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Approximately Continuous Functions in a Measure Space

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Abstract. The notion of approximately continuous functions in a measure space is introduced by using the concept of density topology. Some theorems on properties of such a function are proved.

1. Introduction

Approximately continuous functions are a major source of formulating the concept of density topology, a subject of current interest to many mathematicians. Some basic properties of approximately continuous functions may be seen in [5], [6], [12]. A few generalizations with the domain (sometimes the range also) being abstract spaces may be seen in [7], [11].

The definition of approximately continuous functions basically depends on the structure of open sets in the space. If we consider an arbitrary measure space, then as such, it has no open set. But Martin [8] showed that by using the concept of the density of a set in such a space, one can define in it a topology, known as the density topology. Here, our purpose is to prove some basic theorems on approximately continuous functions in a measure space.

2. Preliminaries

Let (X, S, m) be a complete measure space in which $m(X) = 1$. If $E \subset X$, then the outer measure $m^*(E)$ of E is defined by

$$m^*(E) = \inf\{m(F) : E \subset F \in S\}.$$

Let $\mathscr{K}$ be a collection [8] of sequences $\{K_n\}$ of sets from S such that, for each $x \in X$, there is at least one sequence $\{K_n\} \in \mathscr{K}$ such that

(i) $x \in K_n$ for all n and

(ii) $m(K_n) \to 0$ as $n \to \infty$.

Any sequence of sets satisfying (i) and (ii) is said to converge to x. The collection of all sequences in $\mathscr{K}$ which converge to x is denoted by $\mathscr{K}(x)$.

Assuming the details on the concept of density of sets in X [8], we note that if

$$\mathscr{U} = \{E : E \subset X \text{ and } D^*(X - E, x) = 0 \text{ for all } x \in E\}$$

then $\mathscr{U}$ is a topology, which is known as the density topology or d-topology on X and the sets E are known as d-open sets [8].

A (weak) density theorem is said to hold for $\mathscr{K}$ if almost all the points of every (measurable) set is a point of the outer density (point of density) of the set. Martin [8] observed that a weak density theorem holds if and only if a density theorem holds for $\mathscr{K}$.

A class $\beta \subset S$ is said to cover indefinitely a set $Y \subset X$ if, for each $y \in Y$, there is a sequence $A_n \in \beta$ such that $y \in A_n$ for all n and $m(A_n) \to 0$ as $n \to \infty$.

A family $\mathscr{A} \subset S$ is called m-regular [1], [3] if

(i) $m^*\left(\bigcup_{A \in \mathscr{A}} A\right) < \infty$;

(ii) the set $\rho(A)$ of points outside A and indefinitely covered by subsets A' joint to A has measure zero;

(iii) there exist numbers a and b ($b > a > 1$) such that, for every $A \in \mathscr{A}$,

$$m^*\{\Omega(A)\} < bm(A),$$

where

$$\Omega(A) = \{\bigcap A' : A' \in \mathscr{A},\ A \cap A' = \emptyset, m(A') < am(A)\}.$$

Since $m(X) = 1$, condition (i) holds automatically.

The following theorem, to which we refer here as the Vitali theorem, was proved in [3]; see also [1].

Vitali theorem. *Let $A \in S$ be a set indefinitely covered by the regular family $\mathscr{A}$. Then there exists a sequence $\{A_n\}$ of disjoint sets in $\mathscr{A}$ such that*

$$m\left[A - A \cap \left(\bigcup_{n=1}^{\infty} A_n\right)\right] = 0. \tag{1}$$

Also, for every $\varepsilon > 0$, there is a sequence $\{A_n\}$ of disjoint sets in $\mathscr{A}$ such that

$$m\left[\bigcup_{n=1}^{\infty} A_n\right] < m(A) + \varepsilon. \tag{2}$$

We obtain the following corollary as an application of the Vitali theorem which will be needed here.

Corollary A. *Let $\varepsilon > 0$ be arbitrary. Under the conditions of the Vitali theorem, there is a positive integer k depending on ε such that*

$$m\left[A - \bigcup_{r=1}^{k} A_r\right] < \varepsilon.$$

Proof. Since the sets $\{A_r\}$ are disjoint, $\sum_{r=1}^{\infty} m(A_r) < \infty$. So, corresponding to $\varepsilon > 0$, there is a positive integer k such that

$$\sum_{r=k+1}^{\infty} m(A_r) < \varepsilon.$$

Now,

$$A - \bigcup_{r=1}^{k} A_r = \left(A - \bigcup_{r=1}^{\infty} A_r\right) \cup \left[A \cap \left(\bigcup_{r=k+1}^{\infty} A_r\right)\right]$$
$$\subset \left(A - \bigcup_{r=1}^{\infty} A_r\right) \cap \left(\bigcup_{r=k+1}^{\infty} A_r\right).$$

So

$$m\left[A - \bigcup_{r=1}^{k} A_r\right] \le m\left[A - \bigcup_{r=1}^{\infty} A_r\right] + m\left[A - \bigcup_{r=k+1}^{\infty} A_r\right]$$
$$= \sum_{r=k+1}^{\infty} m(A_r), \quad \text{using (1)}$$
$$\le \varepsilon. \qquad \blacksquare$$

As in [8], we will assume throughout that a density theorem holds for $\mathscr{K}$.

