists $q \in (s,t)$ such that $F(x,y) < \alpha\ \forall y \in [s,q]$. Since $x \notin C([t,b])$ there exists $y_1 \in [t,b]$ satisfying $F(x,y_1) > \alpha$. Hence, $y_1 > q$ and due to A1 we have $F(x,y) \leq \alpha\ \forall y \in [a,q]$, contradicting the definition of s. Since $F(x^0,s) \leq \alpha$ and $x^0 \in W$ it follows that $F(x^0,s) = \alpha$, so x^0 is a minimizer of $F(\cdot,s)$ on W, i.e., a local minimizer on C because W is the complement of a closed set. By hypothesis, x^0 must be a global minimizer on C, so $F(x,s) \geq \alpha\ \forall x \in C$, conflicting with $\alpha > \inf_{x\in C} F(x,s)$. Therefore, we must have $C_\alpha([a,s]) \cap C_\alpha([s,b]) \neq \emptyset$, hence, in particular, (6).

We thus have established (6) in all cases. At the same time, we just saw that in case III ((A2c) is satisfied), $C_\alpha([a,s]) \cap C_\alpha([s,b]) \neq \emptyset$, i.e.,

$$\cap_{y\in\Delta} C_\alpha(y) \neq \emptyset. \tag{8}$$

We now show that (8) holds in the other cases, too. Indeed, if $x \notin C_\alpha([a,y]) \cup C_\alpha([y,b])$, i.e., $F(x,y_1) > \alpha, F(x,y_2) > \alpha$ for some $y_1 \in [a,y], y_2 \in [y,b]$, then by A1 one must have $F(x,y) > \alpha$, i.e., $x \notin C_\alpha(y)$. Therefore, $C_\alpha(y) \subset C_\alpha([a,y]) \cup C_\alpha([y,b])$ and since the converse inclusion is obvious it follows that

$$C_\alpha(y) = C_\alpha([a,y]) \cup C_\alpha([y,b]).$$

Note that all sets in this equality are closed. In case (T2a) $C_\alpha(y)$ is connected, while in case (T2b) the nonempty set $\varphi(y)$ is connected and satisfies $\varphi(y) \subset C_\alpha(y) = C_\alpha([a,y]) \cup C_\alpha([y,b])$. Therefore, in every case one must have $C_\alpha([a,y]) \cap C_\alpha([y,b]) \neq \emptyset$, proving (8). Since Δ is an arbitrary segment this means that the family of sets $C_\alpha(y), y \in D$, which are closed subsets of the compact set C, has the finite intersection property. Hence there exists $\bar{x} \in \cap_{y\in D} \inf_{x\in C} F(x,y)$, i.e., such that $F(\bar{x},y) \leq \alpha\ \forall y \in D$. The latter implies $\min_{x\in C} \sup_{y\in D} \leq \alpha$, and noting that α can be arbitrarily close to η yields $\min_{x\in C} \sup_{y\in D} \leq \eta$, hence (5). The proof of Theorem 5.1 is complete. ■

As an immediate consequence we can state:

Corollary 5.2. *Assume* $Y = \mathbb{R}$, *the set* C *is compact while* $F(x,y)$ *is lsc in* x *and either usc in* y *or lsc in* y. *If there exists* $\alpha^* > \eta := \sup_{y\in D} \inf_{x\in C} F(x,y)$ *such that conditions* (H) *and* (K) *in Theorem* 4.1 *are satisfied for every* $\alpha \in (\eta, \alpha^*)$, *then the minimax equality* (5) *holds. Furthermore, when* $F(x,y)$ *is usc in* y, *instead of assuming* C *compact it suffices to assume the existence of a finite set* $M \subset D$ *such that* $\cap_{y\in M} C_{\alpha^*}(y)$ *is compact.*

Proof. If $Y = \mathbb{R}$ then (K) is equivalent to (T1); if, in addition, $F(x,y)$ is lsc in x and either usc or lsc in y then (H) implies (T2a). When $F(x,y)$ is usc in y (11) holds independently of whether C is compact or not, so if $\cap_{y\in M} C_{\alpha^*}(y)$ is compact, we also have $\cap_{y\in D} C_\alpha(y) \neq \emptyset$, hence (5). ■

Thus, in the special case $Y = \mathbb{R}$ König's Theorem 4.1 still holds if $F(x,y)$ is lsc separately in each variable as assumed in Tuy's Theorem 2.1.

