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Sufficient Conditions for the Existence of a Hamilton Cycle in Cubic $(6, n)$-Metacirculant Graphs

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Abstract. The smallest value of m, for which we still do not know if all connected cubic (m, n)-metacirculant graphs have a Hamilton cycle, is $m = 6$. In this paper, we show that if $G = MC(6, n, \alpha, S_0, S_1, S_2, S_3)$ is a connected cubic $(6, n)$-metacirculant graph such that $\emptyset \neq S_1 = \{s\}$ and $(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)s \equiv 0 \pmod{n}$, then G possesses a Hamilton cycle.

1. Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges. If G is a graph, then $V(G)$, $E(G)$, and $\mathrm{Aut}(G)$ denote its vertex-set, its edge-set, and its automorphism group, respectively. If n is a positive integer, then we write Z_n for the ring of integers modulo n and Z_n^* for the multiplicative group of units in Z_n. A graph G is called vertex-transitive if the action of $\mathrm{Aut}(G)$ on $V(G)$ is transitive.

Let m and n be two positive integers, $\alpha \in Z_n^*$, $\mu = [m/2]$ and let $S_0, S_1, \ldots, S_\mu$ be subsets of Z_n satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$,

(2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$.

(3) If m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Then we define the (m, n)-metacirculant graph $G = MC(m, n, \alpha, S_0, S_1, \ldots, S_\mu)$ to be the graph with vertex-set $V(G) = \{v_j^i : i \in Z_m,\ j \in Z_n\}$ and edge-set $E(G) =$

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$\{v_j^i v_h^{i+r} : 0 \le r \le \mu;\ i \in Z_m;\ h,\ j \in Z_n\};\ (h-j) \in \alpha^i S_r$, where superscripts and subscripts are always reduced modulo m and modulo n, respectively.

The concept of (m,n)-metacirculant graphs was defined by Alspach and Parsons [1]. All (m,n)-metacirculant graphs are vertex-transitive [1]. These graphs were introduced as a logical generalization of the Petersen graph for the primary reason of providing a class of vertex-transitive graphs in which there might be some new non-Hamiltonian connected vertex-transitive graphs. Among these graphs, cubic (m,n)-metacirculant graphs are paid special attention, being at the same time the simplest nontrivial (m,n)-metacirculant graphs and those most likely to be non-Hamiltonian because of their small number of edges.

For $n = p^t$ with p as a prime, connected (m,n)-metacirculant graphs, other than the Petersen graph, have been proven to have a Hamilton cycle [2]. Connected cubic (m,n)-metacirculant graphs, other than the Petersen graph, are also proved to be Hamiltonian for m odd [4], $m = 2$ [3,4], and m divisible by 4 [5, 6]. Thus, the remaining values of m, for which we still do not know if all connected cubic (m,n)-metacirculant graphs have a Hamilton cycle, are of the form $m = 2\mu$ with $\mu \ge 3$ as an odd positive integer, and the smallest among these values of m is $m = 6$. It seems hard to handle the problem of the existence of a Hamilton cycle in a connected cubic (m,n)-metacirculant graph for these remaining values of m, in particular, for $m = 6$. To deal with this problem, it is useful at first to have sufficient conditions for these graphs to be Hamiltonian. For example, thanks to the sufficient conditions obtained in [7], connected cubic (m,n)-metacirculant graphs, other than the Petersen graph, have been proved [7] to have a Hamilton cycle for many pairs (m,n) with m among the remaining values mentioned above.

It is the author's intention to resolve the problem of the existence of a Hamilton cycle in connected cubic $(6,n)$-metacirculant graphs. Here, an attempt will be made to prove several sufficient conditions for connected cubic $(6,n)$-metacirculant graphs to be Hamiltonian, which are expected to be helpful in future investigation of the problem. In this paper, we will prove the following sufficient conditions needed for the proof of the above-mentioned result.

Theorem. *If $G = MC(6,n,\alpha,S_0,S_1,S_2,S_3)$ is a connected cubic $(6,n)$-metacirculant graph such that $\emptyset \ne S_1 = \{s\}$ and $(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5)s \equiv 0 \pmod n$, then G possesses a Hamilton cycle.*

The conditions in this theorem are trivially satisfied if either $S_1 = \{0\}$ or $S_1 \ne \emptyset$ and $\alpha^3 + 1 \equiv 0 \pmod n$. Thus, we immediately have the following corollaries.

Corollary 1. *If $G = MC(6,n,\alpha,S_0,S_1,S_2,S_3)$ is a connected cubic $(6,n)$-metacirculant graph such that $S_1 = \{0\}$, then G possesses a Hamilton cycle.*

Corollary 2. *If $G = MC(6,n,\alpha,S_0,S_1,S_2,S_3)$ is a connected cubic $(6,n)$-metacirculant graph such that $S_1 \ne \emptyset$ and $\alpha^3+1 \equiv 0 \pmod n$, then G possesses a Hamilton cycle.*

2. Proof of the Theorem

Let $G = MC(6,n,\alpha,S_0,S_1,S_2,S_3)$ be a connected cubic $(6,n)$-metacirculant graph such that $\emptyset \neq S_1 = \{s\}$ and $(1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5)s \equiv 0 \pmod{n}$. If $S_0 \neq \emptyset$, then by [4] G has a Hamilton cycle. Therefore, we assume from now on that $S_0 = \emptyset$. Since G is cubic and $S_1 = \{s\}$ with $0 \le s < n$, we must have $S_2 = \emptyset$ and $S_3 = \{k\}$ with $0 \le k < n$.

