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The Bass-Papp Theorem and Some Related Results

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Abstract. In the spirit of the Bass–Papp theorem, this paper is concerned with rings for which any (finite) direct sum of quasi-continuous modules is quasi-continuous or for which any (finite) direct sum of *CS*-modules is *CS*.

1. Introduction

We show that a ring R is right QI if and only if the direct sum of any two quasi-injective right R-modules is quasi-injective. On the other hand, a ring R is semi-prime Artinian if and only if the direct sum of any two quasi-continuous right R-modules is quasi-continuous. A right nonsingular ring R is right Noetherian if and only if every direct sum of injective right R-modules is CS. We also show that if R is a ring with finitely generated right socle such that the direct sum of any two CS right R-module is CS, then every right R-modules is CS and R is right and left Artinian.

All rings are associative and have identity elements and all modules are unital right modules. Let R be any ring. A right R-module M is called a *CS-module* if every submodule is essential in a direct summand ("*CS*" stands for *complements are summands*). Recall also that the module M is called *quasi-continuous* if M is CS and for all direct summands K and L with $K \cap L = 0$, the submodule $K \oplus L$ is also a direct summand of M. The module M is called *continuous* if M is CS and for each direct summand N of M and each monomorphism $\varphi : N \to M$, the submodule $\varphi(N)$ is also a direct summand of M. Finally, the module M is *quasi-injective* if M is M-injective, i.e., for every submodule H of M, any homomorphism $\theta : H \to M$ can be lifted to M. It is well known that the following implications hold for M:

$$M \text{ is injective} \Rightarrow M \text{ is quasi-injective} \Rightarrow M \text{ is continuous}$$
$$\Rightarrow M \text{ is quasi-continuous} \Rightarrow M \text{ is } CS.$$

For these facts and a good account of this area, see [3] or [7].

The Bass–Papp theorem states that a ring R is ring Noetherian if and only if every direct sum of (a countable number of) injective right R-modules is injective (see, for example, [1, Proposition 18.13] or [11, Theorem 4.1]). It is natural to raise the following general question: for which rings R is every direct sum of quasi-injective (respectively, continuous, quasi-continuous, CS) right R-modules quasi-injective (continuous, quasi-continuous, CS)? The purpose of this note is to try to answer these questions.

For any unexplained terminology, please see [1], [3], or [7].

2. Quasi-Injective, Continuous and Quasi-Continuous Modules

In this section, we are concerned with the problem of finding which rings R have the property that the classes of quasi-injective or continuous or quasi-continuous modules are closed under taking direct sums (coproducts). If M is an R-module, then $E(M)$ will denote the injective hull of M, $\mathrm{Soc}(M)$ the socle of M, $Z(M)$ the singular submodule of M, i.e.,

$$Z(M) = \{m \in M : mA = 0 \text{ for some essential right ideal } A \text{ of } R\}$$

and $Z_2(M)$ the second singular submodule of M, i.e., $Z_2(M)$ is the submodule of M, containing $Z(M)$ such that $Z_2(M)/Z(M) = Z(M/Z(M))$. A ring R is called a right QI-ring if every quasi-injective right R-module is injective.

Our first result gives a characterization of right QI-rings.

Proposition 1. *The following statements are equivalent for a ring R:*
(i) *R is a right QI-ring.*
(ii) *The direct sum of any two quasi-injective right R-modules is quasi-injective.*
(iii) *The direct sum of any family of quasi-injective right R-modules is quasi-injective.*

Proof. (i) ⇒ (iii). Let R be a right QI-ring. Every semisimple R-module is quasi-injective and hence injective. By [11, Theorem 4.1], R is right Noetherian. Now (iii) follows by the Bass–Papp theorem.

(iii) ⇒ (ii). Clear.

(ii) ⇒ (i). Let M be any quasi-injective right R-module. Then $E(R_R) \oplus M$ is quasi-injective by hypothesis, and hence M is an injective R-module by [7, Proposition 1.3]. ■

Proposition 2. *The following statements are equivalent for a ring R:*
(i) *Every continuous right R-module is injective.*
(ii) *The direct sum of any two continuous right R-modules is continuous.*
(iii) *The direct sum of any family of continuous right R-modules is continuous.*

Proof. (i) ⇒ (iii). In particular, R is a right QI-ring and hence is right Noetherian. Now (iii) follows by the Bass–Papp theorem.

(iii) ⇒ (ii). Clear.