Turning to the case when Y is a general topological space, it is convenient to prove first an auxiliary proposition which can often be useful for the extension of minimax theorems from the case $Y = \mathbb{R}$ to general topological space. As previously let

$$\gamma := \inf_{x\in C} \sup_{y\in D} F(x,y), \qquad \eta := \sup_{y\in D} \inf_{x\in C} F(x,y).$$

Recall that $C_\alpha(y) := \{x \in C|\ F(x,y) \le \alpha\} \ne \emptyset$ for every $y \in D$ and $\alpha > \eta$.

Lemma 5.3. *Assume the function $F(x,y)$ is lsc in x. The minimax equality* (5) *holds if for every $\alpha > \eta$ the following conditions are satisfied:*

- (i) *For every nonempty finite set $E \subset D$ such that $C' := \{x \in C|\ F(x,y) \le \alpha\ \forall y \in E\} \ne \emptyset$ we have $C'_\alpha(a) \ne \emptyset\ \forall a \in D$;*
- (ii) *There exists a finite set $M \subset C$ such that $C^M := \cap_{y\in M} C_\alpha(y)$ is compact.*

Proof. From assumption (i) it easily follows that

$$\cap_{y\in E} C_\alpha(y) \ne \emptyset \tag{9}$$

for every nonempty finite set $E \subset D$. Indeed, for every pair $a, b \in D$ we have $C_\alpha(a) \cap C_\alpha(b) \ne \emptyset$ by assumption (i) where $C' = C_\alpha(b)$. So (9) is true for $1 \le |E| \le 2$. Supposing (9) true for every set $E \subset D$ such that $2 \le |E| \le k$ consider the case $|E| = k+1$, say $E = \{a^0, a^1, \ldots, a^k\}$. For the nonempty set $C' = \cap_{i=1}^k C_\alpha(a^i)$ we have $C'_\alpha(a^0) \ne \emptyset$, hence $\cap_{i=0}^k C_\alpha(a^i) \ne \emptyset$. So by induction, (9) is true for every finite set $E \subset D$. That is, the family $C_\alpha(y), y \in D$, has the finite intersection property. Since these are closed sets satisfying $\cap_{y\in M} C_\alpha(y) \subset C_M$ with C_M compact by assumption (ii), it follows that $\cap_{y\in D} C_\alpha(y) \ne \emptyset$, i.e., there exists $\bar{x} \in C$ satisfying $F(\bar{x}, y) \le \alpha\ \forall y \in D$. This implies $\min_{x\in C} \sup_{y\in D} F(x,y) \le \alpha$, and since α can be arbitrarily close to η we conclude $\min_{x\in C} \sup_{y\in D} F(x,y) \le \eta$, hence the minimax equality (5). ∎

With the help of Lemma 5.3, from Corollary 5.2 the following theorem can be derived which is a unified and strengthened restatement of Tuy's earlier results in [17].

Theorem 5.4. *Let X, Y be two Hausdorff topological spaces, C a nonempty compact subset of X, D a nonempty closed subset of Y and $F(x,y) : C \times D \to \mathbb{R}$ a real-valued function lsc in x. Assume there exists a real number $\alpha^* > \eta := \sup_{y\in D} \inf_{x\in C} F(x,y)$ such that for every $\alpha \in (\eta, \alpha^*)$ the following conditions are satisfied:*

- (H) *For every nonempty finite set $H \subset D$ the set $\cap_{y\in H}\{x \in C|\ F(x,y) \le \alpha\}$ is either empty or connected;*
- ($T^\flat$) *For every pair $a, b \in D$ there exists a mapping $u_{ab}(t) : [0,1] \to D$ such that the function $F(x, u_{ab}(t))$ is either lsc or usc in t and $u_{ab}(0) = a, u_{ab}(1) = b$ while*

$$C_\alpha(u_{ab}(t)) \subset C_\alpha(u_{ab}(\theta)) \cup C_\alpha(u_{ab}(\theta')) \qquad \forall t \in [\theta,\theta'] \subset [0,1]. \tag{10}$$

