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Maximal Periodic Subgroups of Finite Dimensional Locally Compact Groups

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Abstract. Let G be a finite dimensional locally compact group with compact quotient G/G_0. We prove that if G/G_0 is an abstract periodic group, then the maximal periodic subgroups of G are divided in a finite number of conjugacy classes.

1. Introduction

The periodic subgroups are of interest both in abstract group theory and in topological group theory. Many interesting results concerning periodic groups were obtained in [3], [4], [6], [8]. One of the most important problems is a classification of the maximal periodic subgroups under conjugation.

In [5], V. P. Platonov and Nguyen Quoc Thi studied this problem for the finite dimensional locally compact groups with a finite number of connected components. In this work, we prove the following result, which contains the main result in [5] as a particular case.

Theorem. *Let G be a finite dimensional locally compact group with compact quotient G/G_0, where G_0 denotes the connected component of identity of G. If G/G_0 is an abstract periodic group, then in G there are only a finite number of conjugacy classes of maximal periodic subgroups.*

2. Proof of the Theorem

Following [5], if P is a maximal periodic subgroup of G, then $\bar{P}$ is compact. By virtue of the well-known Cartan–Maltsev–Iwasawa theorem, the maximal compact subgroups of G are conjugate , so we can assume that G is a compact group.

In the case when G/G_0 is a finite group, our theorem is exactly the main theorem in [5].

Now let G be a compact group with infinite abstract periodic quotient G/G_0 where G_0 is a Lie group.

First, we show that there exists in G a normal subgroup K such that G/K is a Lie group. Indeed, following [9] there exists in G a totally disconnected normal subgroup K such that G/K is a Lie group, due to our assumption that $\dim G$ is finite. Pose $H = G_0 \cap K$. Then H is a totally disconnected normal subgroup of G_0, and therefore, H is a finite Abelian group. We prove that K is an abstract periodic group. Suppose that x is an arbitrary element of K. Since G/G_0 is an abstract periodic group, we can assert that $x^n \in G_0$ for each integer n. Since K is a group and $x \in K$, we must have $x^n \in K$, and therefore, $x^n \in H$. As mentioned, H is a finite group, therefore, the order of element x must be finite. Hence, K is an abstract periodic group. According to [5], the maximal periodic subgroups of G/K are divided into a finite number of conjugacy classes.

Suppose M' to be a maximal periodic subgroup of Lie group G/K. Denote $M = \Phi^{-1}(M')$, the preimage of M' under the natural epimorphism $\Phi : G \to G/K$.

Remark first that, since the maximal periodic subgroups M' are divided in a finite number of conjugacy classes in G/K, so are the subgroups of type $M = \Phi^{-1}(M')$ in G.

Suppose that x is an arbitrary element of M. Since M' is an abstract periodic subgroup of G/K, there exists an integer number n such that $x^n \in K$. Then as previously mentioned, since K is an abstract periodic group, there exists an integer l such that $(x^n)^l = e$ and, therefore, the order of element x must be finite. Hence, M is a periodic subgroup. Since M' is a maximal periodic subgroup of G/K, M is a *maximal* periodic subgroup G. Hence, the number of classes of maximal periodic subgroups of G is equal to the number of classes of maximal periodic subgroups of G/K.

Finally, suppose G_0 is a connected compact group such that the quotient G/G_0 is an abstract periodic group. Remark that $\dim G$ is finite. Following [9], there exists in G a totally disconnected normal subgroup J such that G/J is a Lie group. Put $L = G_0 \cap J$. The group L is compact and totally disconnected. Therefore, L is contained in the center of G_0. We have the isomorphic topological groups

$$G_0J/J \approx G_0/(G_0 \cap J)\,.$$

Since G_0J/J is a closed subgroup of a Lie group G/J, $G_0/(G_0 \cap J) = G_0/L$ is a Lie group. It is easy to see that G_0/L is the connected component of the identity of G/L, and in G/L the maximal periodic subgroups are divided into a finite number of conjugacy classes.

Denote by $\psi : G \to G/L$ the natural epimorphism. Suppose N' is a maximal periodic subgroup of group G/L and set $N = \psi^{-1}(N')$.

If N is a totally disconnected group, then $N/N_0 = N/\{e\}$ is an abstract periodic group. Hence, N is a maximal periodic subgroup of G.

If $N_0 \neq \{e\}$, then $\overline{N}_0'$ is a torus of $\overline{N}'$; see [5].

Denote $p = \psi_{|\overline{N}} : \overline{N} \to \overline{N}'$. Since ψ is an open epimorphism, $p(\overline{N}_0)$ is the connected component of identity of N', and therefore, $p(\overline{N}_0) = \overline{N}_0'$. Since N' is a periodic subgroup of G/L, it follows from [5] that $p(\overline{N}_0)$ is a torus.