Let $H = MC(6,n,\alpha',S_0',S_1',S_2',S_3')$ be a cubic $(6,n)$-metacirculant graph such that $\alpha' = \alpha$ and $S_0' = \emptyset$, $S_1' = \{0\}$, $S_2' = \emptyset$, and $S_3' = \{k'\}$ with $k' = k-(1+\alpha+\alpha^2)s$. Let $V(H) = \{x_j^i : i \in Z_6,\ j \in Z_n\}$. Then it is not difficult to verify that the mapping

$$\varphi : V(H) \to V(G) : \begin{cases} x_j^0 \mapsto v_j^0, & \\ x_j^i \mapsto v^i_{j+(1+\alpha+\cdots+\alpha^{i-1})s} & \text{if } i \neq 0 \end{cases}$$

is an isomorphism of H and G. Since G is connected, the graph H is also connected. By [8], $gcd(k',n) = 1$. So $\alpha^3+1 \equiv 0 \pmod{n}$. Now if $F = MC(6,n,\alpha'',S_0'',S_1'',S_2'',S_3'')$ is a cubic $(6,n)$-metacirculant graph such that $\alpha'' = \alpha$, $S_0'' = \emptyset$, $S_1'' = \{0\}$, $S_2'' = \emptyset$, $S_3'' = \{1\}$, and $V(F) = \{y_j^i : i \in Z_6,\ j \in Z_n\}$, then since $gcd(k',n) = 1$, it is not difficult to verify that the mapping

$$\psi : V(F) \to V(H) : y_j^i \mapsto x_{jk'}^i$$

is an isomorphism of F and H. Therefore, without loss of generality, we may assume from now on that G is a cubic $(6,n)$-metacirculant graph $MC(6,n,\alpha,S_0,S_1,S_2,S_3)$ such that

$$S_0 = \emptyset, \quad S_1 = \{0\}, \quad S_2 = \emptyset, \quad \text{and} \quad S_3 = \{1\}. \tag{C1}$$

Since $S_1 = \{0\}$ by (C1), we see that for any $j \in Z_n$,

$$B_j = v_j^0 v_j^1 v_j^2 v_j^3 v_j^4 v_j^5 v_j^0$$

is a cycle of length 6. The direction on B_j from v_j^i to v_j^{i-1} is called negative.

We arrange vertices of G as in Table 1, columns and rows of which are indexed by elements of Z_6 and Z_n, respectively. Thus, the (j,i)-entry of this table, i.e., the element on the intersection of row j and column i, is v_j^i. For $t = 0,2,4$, the subgraph induced by G on vertices in columns t and $t+1$ of Table 1 is denoted by $G^{t/2}$. An edge of G of the type $v_j^i v_j^{i+1}$ is called an S_1-edge, and that of the type $v_j^i v_{j+\alpha^i}^i$ is an S_3-edge. It is easy to see that an S_1-edge joins two vertices of the same cycle B_j, whilst an S_3-edge joins two vertices of two distinct cycles B_j and B_h if $n > 1$. Moreover, every vertex v_j^i of G is incident with just one S_3-edge, $v_j^i v_{j+\alpha^i}^i$, and with two S_1-edges, $v_j^i v_j^{i+1}$ and $v_j^i v_j^{i-1}$. The S_1-edge $v_j^i v_j^{i-1}$ leaves v_j^i in the positive direction of B_j. Meanwhile, the S_1-edge $v_j^i v_j^{i+1}$ leaves v_j^i in the negative direction of B_j.

Table 1.

v_1^0	v_1^1	v_1^2	v_1^3	v_1^4	v_1^5
v_2^0	v_2^1	v_2^2	v_2^3	v_2^4	v_2^5
v_3^0	v_3^1	v_3^2	v_3^3	v_3^4	v_3^5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{n-2}^0	v_{n-2}^1	v_{n-2}^2	v_{n-2}^3	v_{n-2}^4	v_{n-2}^5
v_{n-1}^0	v_{n-1}^1	v_{n-1}^2	v_{n-1}^3	v_{n-1}^4	v_{n-1}^5
v_0^0	v_0^1	v_0^2	v_0^3	v_0^4	v_0^5

Now we describe the procedures $\mathscr{P}$ and $\mathscr{C}$ which produce paths and cycles of G, respectively. These procedures apply to a vertex v_h^t with t even as a starting vertex. The path and the cycle obtained by applying $\mathscr{P}$ and $\mathscr{C}$ to v_h^t are denoted by $\mathscr{P}(v_h^t, v_h^{t+1})$ and $\mathscr{C}(v_h^t)$, respectively. They are constructed as an alternating sequence of adjacent S_3- and S_1-edges of G as follows. Start $\mathscr{P}(v_h^t, v_h^{t+1})$ at v_h^t. Let v_j^i be chosen as a vertex of $\mathscr{P}(v_h^t, v_h^{t+1})$. If i is even, then the following edge of $\mathscr{P}(v_h^t, v_h^{t+1})$ is the unique S_3-edge which v_j^i is incident with. If i is odd, then the following edge of $\mathscr{P}(v_h^t, v_h^{t+1})$ is the S_1-edge which leaves v_j^i in the positive direction of the cycle B_j, i.e., the edge $v_j^i v_j^{i-1}$. Since 3 is an odd number, it is not difficult to see that vertices which have been chosen and are different to v_h^t cannot be chosen again. Therefore, we can continue in this way until reaching v_h^{t+1}, producing the path $\mathscr{P}(v_h^t, v_h^{t+1})$ of G. By joining the edge $v_h^{t+1} v_h^t$ to $\mathscr{P}(v_h^t, v_h^{t+1})$, we obtain the cycle $\mathscr{C}(v_h^t)$. By the definitions of $\mathscr{P}$ and $\mathscr{C}$, it is clear that we immediately have the following property of $\mathscr{P}(v_h^t, v_h^{t+1})$ and $\mathscr{C}(v_h^t)$.

