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Supernilpotent Radicals of Γ_N-Rings

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Abstract. Let M be a Γ-ring in the sense of Nobusawa. The ring $M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix}$ was defined by Kyuno. Booth [3] defined supernilpotent radicals and N-radicals for Γ-rings and established relations between supernilpotent radicals and N-radicals of Γ-ring M and the corresponding radicals of the right operator ring R of M. In this paper, a result of [3] is improved (cf. Proposition 1.4) and the relationships between the supernilpotent radicals of Γ-ring M and the corresponding radicals of $\Gamma_{n,m}$-ring $M_{m,n}$, M-ring Γ, and the ring M_2 are established.

1. Introduction

In [3], the notions of supernilpotent radicals and N-radicals of Γ-rings were introduced and the relationships between the supernilpotent radicals and N-radicals on Γ-ring M and the corresponding radicals of the right (left) operator ring $R(L)$ of Γ-ring M were established. It is well known that for Γ_N-ring M, there are six related rings: the Γ-ring M, the right (left) operator ring $R(L)$ of Γ-ring M, the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, the M-ring Γ, and the ring M_2, and these rings are interelated. However, Booth [3] does not treat relationships between supernilpotent radicals of Γ-ring M and the corresponding radicals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, the M-ring Γ, and the ring M_2. This is the motivation of this note.

Let $\mathscr{R}$ be a supernilpotent radical of rings, and let $\mathscr{R} = \mathscr{U}\mathscr{C}$, where $\mathscr{C}$ is a weakly special class of rings. Let $\bar{\mathscr{C}}$ be the class of all Γ-rings M such that the right operator ring R of M is in $\mathscr{C}$, and $M\Gamma x = 0$ implies $x = 0$ for all $x \in M$. Then $\bar{\mathscr{C}}$ is a weakly special class in Γ-rings, and the upper radical class ${}_{\rho\Gamma}\mathscr{R}$ determined by $\bar{\mathscr{C}}$ is uniquely defined by $\mathscr{R}$. An analogous class of Γ-rings can be defined using the left operator ring (cf. [3], Theorems 2.5 and 2.6).

Throughout this paper, let $\mathscr{P}$ be a special (weakly special) class of rings, and $R_{\mathscr{P}} = \mathscr{U}\mathscr{P}$ be the upper radical determined by $\mathscr{P}$. We say that $\mathscr{P}$ satisfies condition $(*)$ if for every semiprime ring R and any $0 \neq e^2 = e \in R$, $eRe \in \mathscr{P}$ if and only if $R \in \mathscr{P}$. We note that many well-known classes of rings satisfy condition $(*)$: the class of prime rings, the class of primitive rings, the class of prime Levitzki semisimple rings, the class of prime subdirect irreducible rings, the class of primitive rings with nonzero socle, the class of weak primitive rings, the class of k-primitive rings, the class of prime Johnson rings, and the class of nonsingular prime rings (cf. [7, p. 11]).

Following Booth [3], a Γ-ring M is called (right) $\mathscr{P}$-Γ-ring or $\mathscr{P}_{(\Gamma)}$-ring if (i) the right operator ring R of M belongs to $\mathscr{P}$, and (ii) $M\Gamma x = 0$ implies $x = 0$. We will use $\mathscr{P}_{(\Gamma)}$ to denote the class of all $\mathscr{P}$-Γ-rings. $I \trianglelefteq M$ is called a $\mathscr{P}$-ideal if $M/I \in \mathscr{P}_{(\Gamma)}$. For any Γ-ring M, the $\mathscr{P}$-radical of M is defined as the intersection of all $\mathscr{P}$-ideals of M and denoted by $R_{\mathscr{P}(\Gamma)}(M)$. By [2, Proposition 2.7] or [4, Lemma 1.11] and [3, Theorem 2.6], this radical coincides with the supernilpotent radical of Γ-rings defined by the class $\mathscr{P}$ of rings.