3. Approximately Continuous Functions

Let $(Y, \mathscr{T})$ be a topological space. By mappings, we shall always mean mappings (functions) from X into Y, unless otherwise stated. Further, we shall assume the definition of measurability of functions as in [4].

Definition 1. *A mapping f is said to be approximately continuous at $\alpha \in X$ if, for every $G \in \mathscr{T}$ with $f(\alpha) \in G$, there is a set $E \in S$ with $D(E, \alpha) = 1$ such that $f(E) \subset G$.*

If f is approximately continuous at each $\alpha \in X$, then f is called an approximately continuous function.

Definition 2. (cf. [2], [9], [10]) *A mapping f is said to be continuous with respect to the density topology at $x \in X$ if, for every $G \in \mathscr{T}$ with $f(x) \in G$, there is a d-open set E with $x \in E$ and $f(E) \subset G$.*

If f is continuous with respect to the density topology at each $x \in X$, then f is continuous with respect to the density topology.

In the real number case, the statement of Theorem 1 is taken as the definition of approximately continuous function [6], but here we need to establish the equivalence because in Definition 1, α need not be a member of E.

Theorem 1. *A mapping f is approximately continuous if and only if it is continuous with respect to the density topology.*

Proof. We suppose that f is approximately continuous. Let $x \in X$ be arbitrary and $G \in \mathscr{T}$ be such that $f(x) \in G$. There is a set $E \subset S$ with $D(E, x) = 1$ and $f(E) \subset G$. If $T = E \cap \{x\}$, then we observe that T is a measurable set $D(T, x) = 1$ and $f(T) \subset G$. By Lemma 4.8 [8],

$$f\{d - \text{Int}(T)\} = f\{y \in T : D(T, y) = 1\} \subset f(T) \subset G.$$

Thus, $H = d - \text{Int}(T)$ is a d-open set containing x such that $f(H) \subset G$. Hence, f is continuous with respect to the density topology at $x \in X$.

Conversely, let f be continuous with respect to the density topology. Let $x \in X$ be arbitrary and $G \in \mathscr{T}$ be such that $f(x) \in G$. There is a d-open set E containing x such that $f(E) \subset G$. By Corollary 4.4 and Theorem 4.5 [8] and then the density theorem, $E \subset S$ and $D(E, x) = 1$. So f is approximately continuous at x. ■

Corollary. *Any approximately continuous function $f : X \to Y$ has the property of Baire.*

Note 1. We observe that in the present situation, continuity and approximate continuity of $f : X \to Y$ coincide.

Note 2. It follows from Theorem 1 that if f is approximately continuous, then it is measurable because if $G \in \mathscr{T}$, then $f^{-1}(G)$ is d-open and so by Corollary 4.4 [8], $f^{-1}(G) \in S$. However, a stronger result is proved in the following theorem.

Theorem 2. *If f is approximately continuous at almost all points of X, then f is measurable.*

Proof. Let $G \in \mathscr{T}$ and $E = f^{-1}(G)$. If $m^*(E) = 0$, then because m is complete, $E \subset S$. Suppose $m^*(E) > 0$. Let $y \in E$ be a point of approximate continuity. Then there is a measurable set E_0 having y as a point of density such that

$$f(E_0) \subset G, \quad \text{i.e.,} \quad E_0 \subset E.$$

By Lemma 3.4 [8], $D(X - E, y) = 0$ and so by Lemma 3.2 in [8],

$$\bar{D}^*(X - E, y) \leq \bar{D}(X - E_0, y) = 0.$$

This is true for almost all points of E. So, by Theorem 4.3 in [8], $X - E \in S$ and so $E \in S$. Hence, f is measurable. ■

Theorem 3. *If f is measurable and $(Y, \mathscr{T})$ is second countable, then f is approximately continuous at almost all points of X.*