Claim 1. (a) *For any vertex v_j^i of $\mathscr{P}(v_h^t, v_h^{t+1})$ with i even, the vertex-sets of $\mathscr{P}(v_h^i, v_h^{i+1})$ and $\mathscr{P}(v_h^t, v_h^{t+1})$ are the same. Moreover, $\mathscr{C}(v_j^i) = \mathscr{C}(v_h^t)$.*
(b) *If v_j^i with i even is not a vertex of $\mathscr{P}(v_h^t, v_h^{t+1})$, then the vertex-sets of $\mathscr{P}(v_h^i, v_h^{i+1})$ and $\mathscr{P}(v_h^t, v_h^{t+1})$ are disjoint.*

Claim 2. *Let $e = gcd(1 - \alpha + \alpha^2, n)$. Then for any $v_h^t \in V(G)$ with t even, the number of edges of G^i, $i = 0, 1, 2$, contained in $\mathscr{C}(v_h^t)$ is n/e. The subscripts of the vertices incident with these edges are congruent to each other modulo e.*

Proof. Let ρ and τ be the automorphism of G defined by $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Using ρ and τ^2, if necessary, and Claim 1, we may assume that $v_h^t = v_0^0$ and $i = 0$. By the definition of $\mathscr{C}$, we have

$$\mathscr{C}(v_0^0) = P(v_0^0)P(v_q^0)P(v_{2q}^0)P(v_{3q}^0)\ldots$$

where q is $(1 - \alpha + \alpha^2)$ reduced modulo n and

$$P(v_j^0) = v_j^0 v_{j+1}^3 v_{j+1}^2 v_{j+1+\alpha^2}^5 v_{j+1+\alpha^2}^4 v_{j+1+\alpha^2-\alpha}^1.$$

Since $gcd(q, n) = e$, the smallest positive integer b for which $bq \equiv 0 \pmod n$

is $b = n/e$. So, $\mathscr{C}(v_0^0)$ consists of n/e paths $P(v_{jq}^0)$ and edges of G^0 joining the terminal vertex of $P(v_{jq}^0)$ to the initial vertex of $P(v_{(j+1)q}^0)$.

It follows that the number of edges of G^0 contained in $\mathscr{C}(v_0^0)$ is n/e and the subscript of the vertices incident with these edges are congruent to each other modulo e.

Claim 3. *If $\mathscr{C}(v_0^0)$ contains at least two edges of B_0, then $\mathscr{C}(v_0^0)$ is a Hamilton cycle of G.*

Proof. By the definition of $\mathscr{C}(v_0^0)$, the edge $v_0^0v_0^1$ is contained in $\mathscr{C}(v_0^0)$, and among other edges of B_0 only $v_0^2v_0^3$ and $v_0^4v_0^5$ may be contained in $\mathscr{C}(v_0^0)$. Therefore, by the assumption of the claim, either $v_0^2v_0^3$ or $v_0^4v_0^5$ or both of them are contained in $\mathscr{C}(v_0^0)$.

Let ρ and τ be the automorphism of G defined as in the proof of Claim 2, i.e., $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Let $v_0^2v_0^3$ be an edge of $\mathscr{C}(v_0^0)$. By definition, $v_1^2v_1^3$ is also an edge of $\mathscr{C}(v_0^0)$. Since ρ maps the vertex v_0^2 of $\mathscr{C}(v_0^0)$ to the vertex v_1^2 of $\mathscr{C}(v_0^0)$, by the definitions of and by Claim 1, it is not difficult to see that $\rho(\mathscr{C}(v_0^0)) = \mathscr{C}(v_0^0)$. Thus,

$$\rho(v_1^2)\rho(v_1^3) = v_2^2v_2^3,$$

$$\rho(v_2^2)\rho(v_2^3) = v_3^2v_3^3, \ldots$$

are also contained in $\mathscr{C}(v_0^0)$. This implies that all edges of G^1 are contained in $\mathscr{C}(v_0^0)$. From this, by the definition of $\mathscr{C}$ it follows that all edges of G^0 and G^2 are also contained in $\mathscr{C}(v_0^0)$. Thus, all vertices of G are contained in $\mathscr{C}(v_0^0)$ and $\mathscr{C}(v_0^0)$ is a Hamilton cycle of G.