Then the minimax equality (5) *holds.*

Proof. First we show that for every $\alpha > \eta := \sup_{y\in D}\inf_{x\in C} F(x,y)$ and every pair $a,b \in D$ there holds

$$C_\alpha(a) \cap C_\alpha(b) \neq \emptyset. \tag{11}$$

Let $u(t) = u_{ab}(t) : [0,1] \to D$ be the mapping that exists according to ($T^\flat$). Clearly (10) implies that

$$F(x,u(t)) \geq \min\{F(x,u(\theta)), F(x,u(\theta')\} \quad \forall x \in C,\ \forall t \in [\theta,\theta'] \subset [0,1],$$

so for every $x \in C$ the set $\{t \in [0,1]|\ F(x,u(t)) > \alpha\}$ is a segment, i.e., is connected. It is then easily verified that the function $f(x,t) = F(x,u(t)) : [0,1] \to R$ is lsc in x and either usc or lsc in t; furthermore it satisfies conditions (H) and (K) of Theorem 4.1 for $Y = \mathbb{R}$(i.e., condition (T2a) and (T1) of Theorem 5.1):

(H): for every $t \in \Delta := [0,1]$ the set $\{x \in C|\ f(x,t) \leq \alpha\}$ is connected;

(K): for every $x \in C$ the set $\{t \in \Delta|\ f(x,t) > \alpha\}$ is connected.

Since $\alpha > \sup_{t\in\Delta}\inf_{x\in C} F(x,y)$, by Corollary 5.2 applied to the function $f(x,t) : C \times \Delta \to \mathbb{R}$, we have $\emptyset \neq \cap_{t\in\Delta} C_\alpha(t) \subset C_\alpha(0) \cap C_\alpha(1)$, hence (11).

Now let $C' = \cap_{y\in E} C_\alpha(y) \neq \emptyset$ with $E \subset D$. Then, $\eta' := \sup_{y\in D}\inf_{x\in C'} F(x,y) \geq \eta$ and noting that $C'_\alpha(y) = C' \cap C_\alpha(y)$ it can be easily checked that for every $\alpha > \eta'$ conditions (H) and ($T^\flat$) are still satisfied with C replaced by C'. So by the above, $C'_\alpha(a) \cap C_\alpha(b) \neq \emptyset$ for every pair $a,b \in D$; in particular, $C'_\alpha(a) \neq \emptyset\ \forall a \in D$. Therefore, by Lemma 5.3 the minimax equality follows. ■

Remark 5.5. In Theorem 2.1 the function $u_{ab}(t)$ is assumed to be continuous, so Theorem 5.4 is stronger than Theorem 2.1. In particular, when Y is a vector topological space and D is convex (so that $u_{ab}(t) = (1-t)a + tb$), Theorem 5.4 simply requires that $F(x,y)$ is *usc* (*or lsc, resp.*) *in* y *in every line segment*, as defined in [1].

Remark 5.6. Theorem 5.4 has been derived from Corollary 5.2. Conversely, it is not hard to derive Corollary 5.2 from Theorem 2.1. In fact, when $Y = \mathbb{R}$ condition (K) amounts to saying that for every set $K \subset C$ the set $D^K := \{y \in D|\ F(x,y) > \alpha\ \forall x \in K\}$ is a segment, and therefore (K) reduces to (T) by taking $u_{ab}(t) = (1-t)a + tb$. Theorem 2.1 is thus equivalent to Corollary 5.2.

Since Theorem 5.4 is a consequence of Corollary 5.2, the following theorem contains both König's Theorem 4.1 and Tuy's Theorem 2.1:

Theorem 5.7. *Let* X, Y *be two Hausdorff topological spaces,* C *a nonempty closed subset of* X, D *a nonempty closed subset of* Y *and* $F(x,y)$ *a real-valued function on* $C \times D$. *The minimax equality* (5) *holds under the following assumptions:*

(i) *There exists a real number* $\alpha^* > \eta$ *such that for every* $\alpha \in (\eta, \alpha^*)$:

(H) *For every nonempty finite set* $H \subset D$ *the set* $C_H := \cap_{y \in H}\{x \in C|\ F(x,y) \le \alpha\}$ *is either empty or connected;*

(K) *For every set* $K \subset C$ *the set* $D^K := \{y \in D|\ F(x,y) > \alpha\ \forall x \in K\}$ *is either empty or connected.*