For the definition of Γ-rings, weak Γ_N-rings, Γ_N-rings and the operator rings of Γ-rings, we refer to [1], [5], and [6]. M will denote an arbitrary Γ-ring, and $R(L)$ its eight (left) operator rings, with the notion "$I \trianglelefteq M$" meaning that "I is an ideal of M".

Let (M, Γ) be a weak Γ_N-ring and $A \subseteq M, P \subseteq R$, $Q \subseteq L$ and $\Phi \subseteq \Gamma$; then we define

$P* = \{x \in M : [\beta, x] \in P \text{ for all } \beta \in \Gamma\}$,
$Q^+ = \{x \in M : [x, \mu] \in Q \text{ for all } \mu \in \Gamma\}$,
$A^{*\prime} = \{r \in R : Mr \subseteq A\}$, $A^{+\prime} = \{y \in L : yM \subseteq A\}$,
$\Gamma(A) = \{\mu \in \Gamma : M\mu M \subseteq A\}$ and $M(\Phi) = \{x \in M : \Gamma x \Gamma \subseteq \Phi\}$.
Those maps map ideals into ideals.

Radical classes of Γ-rings, special radicals and the upper radical $\mu\mathscr{M}$ determined by a class $\mathscr{M}$ of Γ-rings are defined exactly as for rings. See, for example, [2], [3], and [4].

2. $\mathscr{P}$-Radical of the Right Operator Rings and Matrix Γ-Rings

If M is a Γ-ring, let R be the subring of the endomorphism ring of M (operator being taken to act on the right), consisting of all sums of the form $\sum_{i=1}^{n} [\beta_i, x_i] (\beta_i \in \Gamma, x_i \in M)$, where $[\beta, x]$ is defined by $y[\beta, x] = y\beta x$, for all $y \in M$. The ring R is called the right operator ring of M. A left operator ring L of M is similarly defined.

In the following, only right operator rings are considered. Analogous results for the left operator ring can be proved similarly. We will need the following two lemmas.

Lemma 2.1. *If A is an ideal of the Γ-ring M, R and $[\Gamma, M/A]$ are right operator rings of Γ-ring M and Γ-ring M/A, respectively, then we have $[\Gamma, M/A] \cong R/A^{*\prime}$ under the mapping*

$$\sum_i [\gamma_i, x_i + A] \to \sum_i [\gamma_i, x_i] + A^{*\prime}.$$

Moreover, if (M, Γ) is a weak Γ_N-ring, we have $[\Gamma/\Gamma(A), M/A] \cong R/A^{*\prime}$ under the *mapping*

$$\sum_i [\gamma_i + \Gamma(A), x_i + A] \to \sum_i [\gamma_i, x_i] + A^{*\prime}.$$

Lemma 2.2. *If M is a Γ-ring with the right operator ring R, and P is a prime ideal of R or M has right unity and P is an ideal of R, $[\Gamma, M/P^*]$ is the right operator rings of Γ-ring M/P^*, then we have $[\Gamma, M/P^*] \cong R/P$ under the mapping*

$$\sum_i [\gamma_i, x_i + P^*] \to \sum_i [\gamma_i, x_i] + P.$$

The proof of Lemmas 2.1 and 2.2 can be easily verified by direct computation.

Lemma 2.3. *Let M be a Γ-ring with the operator ring R. Then the mapping $P \to P^*$ defines a one-to-one correspondence between the set of semiprime ideals of R and M. Moreover, $(P^*)^{*\prime} = P$.*

The proof of Lemma 2.3 is an easy modification of [5, Theorem 1].

As an immediate consequence of Lemmas 2.2 and 2.3 and [5, Theorem 1], we have the following result.