Let $v_0^4v_0^5$ be an edge of $\mathscr{C}(v_0^0)$. Since v_0^0 and v_0^4 are vertices of $\mathscr{C}(v_0^0)$ and $\tau^2(v_0^4) = v_0^0$, by the definitions of and by Claim 1, it is also not difficult to see that $\tau^2(\mathscr{C}(v_0^0)) = \mathscr{C}(v_0^0)$. Therefore, $\tau^2(v_0^0)\tau^2(v_0^1) = v_0^2v_0^3$ is also an edge of $\mathscr{C}(v_0^0)$. We return to the case of the preceding paragraph. ■

By Claim 3 we may assume from now on that

$$v_0^2, v_0^3, v_0^4, v_0^5 \notin V(\mathscr{C}(v_0^0)). \tag{C2}$$

By this assumption and by using the automorphism ρ and τ, we immediately have Claim 4.

Claim 4. *For any $j \in Z_n$ and $v_h^t \in V(G)$ with t even, $\mathscr{C}(v_h^t)$ contains at most one edge of B_j.*

Now we consider separately two cases.

Case 1.
$$e = gcd(1 - \alpha + \alpha^2, n) = n. \tag{C3}$$

In this case, $\alpha^2 - \alpha \equiv n - 1$, $\alpha^2 \equiv \alpha - 1$, and $1 + \alpha^2 \equiv \alpha \pmod{n}$. In many computations below we will use these congruences without mentioning them.

By Claim 2, each $G^i, i = 0, 1, 2$, contains one edge of $\mathscr{C}(v_h^t)$. In particular, we have Part 1 of Table 2.

Table 2.

<table>
<tr><th rowspan="2">Part</th><th rowspan="2">Cycles</th><th colspan="3">Subscripts of vertices of G^i contained in C_j</th></tr>
<tr><th>G^0</th><th>G^1</th><th>G^2</th></tr>
<tr><td>1</td><td>$C_0 = \mathscr{C}(v_0^0)$</td><td>0</td><td>1</td><td>α</td></tr>
<tr><td>2</td><td>C_1</td><td>0
$n-1$
$n-\alpha$</td><td>0
$n-\alpha+1$
1</td><td>0
α
$\alpha-1$</td></tr>
<tr><td>2</td><td>C_1</td><td>0
$n-1$
$n-2$
$n-\alpha$
$n-\alpha-1$</td><td>0
$n-1$
$n-\alpha+1$
$n-\alpha$
1</td><td>0
$n-1$
α
$\alpha-1$
$\alpha-2$</td></tr>
</table>

Now we again divide Case 1 into two subcases.

Subcase 1.1. $\alpha \leq n - \alpha$.

In the following three steps, we construct a sequence of cycles $C_0, C_1, C_2, \ldots, C_{n-1}$ of G with the property that all vertices of each cycle C_j with $j < n-1$ are contained in the cycle C_{j+1} following it in the sequence. Denote by H_j^i the set of subscripts of all vertices of G^i contained in C_j, where $i = 0, 1, 2$ and $j = 0, 1, \ldots, n-1$.

Step 0. Set $C_0 = \mathscr{C}(v_0^0)$.

Step 1. The construction of the cycle C_j with $j \geq 1$, when all vertices of $\mathscr{P}(v_{n-j+1}^4, v_{n-j+1}^5)$ are not contained in C_{j-1}.

The cycle C_j is constructed recursively as follows. Let the cycle C_{j-1} be constructed and be such that $v_{n-j+1}^1 v_{n-j+1}^0$ is only the edge of B_{n-j+1} contained in C_{j-1} and all vertices of $\mathscr{P}(v_{n-j+1}^4, v_{n-j+1}^5)$ are not contained in C_{j-1}. If all vertices of $\mathscr{P}(v_{n-j+1}^2, v_{n-j+1}^3)$ are also not contained in C_{j-1}, then C_j is constructed from C_{j-1} by replacing the edge $v_{n-j+1}^1 v_{n-j+1}^0$ by the path $(v_{n-j+1}^1 v_{n-j+1}^2)$ $\mathscr{P}(v_{n-j+1}^2, v_{n-j+1}^3)(v_{n-j+1}^3 v_{n-j+1}^4)\mathscr{P}(v_{n-j+1}^4, v_{n-j+1}^5)(v_{n-j+1}^5 v_{n-j+1}^0)$. If all vertices of $\mathscr{P}(v_{n-j+1}^2, v_{n-j+1}^3)$ are already contained in C_{j-1}, then $C_j = C_{j-1}$.

By Claims 1 and 4 and by introduction on j, it is not difficult to see that the following claim is true.

Claim 5. *Let C_j be one of the cycles constructed in Step 1. Then*
(a) *All vertices of C_{j-1} are contained in C_j.*
(b) *For any $\mathscr{P}(v_h^t, v_h^{t+1})$ with t even, either all vertices of $\mathscr{P}(v_h^t, v_h^{t+1})$ are not contained in C_j.*
(c) *If $j-1 \neq 2b(\alpha-1)$ with b a nonnegative integer and $C_j \neq C_{j-1}$, then vertices of Gi contained in C_j, but not contained in C_{j-1}, have subscripts of the form $h-1$,*

where h is a subscript in H^i_{j-1} such that $h-1 \notin H^i_{j-1}$. Conversely, if $j-1 \neq 2b(\alpha-1)$ with b a nonnegative integer, $C_j \neq C_{j-1}$ and h is a subscript in H^i_{j-1} such that $h-1 \notin H^i_{j-1}$, then all the vertices of G^i having the subscript $h-1$ are contained in C_j, but not in C_{j-1}.
(d) *If $j-1 = 2b(\alpha-1)$ with b a nonnegative integer and $C_j \neq C_{j-1}$, then vertices of G^i contained in C_j, but not in C_{j-1}, have subscripts in the form $h-1$ or $h-\alpha$, where h is a subscript in H^i_{j-1} such that $h-1 \notin H^i_{j-1}$. Conversely, if $j-1 = 2b(\alpha-1)$ with b a nonnegative integer, $C_j \neq C_{j-1}$ and h is a subscript in H^i_{j-1} such that $h-1 \notin H^i_{j-1}$, then all the vertices of G^i having the subscript $h-1$ or $h-\alpha$ are contained in C_j, but not in C_{j-1}.* ■