Proof. Let $\{U_1, U_2, U_3, \ldots\}$ be a countable basis for $\mathscr{T}$. Then $M_n = f^{-1}(U_n) \in S$ for all $n = 1, 2, 3, \ldots$ For each n, let $A_n = \{x \in M_n : D(M_n, x) = 1\}$. By the density theorem and completeness of m, $T_n = M_n - A_n \in S$ and $m(T_n) = 0$. If $y \in A_n$, then by Lemma 3.4 in [8],

$$D(A_n, y) = D(M_n - T_n, y) = D(M_n, y) - D(T_n, y) = 1.$$

Let $T = \bigcup_{n=1}^{\infty} T_n$. Then $T \in S$ and $m(T) = 0$. Let $X \in X - T$ and $U \in \mathcal{T}$ be such that $f(x) \in U$. Then there is a positive integer k such that $f(x) \in U_k \subset U$. So, $x \in M_k$. Since $x \notin T$, $x \in A_k$. Also, $f(A_k) \subset U_k$. Thus, A_k is a measurable set having x as a point of density such that

$$f(A_k) \subset U_k \subset U.$$

So, f is approximately continuous at x. ■

Combining Theorems 2 and 3 we obtain the following.

Theorem 4. *If $(Y, \mathcal{T})$ is second countable, then a mapping f is measurable if and only if it is approximately continuous at almost all points of X.*

Note 3. Theorem 4 is a generalization of the corresponding classical result when $X = Y = R$, the real number space [12, p. 132]. The method of proof needed here has no connection with the classical case.

In the following theorem, we observe that the range of an approximately continuous function has some special property under certain conditions (cf. [6]).

Theorem 5. *Let $(Y, \mathcal{T})$ be a metric space with metric ρ and $f: X \rightarrow Y$ be approximately continuous. Then $f(X)$ is a separable subspace of Y, provided that $\bigcup_{x \in X} \mathcal{K}(x)$ is m-regular.*

Proof. Let $\alpha \in X$ and $\varepsilon > 0$ be arbitrary. Then there is a set $E \in S$ with $D(E, \alpha) = 1$ and $\rho(f(x), f(\alpha)) < \varepsilon$ if $x \in E$.

Let $K_n^{(\alpha)} \in \mathcal{K}(\alpha)$. There is a positive integer N such that

$$\frac{m[E \cap K_n^{(\alpha)}]}{m[K_n^{(\alpha)}]} > 1 - \varepsilon \quad \text{if } n \geq N.$$

Thus, for all large n, $\rho(f(x), f(\alpha)) < \varepsilon$ for a set of points x in $K_n^{(\alpha)}$ whose relative measure with respect to $K_n^{(\alpha)}$ is less than ε.

Such a sequence exists for each $\alpha \in X$. Hence, by Corollary A, there exists a finite disjoint sequence $\{K_{n_i}^{(x_i)}\}$ $i = 1, 2, \ldots, k$ such that

$$m\left[X - \bigcup_{i=1}^{k} K_{n_i}^{(x_i)}\right] < \varepsilon.$$

The set $K_{n_i}^{(x_i)}$ has the property that, for all y in $K_{n_i}^{(x_i)}$, except for a set whose relative measure with respect to $K_{n_i}^{(x_i)}$ is less than ε, we have

$$\rho(f(x_i), f(y)) < \varepsilon.$$

Let $B_\varepsilon = \{f(x_1), f(x_2), \ldots, f(x_k)\}$. So, except for a set of measure less than

$$\varepsilon m\{K_{n_1}^{(x_1)}\} + \varepsilon m\{K_{n_2}^{(x_2)}\} + \cdots + \varepsilon m\{K_{n_k}^{(x_k)}\} + m\left[X - \bigcup_{r=1}^{k} K_{n_r}^{(x_r)}\right]$$
$$< \varepsilon \sum_{r=1}^{k} m\{K_{n_r}^{(x_r)}\} + \varepsilon \leq [m(X) + 1] = 2\varepsilon$$

for all points q in X, $f(q)$ is at a distance less than ε from B_ε.

Let $\{\varepsilon_\nu\}$ be a sequence such that $\varepsilon_\nu > 0$, $\varepsilon_{\nu+1} \leq \varepsilon_\nu$ and $\lim_{\nu\to\infty} \varepsilon_\nu = 0$ and $B = \bigcup_{\nu=1}^{\infty} B_{\varepsilon_\nu}$. Then B is an enumerable subset of $f(X)$ and dense in $f(X - M)$ where M is a subset of measure zero. Let $\xi \in M$ and $\delta > 0$ be arbitrary. From the approximate continuity at ξ, we obtain that there is a $\xi_1 \in X - M$ such that $\rho(f(\xi_1), f(\xi)) < \delta/2$. So, there is a $\eta \in B$ such that $\rho(f(\xi_1), \eta) < \delta/2$ and hence, $\rho(f(\xi), \eta) < \delta$. So, B is dense in $f(X)$. ■

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