Theorem 2.4. *Let M be a Γ-ring with the right operator ring R. Then the mapping $P \to P^*$ defines a one-to-one correspondence between the $\mathscr{P}$-ideals of R and that of M. Moreover, $(P^*)^{*\prime} = P$.*

Let $\mathscr{R} = \mathscr{U}\mathscr{C}$ be a supernilpotent radical of rings; for any Γ-ring M with right operator ring R, Booth [3, Theorem 2.3] proved that $\mathscr{R}(R)^* \subseteq \mathscr{R}_{\mathscr{C}(\Gamma)}(M)$. However, we prove the following.

Proposition 2.5. *Let M be a Γ-ring. Then $R_{\mathscr{P}}(R)^* = R_{\mathscr{P}(\Gamma)}(M)$.*

Proof. By Theorem 2.4, we have

$$\begin{aligned} R_{\mathscr{P}(\Gamma)}(M) &= \bigcap \{P^* \mid P \text{ is a } \mathscr{P}\text{-ideal of } R\} \\ &= (\bigcap \{P \mid P \text{ is a } \mathscr{P}\text{-ideal of } R\}) \\ &= [R_{\mathscr{P}}(R)]^*. \end{aligned}$$

A Γ-ring M is said to have right unity if there exists an element $\sum_{i=1}^{n} [\delta_i, a_i] \in R$ such that $\sum_{i=1}^{n} x b_i a_i = x$ for all $x \in M$. It is easily verified in this case that $\sum_{i=1}^{n} [\delta_i, a_i]$ is the unity of the ring R. ■

We now prove the next theorem which indicates one way to construct new $\mathscr{P}$-Γ-rings from given ones.

Theorem 2.6. *Let M be a Γ-ring with right unity and $\mathscr{P}$ satisfies $(*)$; then $M \in \mathscr{P}_{(\Gamma)}$ if and only if $M_{m,n} \in \mathscr{P}_{(\Gamma)}$.*

Proof. Suppose $M \in \mathscr{P}_{(\Gamma)}$. Then, $R = [\Gamma, M] \in \mathscr{P}$ and M satisfies $M\Gamma x = 0$ which implies $x = 0$. Denote the right operator ring of $M_{m,n}$ by $[\Gamma_{n,m}, M_{m,n}]$. Recall that $[\Gamma_{n,m}, M_{m,n}] \cong R_n$ (see [5, p. 376]) and R_n is semiprime and $e_{11}R_ne_{11} \cong R$. We have that $[\Gamma_{n,m}, M_{m,n}] \in \mathscr{P}$. Also, if $M_{m,n}\Gamma_{n,m}(x_{i,j}) = 0$, $(x_{i,j}) \in M_{m,n}$, then for all $m \in M$, $\gamma \in \Gamma$, we have

$$0 = (me_{ik})(\gamma e_{kj})(x_{st}) = \begin{pmatrix} 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ m\gamma x_{j1} & \cdots & m\gamma x_{jn} \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} (i).$$

Therefore, $M\Gamma x_{ij} = 0$ for all $1 \leq i \leq m$, $1 \leq j \leq n$ and, consequently, $x_{ij} = 0$ and $(x_{ij}) = 0$. Hence, the $\Gamma_{n,m}$-ring $M_{m,n} \in \mathscr{P}_{(\Gamma)}$.

Conversely, suppose $\Gamma_{n,m}$-ring $M_{m,n} \in \mathscr{P}_{(\Gamma)}$. Then $[\Gamma_{n,m}, M_{m,n}] \cong R_n \in \mathscr{P}$. Hence, $R \in \mathscr{P}$. Also, if $M\Gamma x = 0, x \in M$, then $M_{m,n}\Gamma_{n,m}xe_{11} = 0$ and so $xe_{11} = 0$, i.e. $x = 0$. Hence, $M \in \mathscr{P}_{(\Gamma)}$. ■

Lemma 2.7. *If $I \trianglelefteq M$, then the matrix $\Gamma_{n,m}$-ring $(M/I)_{m,n}$ is isomorphic to the $\Gamma_{n,m}$-ring $M_{m,n}/I_{m,n}$.*

By easy modifications of the proof of [5, Theorem 2], we can prove the following lemma.