In Parts 2 and 3 of Table 2, as an example, we list the subscripts of vertices of C_1 and C_2, using Claim 5 and H^i_0 listed in Part 1 of this table.

Constructions of C_j in this step must be repeated until we get in the first time the cycle C_j with the property that all vertices of $\mathscr{P}(v^4_{n-j}, v^5_{n-j})$ are contained in C_j. By Claims 1, 4, and 5 and by introduction on j, we can easily prove that constructions of Step 1 can be used until $j = n-\alpha$ and all vertices of all $\mathscr{P}(v^4_{n-j}, v^5_{n-j})$ with $n-\alpha \leq j \leq n-1$ are contained in $C_{n-\alpha}$. From $j = n-\alpha+1$, we use constructions in Step 2.

Step 2. The construction of the cycle C_j with $j \geq n-\alpha+1$, when all vertices of $\mathscr{P}(v^4_{n-j+1}, v^5_{n-j+1})$ are already contained in C_{j-1}.

The cycle C_j is constructed recursively as follows. Let the cycle C_{j-1} be constructed such that all vertices of $\mathscr{P}(v^4_{n-j+1}, v^5_{n-j+1})$ are contained in C_{j-1}. If the edge $v^5_{n-j+1}v^4_{n-j+1}$ is the only edge of B_{n-j+1} contained in C_{j-1}, then C_j is obtained from C_{j-1} by replacing $v^5_{n-j+1}v^4_{n-j+1}$ by the path $(v^5_{n-j+1}v^0_{n-j+1})\mathscr{P}(v^0_{n-j+1}, v^1_{n-j+1})(v^1_{n-j+1}v^2_{n-j+1})\mathscr{P}(v^2_{n-j+1}, v^3_{n-j+1})(v^3_{n-j+1}v^4_{n-j+1})$. If the edge $v^5_{n-j+1v}v^4_{n-j+1}$ is not the only edge of B_{n-j+1} contained in C_{j-1}, then $C_j = C_{j-1}$.

By Claims 1, 4, and 5 and by introduction on j, it is not difficult to see that the following claim is true.

Claim 6. *Let C_j be one of the cycles constructed in Step 2. Then*
(a) *All vertices of C_{j-1} are contained in C_j.*
(b) *For any $\mathscr{P}(v^t_h, v^{t+1}_h)$ with t even, either all vertices of $\mathscr{P}(v^t_h, v^{t+1}_h)$ are contained in C_j or all vertices of $\mathscr{P}(v^t_h, v^{t+1}_h)$ are not contained in C_j.*
(c) *If $C_j \neq C_{j-1}$, then vertices of G^i contained in C_j, but not contained in C_{j-1}, have subscripts of the form $h-1$ or $h-2$, where h is a subscript in H^i_{j-1} such that $h-1 \notin H^i_{j-1}$. Conversely, if $C_j \neq C_{j-1}$ and h is a subscript in H^i_{j-1} such that $h-1 \notin H^i_{j-1}$, then all the vertices of G^i having the subscripts $h-1$ or $h-2$ are contained in C_j, but not contained in C_{j-1}.* ■

We have noted earlier that all vertices of all $\mathscr{P}(v^4_{n-j}, v^5_{n-j})$ with $n-\alpha \leq$

$j \leq n-1$ are contained in $C_{n-\alpha}$. Therefore, constructions in Step 2 can be used until $j = n-1$.

Claim 7. *If $\alpha \leq n-\alpha$, then C_{n-1} is a Hamilton cycle of G.*

Proof. If $\alpha = 1$, then $1-\alpha+a^2 = 1$. If $\alpha = 2$, then $1-\alpha+\alpha^2 = 3$. Since $gcd(1-\alpha+\alpha^2, n) = n$, we must have $n = 1$ in the former case and $n = 3$ in the latter case. This contradicts the assumption that $\alpha \leq n-\alpha$. Thus, $\alpha > 2$. We have noted after Claim 5 that constructions in Step 1 can be used until we get $C_{n-\alpha}$. Moreover, since $2 < \alpha \leq n-\alpha$, $C_1 \neq C_2 \neq \cdots \neq C_{\alpha-1} = C_\alpha = C_{\alpha+1} = \cdots = C_{2(\alpha-1)} \neq C_{2\alpha-1} \neq C_{2\alpha} \neq \cdots \neq C_{3(\alpha-1)} = C_{3\alpha-2} = C_{3\alpha-1} = \dots$. In a word, the sequence $C_1, C_2, \dots, C_{n-\alpha}$ is an alternating sequence of subsequences of distinct successive cycles C_j and subsequences of equal successive cycles C_j. All subsequences of the sequence, other than the last one, have $\alpha-1$ successive cycles C_j. Moreover, the first subsequence is a subsequence of $\alpha-1$ distinct successive cycles. Let S be the last subsequence and d the number of its terms. Then we trivially have $0 < d \leq \alpha-1$ and $d = (n-\alpha) - t(\alpha-1)$ with t even if S is a subsequence of distinct successive cycles C_j and odd if S is a subsequence of equal successive cycles C_j.