(ii) ⇒ (i). Let M be a continuous R-module. The $E(R_R \oplus M)$ is continuous by hypothesis. Hence, M is an injective R-module by [7, Propositions 1.3 and 2.10]. ■

Propositions 1 and 2 raise the following natural question: if R is a right QI-ring, is every continuous right R-module injective?

Theorem 3. *The following statements are equivalent for a ring R:*
(i) *R is semiprime Artinian.*
(ii) *Every quasi-continuous right R-module is injective.*
(iii) *The direct sum of any two quasi-continuous right R-modules is quasi-continuous.*
(iv) *The direct sum of any family of quasi-continuous right R-modules is quasi-continuous.*

Proof. (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (ii). By the proof of Proposition 2 (ii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (i). Suppose every quasi-continuous R-module is injective. Then R is a right QI-ring and hence is right Noetherian. By [11, Theorem 4.4], $E(R_R) = \oplus_{i \in I} E_i$, where E_i is indecomposable injective for each $i \in I$. Let $i \in I$. Let $0 \neq u \in E_i$. Then uR is uniform, whence quasi-continuous. By hypothesis, uR is injective. Hence, uR is a direct summand of E_i and we have $E_i = uR$. Thus E_i is simple. It follows that $E(R_R)$ is semisimple and hence so too is R_R. ■

It is not the case that every right QI-ring is semiprime Artinian (see for example, [4, 19.49]).

3. Direct Sums of Injectives

A ring R is right Noetherian if and only if the direct sum of every (countable) collection of injective right R-modules is quasi-continuous. First, suppose that the direct sum of every countable collection of injective right R-modules is quasi-continuous. Let $E_i (i \in I)$ be any countable collection of injective R-modules. Then $(\oplus_I E_i) \oplus E(R_R)$ is quasi-continuous and hence, by [7, Proposition 2.10], $\oplus_I E_i$ is an injective R-module. By [11, Theorem 4.1], R is right Noetherian. The converse also clearly follows from [11, Theorem 4.1].

We shall show that a right nonsingular ring R is right Noetherian if and only if every direct sum of injective right R-modules is CS.

Lemma 4. *Let R be a ring with right socle S.*
(i) *If the direct sum of every countable collection of injective hulls of singular simple right R-modules is injective, then R/S is right Noetherian.*
(ii) *If R/S is right Noetherian, then every direct sum of singular injective right R-modules is injective.*

Proof. (i) By a result of Goodearl [5, Proposition 3.6], it is sufficient to prove that the R-module R/E is Noetherian for any essential right ideal E of R. This is done by adapting the proof of [11, Theorem 4.1].

(ii) By [9, Corollary 11].

Lemma 5. *Let R be a right nonsingular ring right socle S such that the direct sum of* *every countable collection of injective right R-modules is CS. Then the right R/S is right Noetherian.*

Proof. Let S_i $(i \in I)$ be singular simple right R-modules. Let $E = E(S_1) \oplus E(S_2) \oplus E(S_3) \oplus \cdots$ and let $M = E(R_R) \oplus E$. By hypothesis, M is a CS-module. By [6, Theorem 1], because $E(R_R)$ is nonsingular and $E = Z_2(M)$, we deduce that E is $E(R_R)$-injective. Hence, E is an injective R-module. By Lemma 4, R/S is right Noetherian. ■

Recall that a ring R *has finite right uniform dimension* if it contains no infinite direct sums of nonzero right ideals.

Corollary 6. *A right nonsingular ring R is right Noetherian if and only if R has finite right uniform dimension and the direct sum of every (countable) collection of indecomposable injective right R-modules is CS.*

Proof. By Lemma 5.

Let R be a right nonsingular ring and let Q denote the maximal right quotient ring of R. Recall that Q is a von Neumann regular right self-injective ring, R is a subring of Q, and the right R-module Q is the injective hull of the right R-module R. Let X be a right Q-module such that as a right R-module, X is nonsingular and CS. Let Y be any submodule of X_Q. There exist submodules Z, Z' of X_R such that $X = Z \oplus Z'$ and Y is essential in Z. Since X_R is nonsingular, it is easy to check that $X = (ZQ) \oplus (Z'Q)$. Moreover, Z is essential in $(ZQ)_R$ so that Y is essential in $(ZQ)_R$. But this implies that Y is essential in $(ZQ)_Q$. It follows that the Q-module X is CS. ■

We have proved the following.