Lemma 2.8. *Let M be an arbitrary Γ-ring. Then, the semiprime ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are precisely the sets $P_{n,m}$, where P is a semiprime ideal of the Γ-ring M.*

As a consequence of Lemma 2.8 and Theorem 2.6 we have the following.

Theorem 2.9. *Let M be a Γ-ring with right unity and $\mathscr{P}$ satisfies $(*)$. Then the $\mathscr{P}$-ideals of the matrix $\Gamma_{n,m}$-ring $M_{m,n}$ are precisely the sets $P_{n,m}$, where P is a $\mathscr{P}$-ideal of the Γ-ring M.*

Theorem 2.10. *If M is a Γ-ring with right unity and $\mathscr{P}$ satisfies $(*)$, then $\mathscr{R}_{\mathscr{P}(\Gamma)}(M_{m,n}) = (\mathscr{R}_{\mathscr{P}(\Gamma)}(M))_{m,n}$.*

Proof. By Theorem 2.9, we have that

$$\begin{aligned} R_{\mathscr{P}(\Gamma)}(M_{m,n}) &= \bigcap\{(I)_{m,n} | I \text{ is a } \mathscr{P}\text{-ideal of } M\} \\ &= (\bigcap\{I | I \text{ is a } \mathscr{P}\text{-ideal of } M\})_{m,n} \\ &= [R_{\mathscr{P}(\Gamma)}(M)]_{m,n}. \end{aligned}$$ ■

3. $\mathscr{P}$-Radical of M-Ring Γ and the Ring M_2

Let (M, Γ) be a Γ_N-ring. Let R and L denote, respectively, the right and left operator rings of the Γ-ring M. The set

$$M_2 = \begin{pmatrix} R & \Gamma \\ M & L \end{pmatrix} = \left\{ \begin{pmatrix} r & \gamma \\ m & l \end{pmatrix} \middle| r \in R, \gamma \in \Gamma, m \in M, l \in L \right\}$$

is a ring with respect to the obvious operations of matrix multiplication and addition. For details, see [1, 6].

The proof of the following lemma can be easily verified by direct computation.

Lemma 3.1. *Let (M, Γ) be a Γ_N-ring. Then the left operator ring L' of the M-ring Γ is isomorphic to $[\Gamma, M]/K$, where $K = \{x \in [\Gamma, M] | x\Gamma = 0\}$. In particular, if M is a semiprime Γ-ring, $K = 0$.*

Let (M, Γ) be a Γ_N-ring. It is easily verified that an ideal P of M is a $\mathscr{P}$-ideal if and only if M/P is a $\mathscr{P}$-$\Gamma/\Gamma(P)$-ring. From Lemmas 2.1 and 3.1, and [1, Theorem 3.3], we have the following.

Proposition 3.2. *Let (M, Γ) be a weak Γ_N-ring. Then the mapping $A \to \Gamma(A)$ defines a one-to-one correspondence between the sets of $\mathscr{P}$-ideals of Γ-ring M and that of M-ring Γ. Moreover, $M(\Gamma(A)) = A$.*

Corollary 3.3. *Let (M, Γ) be a weak Γ_N-ring. Then $R_{\mathscr{P}(M)}(\Gamma) = \Gamma(R_{\mathscr{P}(\Gamma)}(M))$.*

Proof. By Proposition 3.2, we have that

$$\begin{aligned} R_{\mathscr{P}(M)}(\Gamma) &= \bigcap \{\Gamma(A) | A \text{ is a } \mathscr{P}\text{-ideal of } M\} \\ &= \Gamma(\bigcap \{A | A \text{ is a } \mathscr{P}\text{-ideal of } M\}) \\ &= \Gamma(R_{\mathscr{P}(\Gamma)}(M)). \end{aligned}$$ ■

We need the following results.