We have noted after Claim 5 that all vertices of all $\mathscr{P}(v^4_{n-j}, v^5_{n-j})$ with $n-\alpha \leq j \leq n-1$ are already contained in $C_{n-\alpha}$. Therefore, we can proceed to construct the remaining cycles C_j, $j = n-\alpha+1, n-\alpha+2, \dots, n-1$, in Step 2. Let z be the number of subscripts of vertices of G^1, not contained in $H^1_{n-\alpha}$. We show now that z is always even. There are two cases to consider.
(i) S consists of distinct successive cycles C_j.

In this case, by Claim 5, it is not difficult to see that

$$z = (\alpha-1) - d.$$

Recall that $gcd(\alpha, n) = 1$. Therefore, since $gcd\,(1-\alpha+\alpha^2, n) = n$, the number n must be odd. If α is also odd, then both $\alpha-1$ and $n-\alpha$ are even. This implies that d is even. So, z is even in this case and d is odd. So, z is even.
(ii) S consists of equal successive cycles C_j.

In this case, also by Claim 5, we have

$$z = (\alpha-1) - [(\alpha-1) - d] = d.$$

As in the earlier case, since $gcd(1-\alpha+\alpha^2, n) = n$, the number n must be odd. If it is odd, then both $\alpha-1$ and $n-\alpha$ are even. Hence, $z = d$ is even. If it is even, then both $\alpha-1$ and $n-\alpha$ are odd. Since t is odd in this case, $z = d$ is even.

From the constructions of C_j in Step 1, it follows that there is the unique subscript h in $H^1_{n-\alpha}$ satisfying $h-1 \notin H^1_{n-\alpha}$ and all the vertices of G^0 and G^1 not contained in $C_{n-\alpha}$ can be listed as in Table 3 which is a subtable of Table 1.

Table 3.

v^0_{h-z-1}	v^1_{h-z-1}		
v^0_{h-z}	v^1_{h-z}	v^2_{h-z}	v^3_{h-z}
v^0_{h-z+1}	v^1_{h-z+1}	v^2_{h-z+1}	v^3_{h-z+1}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
v^0_{h-3}	v^1_{h-3}	v^2_{h-3}	v^3_{h-3}
v^0_{h-2}	v^1_{h-2}	v^2_{h-2}	v^3_{h-2}
		v^2_{h-1}	v^3_{h-1}

Recall that all vertices of all $\mathscr{P}(v^4_{n-j}, v^5_{n-j})$ with $n-\alpha \leq j \leq n-1$ are already contained in $C_{n-\alpha}$. Therefore, by Table 3 and the constructions of C_j in Step 2, we see that $C_{n-\alpha} = C_{n-\alpha+1} = \cdots = C_{n-h+1} = C_{n-h+2}$. By Claim 6, v^0_{h-2}, v^0_{h-3}, v^1_{h-2}, v^1_{h-3}, v^2_{h-1}, v^2_{h-2}, v^3_{h-1}, and v^3_{h-2} are vertices of $C_{n-h+3} = C_{n-h+4}$; v^0_{h-4}, v^0_{h-5}, v^1_{h-4}, v^1_{h-5}, v^2_{h-3}, v^2_{h-4}, v^3_{h-3}, and v^3_{h-4} are vertices of $C_{n-h+5} = C_{n-h+6} \cdots$. We have shown earlier that z is even. Therefore, $v^0_{h-z}, v^0_{h-z-1}, v^1_{h-z}, v^1_{h-z-1}, v^2_{h-z+1}, v^2_{h-z}$, and v^3_{h-z} are vertices of $C_{n-h+z+1}$. Thus, all vertices in Table 3 are contained in $C_{n-h+z+1}$. By the definition of Table 3, this means that all vertices of G^0 and G^1 are contained in $C_{n-h+z+1}$. So, $C_{n-h+z+1} = \cdots = C_{n-1}$ and C_{n-1} contains all vertices of G^0 and G^1. By Claim 6 and the definition of the procedure $\mathscr{P}$, it follows that all vertices of G are contained in C_{n-1} and therefore, C_{n-1} is a Hamilton cycle of G. ■

Subcase 1.2. $\alpha > n - \alpha$.

As in Subcase 1.1, here we also construct a sequence of cycles $D_0, D_1, D_2, \ldots, D_{n-1}$ of G with the property that all vertices of each cycle D_j with $j < n-1$ are contained in the cycle D_{j+1} following it in the sequence. In some sense, these constructions are dual to those of Subcase 1.1. Therefore, we only describe the constructions of D_j here, leaving to the reader with the detailed proof of the fact that they indeed produce a Hamilton cycle D_{n-1} of G.

Step 0. Set $D_0 = \mathscr{C}(v^2_1) = C_0$.

Step 1. The construction of the cycle D_j with $j \geq 1$, when all vertices of $\mathscr{P}(v^4_j, v^5_j)$ are not contained in D_{j-1}.