(ii) *One of the following conditions is satisfied:*

(A) $F(x,y)$ *is lsc in* x, *usc in* y *while there exists a finite set* $M \subset D$ *such that* $\cap_{y \in M} C_{\alpha^*}(y)$ *is compact;*

(B) $F(x,y)$ *is lsc jointly in* (x,y) *while* C *is compact;*

(C) $Y = \mathbb{R}$, $F(x,y)$ *is lsc separately in* x, y *while* C *is compact.*

From the above discussion it appears that Tuy's Theorem 2.1 is merely a variant of Theorem 5.7 when $Y = \mathbb{R}$ (assumption (C)). The question that arose with König's Theorem 4.1 and has remained open until today is thus simply: can the assumption $Y = \mathbb{R}$ in assumption (C) be removed?

The next proposition is obtained from Theorem 5.7 by interchanging the roles of x, y and noting that $\inf_{x \in C} \sup_{y \in D} F(x,y) = -\sup_{x \in C} \inf_{y \in D} F(x,y)$.

Theorem 5.8. *With* X, Y, C, D *and* $F(x,y)$ *as above, the minimax equality* (5) *holds under the following assumptions:*

(i) *There exists a real number* $\alpha_* < \gamma$ *such that for every* $\alpha \in (\alpha_*, \gamma)$*:*

(H*) *For every nonempty finite set* $H \subset C$ *the set* $D_H := \{y \in D|\ F(x,y) \ge \alpha\ \forall x \in H\}$ *is either empty or connected;*

(K*) *For every set* $K \subset D$ *the set* $C^K := \{x \in C|\ F(x,y) < \alpha\ \forall y \in K\}$ *is either empty or connected;*

(ii) *One of the following conditions is satisfied:*

(A*) $F(x,y)$ *is lsc in* x, *and usc in* y *while there exists a finite set* $N \subset C$ *such that the set* $\cap_{x \in N} D_{\alpha_*}(x)$ *is compact;*

(B*) $F(x,y)$ *is usc jointly in* (x,y) *while* D *is compact;*

(C*) $X = \mathbb{R}$, $F(x,y)$ *is usc separately in* x, y *while* D *is compact.*

6. Minimax theorems in vector topological spaces

Although the α-connectedness assumed in Theorem 2.1 looks rather special, this concept is more useful than general connectedness when working in vector topological spaces.

Specifically, let X, Y be vector topological spaces and C, D convex closed subsets of X, Y, respectively. Following [1] we say that a function $F(x,y)$ is *usc* (lsc, resp.) *in* y *in every line segment* if for any two points $a, b \in D$ the univariate function $t \mapsto (1-t)a + tb$ is usc (lsc, resp.) on the segment $0 \le t \le 1$. The next proposition, a weaker version of which was proved in [1], cannot easily be derived from Theorem 4.1 but follows almost in a straightforward manner from Theorem 2.1 (more precisely from Theorem 5.4 which is a strengthened version of Theorem 2.1).

Theorem 6.1. *Let C be a nonempty closed convex subset of a Hausdorff vector topological spaces X, D a nonempty closed convex subset of a Hausdorff vector topological Y, and $F(x,y) : C \times D \to \mathbb{R}$ a function quasiconvex in x, quasiconcave in y. The minimax equality* (5) *holds if either of the following conditions is satisfied:*

(i) *$F(x,y)$ is lsc in x, usc in y in every line segment of D, and there exist a finite set $M \subset D$ and $\alpha^* > \eta$ such that the set $\cap_{y \in M}\{x \in C|\ F(x,y) \le \alpha^*\}$ is compact;*

(ii) *$F(x,y)$ is lsc in x, lsc in y in every line segment of D, and C is compact.*

This theorem can be referred to as the strongest version of Sion's theorem. For more details on minimax propositions in vector topological spaces that can be strengthened by using Theorem 5.4 and Remark 5.5, see [18], [22].

On the other hand, while the intersection of an arbitrary family of convex sets is either empty or convex (hence connected), it is easy to give an example of two connected sets in $\mathbb{R}^n$ whose intersection is neither empty nor connected. One may then wonder whether aside from quasiconvex-quasiconcave functions $F(x,y)$ there exist other classes of functions $F(x,y)$ satisfying both (H) and (K). In answer to this question, we close the paper by a minimax theorem for monotonic functions analogous to Theorem 6.1.