Suppose T is now a maximal torus of $\overline{N}_0$, then $p(T)$ is a maximal torus of $\overline{N}_0'$, and therefore,

$$p(T) = \overline{N}'_0 = p(\overline{N}_0)\,.$$

Remark that $\operatorname{Ker} p \cap \overline{N}_0 = \overline{N}_0 \cap L$. It is easy to see that $\overline{N}_0 \cap L$ is a compact and totally disconnected subgroup and therefore, is contained in the center of $\overline{N}_0$. Hence, $T = \overline{N}_0$, so $\dim T = \dim \overline{N}_0$ is finite and $T/(L \cap T) = \overline{N}_0/(\overline{N}_0 \cap T)$ is a Lie group. Since $L \cap T = L \cap \overline{N}_0$ is a compact totally disconnected subgroup of the finite dimensional torus T, $L \cap T$ is a finite group and therefore, T is a Lie group.

In this case, the connected component of identity of $\overline{N}$ is a Lie group $\overline{N}_0 = T$, and hence, the maximal periodic subgroup of $\overline{N}$ are divided into a finite number of conjugacy classes.

Note that if N'_1 and N'_2 are conjugate in G/L then $\overline{\psi^{-1}(N'_1)}$ and $\overline{\psi^{-1}(N'_2)}$ are also conjugate in G. Hence, the maximal periodic subgroups of G are divided into a finite number of conjugacy classes. The proof is complete.

3. The Maximal Periodic Subgroups Satisfying the Minimal Condition for Closed Abelian Subgroups

In this section, we consider the following condition, which will be called the minimal condition:

Every strictly diminishing sequence of closed Abelian subgroups

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

stabilized in some finite step.

Corollary 1. *Let C be a locally compact group with compact quotient G/G_0. If G satisfies the minimal condition for closed Abelian subgroups, then the maximal periodic subgroups of G are divided into a finite number of conjugacy classes.*

Proof. Let g be an arbitrary element of G. Because G is a compact group satisfying the minimal condition for closed Abelian subgroups, we conclude that $\overline{\{g\}}/\overline{\{g\}}_0$ is also a compact Abelian totally disconnected group satisfying the minimal condition for closed Abelian subgroups. Since in each totally disconnected compact Abelian group there exists a topological base at identity, consisting of open normal subgroups, the group $\overline{\{g\}}/\overline{\{g\}}_0$ must be finite. There exists, therefore, an integer m such that $g_m \in G_0$. Hence, G/G_0 is a periodic group. According to [7], G_0 is a Lie group and hence $\dim G_0$ is finite. Since G/G_0 is a totally disconnected compact group, $\dim(G/G_0) = 0$. Hence, $\dim G$ must be finite. Following our theorem, the maximal periodic subgroups are divided into a finite number of conjugacy classes. The corollary is proved.

Proposition. *Let G be a finite dimensional locally compact with periodic compact quotient G/G_0. If G is a solvable projective group, then the maximal periodic subgroups of G are conjugate.*

Proof. By [5] and the well-known theorem of Cartan–Maltsev–Iwasawa, we can assume that G is a compact group.

First, we prove that G_0 is a solvable group.

By virtue of the Yamabe theorem [10],

$$G_0 = \lim_{\longleftarrow} (G^0_\beta, \varphi_{\alpha\beta}, \alpha < \beta) ,$$

where G^0_β are the connected solvable projective Lie groups. Therefore, G^0_β are the solvable groups and hence, G_0 is the solvable group. Moreover, since G is a compact group, G_0 is also a compact group and hence, G_0 is a torus. Pose $G_0 = T$.

Let G/T be a finite group. Suppose T_M is the maximal periodic subgroup of T. Consider $G^* = G/T_M$ and $T^* = T/T_M$. If $G^*/T^* \approx G/T$, then G^*/T^* is a finite group. Since G^*/T^* has no nontrival periodic element, we have that $G^* = M^*.T^*$, $M^* \cap T^* = \{e^*\}$, and maximal periodic subgroups are conjugate in G^*. If $M = \varphi^{-1}(M^*)$ is a maximal periodic subgroup of G, where $\varphi : G \to G^*$ is a canonical projection, then the maximal periodic subgroups of G are conjugate [6].

Let G be a Lie group. If G/G_0 is a totally disconnected Lie group, then G/G_0 must be a finite group. Therefore, maximal periodic subgroups are conjugate.

In the general case, we remark that dim$G$ is finite. According to [9], there exists in G a totally disconnected normal subgroup K such that G/K is a Lie group. By similar arguments (see the proof of the theorem), we conclude that the maximal periodic subgroups of G are conjugate. ■

From this proposition and Corollary 1, we derive the following.

Corollary 2. *Let G be a locally compact group with compact quotient G/G_0. If G is a solvable projective group satisfying the minimal condition for closed Abelian subgroups, then the maximal periodic subgroups of G are conjugate.*

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