The cycle D_j is constructed recursively as follows. Let the cycle D_{j-1} be constructed such that $v^3_j v^2_j$ is only the edge of B_j contained in D_{j-1} and all vertices of $\mathscr{P}(v^4_j, v^5_j)$ are also not contained in D_{j-1}. If all vertices of $\mathscr{P}(v^0_j, v^1_j)$ are already contained in D_{j-1}, then $D_j = D_{j-1}$.

Constructions of D_j in this step must be repeated until we get in the first time the cycle D_j with the property that all vertices of $\mathscr{P}(v^4_{j+1}, v^5_{j+1})$ are contained in D_j. By introduction, we can prove that constructions in Step 1 can be used until $j = \alpha - 1$ and all vertices of all $\mathscr{P}(v^4_j, v^5_j)$ with $\alpha \leq j \leq n-1$ are contained in $D_{\alpha-1}$. From $j = \alpha$, we use constructions in Step 2.

Step 2. The construction of the cycle D_j with $\supset \geq \alpha$, when all vertices of $\mathscr{P}(v_j^4, v_j^5)$ are already contained in D_{j-1}.

The cycle D_j is constructed recursively as follows. Let the cycle D_{j-1} be constructed such that all vertices of $\mathscr{P}(v_j^4, v_j^5)$ are contained in D_{j-1}. If the edge $v_j^5 v^4$ is only the edge of B_j contained in D_{j-1}, then D_j is obtained from D_{j-1} by replacing $v_j^5 v_j^4$ with the path $(v_j^5 v_j^4)$ $\mathscr{P}(v_j^0, v_j^1)\, v_j^1 v_j^2\, \mathscr{P}(v_j^2, v_j^3)\, v_j^3 v_j^4$. If the edge $v_j^5 v_j^4$ is not the only edge of B_j contained in D_{j-1}, then $D_j = D_{j-1}$.

We have noted above that all vertices of all $\mathscr{P}(v_j^4, v_j^5)$ with $\alpha \leq j \leq n-1$ are contained in $D_{\alpha-1}$. Therefore, constructions in this step can be used until $j = n-1$.

Claim 8. *If $\alpha > n - \alpha$, then D_{n-1} is a Hamilton cycle of G.*

Proof. The proof of this claim is similar to that of Claim 7. Therefore, we leave it to the reader. ■

Case 2. $e = gcd(1 - \alpha + \alpha^2, n) < n$.

Let ρ be the automorphism of G defined by $\rho(v_j^i) = v_{j+1}^i$. Then $\beta = \rho^{(1-\alpha+\alpha^2)}$ is semiregular, i.e., all cycles in the disjoint cycle decomposition of β have the same length. We define the quotient graph G/β of G as follows. The vertices of G/β are the orbits of the subgroup $\langle\beta\rangle$ generated by β and two such vertices are adjacent if and only if there is an edge in G joining a vertex of one corresponding orbit to a vertex in the other orbit. It is not difficult to verify that G/β is isomorphic to the cubic $(6, e)$-metacirculant graph $\bar{G} = MC(6, e, \bar{\alpha}, \bar{S}_0, \bar{S}_1, \bar{S}_2, \bar{S}_3)$ where $0 \leq \bar{\alpha} < e$ and $\bar{\alpha} \equiv \alpha \pmod{e}$, $\bar{S}_0 = \emptyset$, $\bar{S}_1 = \{0\}$, $\bar{S}_2 = \emptyset$, $\bar{S}_3 = \{1\}$. Therefore, we can identify G/β with the graph $\bar{G} = MC(6, e, \bar{\alpha}, \bar{S}_0, \bar{S}_1, \bar{S}_2, \bar{S}_3)$. Since G is connected, it is not difficult to see that $\bar{G}$ is also connected.

Let $\alpha = \bar{\alpha} + t_1 e$ and $1 - \alpha + \alpha^2 = t_2 e$ with t_1 and t_2 appropriate integers. We have

$$t_2 e = 1 - \alpha + \alpha^2 = 1 - \bar{\alpha} - t_1 e + (\bar{\alpha} + t_1 e)^2$$
$$= (1 - \bar{\alpha} + \bar{\alpha}^2) + (t_1^2 e + 2t_1\bar{\alpha} - t_1)e.$$

Therefore, $1 - \bar{\alpha} + \bar{\alpha}^2 = t_3 e$ for some integer t_3. This implies that $gcd(1 - \bar{\alpha} + \bar{\alpha}^2, e) = e$. Thus, $\bar{G}$ satisfies Conditions (C1) and (C3). Since G satisfies Condition (C2), by Claim 2 and the definition of $\bar{G}$, it is not difficult to see that $\bar{G}$ also satisfies Condition (C2) and therefore, all Conditions (C1)–(C3) are satisfied for $\bar{G}$. By Case 1, either $\bar{\alpha} \leq e - \bar{\alpha}$ and $\bar{G}$ possesses the Hamilton cycle $\bar{C}_{e-1}$ constructed by Steps 0–2 in Subcase 1.1, or $\bar{\alpha} > e - \bar{\alpha}$ and $\bar{G}$ possesses the Hamilton cycle $\bar{D}_{e-1}$ constructed by Steps 0–2 in Subcase 1.2.