Let us first recall some concepts from monotonic optimization [20] (see also [19], [21]). For any two vectors $x', x'' \in \mathbb{R}^n$ we write $x'' \ge x'$ to mean that $x''_i \ge x'_i$ for every $i = 1, \dots, n$; we write $u = x' \wedge x'',\ v = x' \vee x''$ to mean that $u_i = \min(x'_i, x''_i),\ v_i = \max(x'_i, x''_i)\ \forall i = 1, \dots, n$. A set $C \subset \mathbb{R}^n$ is said to be *downward* if $x' \in C,\ x'' \le x'$ implies that $[x'', x'] \subset C$, where $[x'', x'] := \{x \in \mathbb{R}^n|\ x'' \le x \le x'\}$; hence, $x', x'' \in C$ implies that $u = x' \wedge x'' \in C, [u, x'] \cup [u, x''] \subset C$. A set $H \subset \mathbb{R}^n$ is said to be *upward* if $x' \in H,\ x' \le x''$ implies that $[x', x''] \subset H$; hence, $x', x'' \in H$ implies that $v = x' \vee x'' \in H, [x', v] \cup [x'', v] \subset H$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *increasing* if $f(x') \ge f(x)$ whenever $x' \ge x$; *decreasing* if $f(x') \le f(x)$ whenever $x' \ge x$.

Theorem 6.2. *Let C be a nonempty closed upward subset of $X = \mathbb{R}^n$, D a nonempty closed downward subset of $Y = \mathbb{R}^m$ and $F(x,y) : C \times D \to \mathbb{R}$ a function decreasing in x and increasing in y. The minimax equality* (5) *holds if either of the following conditions is satisfied:*

(i) *$F(x,y)$ is lsc in x, usc in y and there exist a finite set $M \subset D$ and $\alpha^* > \eta$ such that the set $\cap_{y \in M}\{x \in C|\ F(x,y) \le \alpha^*\}$ is compact;*

(ii) $F(x, y)$ *is lsc in each variable* x, y *and* C *is compact.*

Proof. Consider any $\alpha \in (\eta, \alpha^*)$ and any finite set $H \subset D$. If $y \in H$ and $x', x'' \in C_\alpha(y) = \{x \in C|\ F(x, y) \leq \alpha\}$ then $x', x'' \in C$, hence, by upwardness of C, $v = x' \vee x'' \in C$; further, for every $x \in [x', v] \cup [x'', v]$ we have either $x \geq x'$, or $x \geq x''$, hence $F(x, y) \leq \min\{F(x', y), F(x'', y)\} \leq \alpha$ by decreasing property of the function $x \mapsto F(x, y)$. Thus, if $x', x'' \in C_H := \cap_{y \in H} C_\alpha(y)$, then $[x', v] \cup [x'', v] \subset C_H$ and so any two points $x', x'' \in C_H$ are connected by the line $\Gamma_{x'x''} \subset C_H$ made up of two segments: one joining x' to v, the other joining v to x''. This proves property (H) .

Now let K be a subset of C. If $x \in K$ and $y', y'' \in D_\alpha(x) := \{y \in D|\ F(x, y) > \alpha\}$ then $y', y'' \in D$, hence by downwardness of D, $u = x' \wedge x'' \in D$; further, for every $y \in [u, y'] \cup [u, y'']$ we have either $y \leq y'$ or $y \leq y''$, hence, $F(x, y) \geq \max\{F(x, y'), F(x, y'')\} > \alpha$ by increasing property of the function $y \mapsto F(x, y)$. Thus, if $y', y'' \in D^K := \cap_{x \in K}\{y \in D|\ F(x, y) > \alpha\}$ then $[u, y'] \cup [u, y''] \subset D^K$, and so any two points $y', y'' \in D^K$ are connected by the line $\Delta_{y'y''} \subset D^K$ made up of two segments: one joining y' to u, the other joining u to y''. This proves (K).

So all conditions of Theorem 5.7 are fulfilled, hence the conclusion. ■

Remark 6.3. As one may have noticed, Theorem 6.2 resembles Theorem 6.1 in many respects: it becomes almost identical to the latter by replacing the words upward, downward, increasing, decreasing by convex, convex, quasiconvex, quasiconcave respectively. This illustrates the parallel between monotonic optimization and dc optimization that has been noticed in [19], [18], [22] and particularly in [20] and [23], where both convex functions and increasing functions were shown to belong to the class of real-valued functions representable as differences of two functions of a class $\mathcal{C}$ such that:

1) $\mathcal{C}$ is a convex cone;

2) $\mathcal{C}$ is stable under the operation of lower envelope.