Proposition 3.4. *Let (M, Γ) be a Γ_N-ring. Then M_2 is a semiprime ring if and only if M is a semiprime Γ-ring..*

Proposition 3.5. *Let (M, Γ) be a Γ_N-ring. Then a subset P_2 of M_2 is a semiprime ideal if and only if*

$$P_2 = \begin{pmatrix} A^{*\prime} & \Gamma(A) \\ A & A^{+\prime} \end{pmatrix},$$

where A is a semiprime ideal of Γ-ring M.

Propositions 3.4 and 3.5 can be proved by easy modifications of [1, Theorems 3.5 and 3.6].

Theorem 3.6. *Let (M,Γ) be a Γ_N-ring and the Γ-ring M has right unity and $\mathscr{P}$ satisfies $(*)$. Then $M_2 \in \mathscr{P}$ if and only if $M \in \mathscr{P}_{(\Gamma)}$.*

Proof. Suppose the Γ-ring $M \in \mathscr{P}_{(\Gamma)}$. Then $R \in \mathscr{P}$ and M is a semiprime Γ-ring. By Proposition 3.4, M_2 is a semiprime ring. Since $eM_2e \cong R \in \mathscr{P}$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, thus, $M_2 \in \mathscr{P}$.

Conversely, suppose $M_2 \in \mathscr{P}$. Since $eM_2e \cong R$, $R \in \mathscr{P}$. By the semiprimeness of M_2 and Proposition 3.4, M is a semiprime Γ-ring, thus, $M\Gamma x = 0$, $x \in M$ implies $x = 0$. Therefore, $M \in \mathscr{P}_{(\Gamma)}$. ■

Proposition 3.7. *Let (M,Γ) be a Γ_N-ring and the Γ-ring M has right unity and $\mathscr{P}$ satisfies $(*)$. Then a subset P_2 of M_2 is a $\mathscr{P}$-ideal if and only if*

$$P_2 = \begin{pmatrix} A^{*\prime} & \Gamma(A) \\ A & A^{+\prime} \end{pmatrix},$$

where A is a $\mathscr{P}$-ideal of M.

Proof. Suppose P_2 is a $\mathscr{P}$-ideal of M_2. Then P_2 is a semiprime ideal of M_2. Hence, by Proposition 3.5,

$$P_2 = \begin{pmatrix} A^{*\prime} & \Gamma(A) \\ A & A^{+\prime} \end{pmatrix},$$

for some semiprime ideal of M. Now

$$M_2/P_2 \cong \begin{pmatrix} R/A^{*\prime} & \Gamma/\Gamma(A) \\ M/A & L/A^{+\prime} \end{pmatrix} \in \mathscr{P}.$$

Hence, by Theorem 3.6, $M/A \in \mathscr{P}_{(\Gamma)}$, i.e., A is a $\mathscr{P}$-ideal of M.

Conversely, if A is a $\mathscr{P}$-ideal of M, then $M/A \in \mathscr{P}_{(\Gamma)}$. Hence, by Theorem 3.6, $M_2/P_2 \cong (M/A)_2 \in \mathscr{P}$, i.e., P_2 is a $\mathscr{P}$-ideal of M_2. ■

The following result encompasses and generalizes the corresponding result of Kyuno [6].

Theorem 3.8. *Let (M,Γ) be a Γ_N-ring and Γ-ring M has right unity and $\mathscr{P}$ satisfies $(*)$. Then*

$$R_{\mathscr{P}}(M_2) = \begin{pmatrix} R_{\mathscr{P}}(R) & R_{\mathscr{P}(M)}(\Gamma) \\ R_{\mathscr{P}(\Gamma)}(M) & R_{\mathscr{P}}(L) \end{pmatrix}.$$

Note. It is still unknown whether or not Theorems 2.9 and 3.8 are true for arbitrary Γ-ring or arbitrary weakly special class.

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