Lemma 7. *Let R be a right nonsingular ring with maximal right quotient ring Q. Let X be a right Q-module such that the right R-module X is nonsingular CS. Then the right Q-module X is CS.*

Corollary 8. *Let R be a right nonsingular ring with maximal right quotient ring Q. Then R has finite right uniform dimension if and only if the right R-module* $Q^{(I)}$ *is CS for any index set I.*

Proof. Suppose that R has finite right uniform dimension. Clearly, Q is a nonsingular right R-module. Let A be a right ideal of R and let $\varphi : A \to Q^{(I)}$ be a holomorphism. There exists a finitely generated right ideal B of R such that B is an essential submodule of A_R. There exists a finite subset J of I such that $\varphi(B) \subseteq Q^{(J)}$. Since Q_R is nonsingular, it follows that $\varphi(A) \subseteq Q^{(J)}$ also. But $Q^{(J)}$ is injective, so that there exists $q \in Q^{(J)} \subseteq Q^{(I)}$ with $\varphi(a) = qa$ for all a in A. Thus, $Q^{(I)}$ is injective and hence CS. Conversely, if the right R-module $Q^{(I)}$ is CS for any index set I, then the right Q-module $Q^{(I)}$ is CS (Lemma 7). Thus, Q is right Artinian by [8, Theorem II] (or see [3, 11.13]). It follows that R has finite right uniform dimension. ■

Combining Lemma 5 and Corollary 8 we have at once:

Theorem 9. *A right nonsingular ring R is right Noetherian if and only if the direct sum of every collection of injective right R-modules is CS.*

4. Direct Sums of CS-Modules

In this section, our concern is with rings R having the property that any direct sum of CS-modules is CS. It turns out that if the direct sum of any two CS-modules is CS, then in many cases the direct sum of any collection of CS-modules is CS. Recall that any uniform module is CS. Note that if R is any ring, then the following statements are equivalent:
(i) the direct sum of any two uniform right R-modules is CS;
(ii) the direct sum of any collection of uniform right R-modules is CS (see [3, 13.1]).

A ring R is a *right V-ring* if every simple right R-module is injective. It is well known that a commutative ring R is a V-ring if and only if R is von Neumann regular. A ring R is called *right semi-Artinian* if every nonzero right R-module has nonzero socle. For examples of such rings, see [2].

Proposition 10. *Let R be a ring which is either (a) a commutative von Neumann regular ring or (b) a right semi-Artinian, right V-ring. Then every uniform right R-module is simple and every direct sum of uniform right R-modules is CS.*

Proof. First, suppose that R is a commutative von Neumann regular ring. Let U be a uniform R-module. Let $0 \neq u \in U$. Let $r \in R$ with $ur \neq 0$. Then $rR = eR$ for some idempotent e in R. Also, $0 \neq urR = ueR \subseteq Ue$. Now $U = Ue \oplus U(1-e)$ gives $U(1-e) = 0$ and hence $u(1-e) = 0$. Thus, $u = ue \in urR$. It follows that uR is simple and hence injective. Thus, $U = uR$. Hence, U is simple.

Now suppose that R is a right semi-artinian right V-ring. Let U be any uniform right R-module. Then U contains a simple submodule S and S is injective, so that $U = S$.

Thus in either case uniform modules are simple and clearly direct sums of uniform modules are CS, because semisimple modules are CS. ■

Lemma 11. *Let R be any ring, M a semisimple right R-module, and M_2 a right R-module with zero socle such that $M = M_1 \oplus M_2$ is a CS-module. Then M_1 is M_2-injective.*

Proof. Clearly, $M_1 = \mathrm{Soc}(M)$. Let N be any submodule of M_2 and let $\varphi : N \to M_1$ be a homomorphism. Let $L = \{x - \varphi(x) : x \in N\}$. Then L is a submodule of M and $L \cap M_1 = 0$. There exist submodules K, K' of M such that $M = K \oplus K'$ and L is an essential submodule of K. Note that $\mathrm{Soc}(K) = K \cap M_1 = 0$, so that $M_1 = \mathrm{Soc}(M) \subseteq K'$. Thus, $K' = M_1 \oplus (K' \cap M_2)$ and $M = K \oplus M_1 \oplus (K' \cap M_2)$. Let $\pi : M \to M_1$ denote the projection with kernel $K \oplus (K' \cap M_2)$. Let θ denote the restriction of φ to M_2. Then $\theta : M_2 \to M_1$ and $\theta(x) = \varphi(x)$ for all x in N. It follows that M_1 is M_2-injective.