References

1. J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*, John Wiley & Sons, NewYork, 1984.
2. O. Yu. Borenshtein and V. S. Shul'man, A minimax theorem, *Math. Notes* **50** (1991), 752–754.
3. M. A. Geraghty and B.-L. Lin, Topological minimax theorems, *Proc. Amer. Math. Soc.* **91** (1984), 337–380.
4. H. König, A general minimax theorem based on connectedness, *Arch. Math.* **59** (1992), 55–64.
5. H. König, Addendum to "A general minimax theorem based on connectedness", *Arch. Math.* **64** (1995), 139–143.
6. J. von Neumann, Zur theorie der gesellschaftsspiele, *Math. Ann.* **100** (1928), 295–320.
7. H. Nikaido, On von Neumann's minimax theorem, *Pacific J. Math.* **4** (1954), 65–72.

8. J. Ponstein, *Approaches to the Theory of Optimization*, Cambridge University Press, London/New York, 1980.
9. B. Ricceri, On a topological minimax theorem and its applications, in: *Minimax Theory and Applications,* B. Ricceri and S. Simons (eds.), Kluwer Academic Publisher, Boston, 1998, pp. 191–216.
10. B. Ricceri, A further improvement of a minimax theorem of Borhenshtein and Shul'man, *J. Nonlinear Convex Anal.* **2** (2001), 279–283.
11. B. Ricceri, Well-posedness of constrained minimization problems via saddle-points, *J. Glob. Optim.* **40** (2008), 389–397.
12. B. Ricceri, Recent advances in minimax theory and applications, in: *Pareto Optimality, Game Theory and Equilibria*, A. Chinchuluun, A. Migdalas, P. M. Pardalos, L. Pitsoulis (eds.), Springer, Berlin, 2008, pp. 23–52.
13. B. Ricceri, Minimax theorems for functions involving a real variable and applications, *Fixed Point Theory* **9** (2008), 275–291.
14. J. Saint Raymond, On a minimax theorem, *Arch. Math.* **74** (2000), 432–437.
15. S. Simons, Minimax theorems, In: *Encyclopedia of Optimization*, C. Floudas, P. Pardalos (eds.), Vol. III, Kluwer Academic Publisher, Netherlands, 2001, pp. 284–291.
16. M. Sion, On general minimax theorems, *Pacific J. Math.* **8** (1958), 171–176.
17. H. Tuy, On a general minimax theorem, *Dokl. Akad. Nauk SSSR* **219** (1974), No. 4 (in Russian); English translation: Soviet Math. Dokl. 15(1974), 1689–1693.
18. H. Tuy, Minimax theorems revisited, *Acta Math. Vietnam.* **29** (2004), 217–229.
19. H. Tuy, Monotonic optimization: Problems and solution approaches, *SIAM J. Optim.* **11** (2000), 464–494.
20. H. Tuy, Monotonicity in the framework of generalized convexity, in: *Generalized Convexity, Generalized Monotonicity and Applications*, A. Eberhard, N. Hadjisavvas, D. T. Luc (eds.), Springer, Berlin, 2005, 61–85.
21. H. Tuy, M. Minoux and N. T. Hoai Phuong, Discrete monotonic optimization with application to a discrete location problem, *SIAM J. Optim.* **17** (2006), 78–97.
22. H. Tuy, Parametric minimax theorems with application, *Nonlinear Anal. Forum* **12** (2007), 1–16.
23. H. Tuy, $\mathcal{DC}$-optimization and robust global optimization, *J. Glob. Optim.* **47** (2010), 485–501; DOI 10.1007/s10898-009-9475-2.
24. H. Tuy, A new topologial minimax theorem with application, *J. Glob. Optim.* **50** (2011), 371–378; DOI 10.1007/s10898-010-9591-z.
25. N. N. Vorobiev, *Foundations of Game Theory: Noncooperative Games*, Nauka, Moscow, 1984 (in Russian).
26. Wu Wen-tsün, A remark on the fundamental theorem in the theory of games, *Science Record, n. Ser.* **5** (1959), 229–233.