Let $\bar{\alpha} \leq e - \bar{\alpha}$, $V(\bar{G}) = \{\bar{v}_j^i : i \in Z_6, j \in Z_e\}$ and $C_0, C_1, \ldots, C_{e-1}$ be the sequence of cycles of $\bar{G}$ constructed by Steps 0–2 in Subcase 1.1. We naturally lift the path $\mathscr{P}(\bar{v}_h^t, \bar{v}_h^{t+1})$ with t even of $\bar{G}$ to the path $\mathscr{P}(v_h^t, v_h^{t+1})$ of G. A vertex v_j^i of G is said to be contained in a vertex $\bar{v}_h^t$ of $\bar{G}$ if v_j^i is contained in the orbit of

$\langle\rho^{(1-\alpha+\alpha^2)}\rangle$ which corresponds to $\bar{v}_h^t$. By Claim 2 and the definition of $\bar{G}$, we immediately have the following result.

Claim 9. *For any $\mathscr{P}(\bar{v}_h^t, \bar{v}_h^{t+1})$ with t even, all the vertices of G are contained in one of the vertices of $\mathscr{P}(\bar{v}_h^t, \bar{v}_h^{t+1})$ and only they are vertices of $\mathscr{P}(v_h^t, v_h^{t+1})$.*

Now we lift each cycle $\bar{C}_j$, $j = 0, 1, \ldots, e-1$, to a cycle C_j of G with the property that all the vertices of G are contained in one of the vertices of $\bar{C}_j$ and only they are vertices of C_j. The cycle $\bar{C}_0 = \mathscr{C}(\bar{v}_0^0)$ is lifted to the cycle $\mathscr{C}(v_0^0)$ of G which is the cycle C_0. By Claim 9, C_0 has the desired property. Suppose that the cycle $\bar{C}_{j-1}$ has lifted to a cycle C_{j-1} of G with the desired property. If $\bar{C}_j = \bar{C}_{j-1}$, then we set $C_j = C_{j-1}$. If $\bar{C}_j \neq \bar{C}_{j-1}$, then by constructions in Steps 1 and 2 of Subcase 1.1, there exists an edge of $\bar{C}_{j-1}$ of the form $\bar{v}_{e-j+1}^{t+1}\bar{v}_{e-j+1}^t$ where $t = 0$ or 4, such that it is the only edge of $\bar{B}_{e-j+1}$ contained in $\bar{C}_{j-1}$. Moreover, the cycle $\bar{C}_j$ is obtained from $\bar{C}_{j-1}$ by replacing the edge $\bar{v}_{e-j+1}^{t+1}\bar{v}_{e-j+1}^t$ by the path $(\bar{v}_{e-j+1}^{t+1}\bar{v}_{e-j+1}^{t+2})$ $\mathscr{P}(\bar{v}_{e-j+1}^{t+2}, \bar{v}_{e-j+1}^{t+3})$ $(\bar{v}_{e-j+1}^{t+3}\bar{v}_{e-j+1}^{t+4})$ $\mathscr{P}(\bar{v}_{e-j+1}^{t+4}, \bar{v}_{e-j+1}^{t+5})$ $(\bar{v}_{e-j+1}^{t+5}\bar{v}_{e-j+1}^t)$. By the property of C_{j-1}, $(\bar{v}_{e-j+1}^{t+1}\bar{v}_{e-j+1}^t)$ is also the only edge of B_{e-j+1} contained in C_{j-1}. By Claims 5 and 9 and the property of C_{j-1}, it is not difficult to see that all vertices of $\mathscr{P}(v_{e-j+1}^{t+2}, v_{e-j+1}^{t+3})$ and $\mathscr{P}(v_{e-j+1}^{t+4}, v_{e-j+1}^{t+5})$ are not contained in C_{j-1}. Therefore, we can construct the cycle C_j from C_{j-1} by replacing the edge $v_{e-j+1}^{t+1}v_{e-j+1}^t$ by the path $(v_{e-j+1}^{t+1}, v_{e-j+1}^{t+2})$ $\mathscr{P}(v_{e-j+1}^{t+2}, v_{e-j+1}^{t+3})$ $(v_{e-j+1}^{t+3}, v_{e-j+1}^{t+4})$ $\mathscr{P}(v_{e-j+1}^{t+4}, v_{e-j+1}^{t+5})$ $(v_{e-j+1}^{t+5}, v_{e-j+1}^t)$. By Claim 9 and the property of C_{j-1}, it is clear that in both cases C_j also possesses the property that all the vertices of G are contained in one of the vertices of $\bar{C}_j$ and only these vertices are vertices of C_j. Thus, by induction, we can construct a sequence $C_0, C_1, C_2, \ldots, C_{e-1}$ of cycles C_j of G from the sequence $\bar{C}_0, \bar{C}_1, \bar{C}_2, \ldots, \bar{C}_{e-1}$ of cycles $\bar{C}_j$ of $\bar{G}$ with the property that all the vertices of G contained in one of the vertices of $\bar{C}_j$ are contained in C_j. Since $\bar{C}_{e-1}$ is a Hamilton cycle of $\bar{G}$, it is easy to see that C_{e-1} is a Hamilton cycle of G.

Now let $\bar{\alpha} > e - \bar{\alpha}$ and $\bar{D}_0, \bar{D}_1, \bar{D}_2, \ldots, \bar{D}_{e-1}$ be the sequence of cycles of $\bar{G}$ constructed by Steps 0–2 in Subcase 1.2. By arguments similar to those used above, we can construct a Hamilton cycle D_{e-1} of G.

The proof of the theorem is complete. ■

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