Lemma 12. *Let R be a ring such that the direct sum of any two CS right R-module is CS. Then R is right semi-Artinian.*

Proof. Suppose first that R has zero right socle. Let X be any semisimple right R-module and let $E = E(R_R)$. By hypothesis, $M = X \oplus E$ is CS and hence, by Lemma 11, X is E-injective. It follows that every semisimple right R-module is injective. By [11, Theorem 4.1], R is right Noetherian. Now [3, 13.3] gives that every right R-module is CS and [3, 13.5] gives R right Artinian.

In general, let $0 = S_0(R) \subseteq S_1(R) \subseteq \cdots \subseteq S_\alpha(R) \subseteq S_{\alpha+1}(R) \subseteq \cdots$ denote the right Loewy series of R, i.e., for each ordinal $\alpha > 0$, $S_{\alpha+1}(R)/S_\alpha(R)$ is the right socle of the ring $R/S_\alpha(R)$ and $S_\alpha(R) = U_{0<\beta<\alpha} S_\beta(R)$ if α is a limit ordinal. There exists an ordinal $\rho > 0$ such that $S_\rho(R) = S_{\rho+1}(R)$, i.e., the ring $R/S_\rho(R)$ has zero right socle. By the above argument, $R/S_\rho(R)$ is right Artinian and hence $S_\rho(R) = R$. It follows that R is semi-Artinian (see, for example, [2, Proposition 1]).

Theorem 13. *The following statements are equivalent for a ring R with Jacobson radical J.*

(i) *R has finite right uniform dimension and the direct sum of any two uniform right R-modules is CS.*

(ii) *R has finite left uniform dimension and the direct sum of any two uniform left R-modules is CS.*

(iii) *R has finitely generated right socle and the direct sum of any two CS right R-modules is CS.*

(iv) *R has finitely generated left socle and the direct sum of any two CS left R-modules is CS.*

(v) *Every right R-module is CS.*

(vi) *Every left R-module is CS.*

(vii) *R is (right and left) Artinian serial and* $J^2 = 0$.

Proof. (v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii). By [3, 13.5].

(v)$\Rightarrow$(i). Clear.

(i)$\Rightarrow$(v). There exist a positive integer n and indecomposable injective right R-modules E_i $(1 < i < n)$ such that $E(R_R) = E_1 \oplus \cdots \oplus E_n$. By [3, 13.1] E_i is Artinian for each i and hence R is right Artinian. Clearly, every cyclic right R-module has finite uniform dimension. By [3, 13.3], every right R-module is CS.

(ii)$\Leftrightarrow$(v). As for (i)$\Leftrightarrow$(v).

(v)$\Rightarrow$(iii). Clear.

(iii)$\Rightarrow$(i). By Lemma 12.

(iv)$\Leftrightarrow$(v). Similar to (iii)$\Leftrightarrow$(v). ■

Two questions spring to mind at this point. First of all, if R is a ring such that the direct sum of any two CS-modules is CS, then is any direct sum of CS-modules CS? Related to this, we also ask: if R is a ring such that the direct sum of any two CS-modules is CS, is any R-module CS?

Finally, we give an example of a commutative ring for which the direct sum of every collection of uniform modules is *CS* but not every direct sum of *CS*-modules is *CS*.

Example. Let R denote the group algebra $K[G]$ where K is a field of characteristic O and G is the direct product of an infinite family of cyclic groups. Then
(i) R is a commutative von Neumann regular ring (and hence is nonsingular).
(ii) R has zero socle (and hence R is not semi-Artinian).
(iii) Every direct sum of uniform R-modules is CS.
(iv) There exist CS R-modules X and Y such that $X \oplus Y$ is not CS.

Proof. (i) by [10, Theorem 3.1.5].

(ii) Suppose R has nonzero socle. Let U be a minimal ideal. Let $0 \neq u \in U$. Then there exist a finite subgroup G_1 and a nontrivial subgroup G_2 with $G = G_1 \times G_2$ and $u \in K[G_1]$. Let $1 \neq g \in G_2$. Then $u(1-g) \neq 0$ and hence, $u = u(1-g)r$ for some $r \in R$. Suppose g has order n. Then $u(1+g+\cdots+g^{n-1}) = 0$ and hence, $u + ug + \cdots + ug^{n-1} = 0$. It follows that $u = 0$, a contradiction.

(iii) By Proposition 10.

(iv) By (ii) and Lemma 12. ■

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