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Extending Property for Finitely Generated Submodules

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Abstract. A module is called extending if every submodule is essential in a direct summand. More generally, the extending property can be restricted to certain classes of submodules, e.g. uniform submodules, semisimple submodules. In this paper, we consider the extending property for the class of (essentially) finitely generated submodules. We study properties of this type of modules, decompositions and the relationship to finitely presented modules.

1. Preliminaries

Throughout this paper, R will denote an associative ring with unit and R-Mod the category of unital left R-modules. Morphism of left modules are written on the right.

For an R-module M, $\sigma[M]$ denotes the full subcategory of R-Mod whose objects are submodules of M-generated modules.

A module M is said to have *finite uniform dimension* if M does not contain an infinite direct sum of nonzero submodules. A submodule K of M is called *essential* in M if $K \cap L \neq 0$ for every nonzero submodule L of M. In this case, M is called an *essential extension* of K. A submodule C of M is closed in M if and only if C is the only essential extension of C in M.

A module M is called *extending* provided every closed submodule of M is a direct summand of M, or equivalently, every submodule of M is essential in a direct summand of M. M is called *uniform-extending* if every uniform submodule is essential in a direct summand of M.

Now we introduce some new notions which generalize the concept of extending modules.

Recall that a module which has a finitely generated essential submodule is said to be *essentially finitely generated* (*essentially finite* for short). In particular, this includes all finitely generated modules.

Definition 1. *A module M is called* ef-extending *if every closed essentially finite submodule is a direct summand of M.*

Definition 2. *A module M is called* f-extending *if every finitely generated submodule of M is essential in a direct summand of M.*

Trivially, every extending module is ef-extending (f-extending). Moreover, every module whose finitely generated submodules are direct summands is ef-extending. In particular, every projective module over a von Neumann regular ring R is ef-extending. Hence, the obvious implications

$$\text{extending} \Rightarrow \text{ef-extending} \Rightarrow \text{f-extending} \Rightarrow \text{uniform extending}$$

are not reversible.

The converse implications hold for modules with finite uniform dimension since for such modules uniform-extending implies extending (see [5, 7.8]).

A ring R is *left PF (pseudo-Frobenius)* provided R is an injective cogenerator in R-Mod. Equivalently, R is a semiperfect left self-injective ring with essential socle.

A module M is *continuous* if M is extending and *direct-injective*, i.e., every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

In [2], a module M is called *finitely continuous* (for short *f-continuous*) if M is f-extending and direct-injective.

In [4], Dischinger and Müller construct a local left PF-ring R which is not right PF. By [2, Theorem 1.5], the ring $M_2(R)$ of 2×2 matrices over R is not right continuous. Therefore, not all left PF-rings are two-sided continuous. However, left PF-rings are (left and) right f-continuous (see [1, Propositions 3.11 and 3.12]).

General background material can be found in Anderson and Fuller [1], Dung *et al.* [5], and Wisbauer [13].

2. Decompositions of ef-Extending Modules

We begin with some elementary observations.

Lemma 2.1. *Any closed submodule of an ef-extending module is also an ef-extending module.*

Proof. Suppose N is a closed submodule of M and M is ef-extending. Let X be a closed essentially finite submodule in N. Since N is a closed submodule in M, so by [5, 1.10], X is a closed submodule in M. Since M is an ef-extending module, X is a direct summand of M and hence in N. ■

Now we consider the properties of decompositions of an ef-extending module.

Lemma 2.2. *An indecomposable module is f-extending (ef-extending, extending) if and only if it is uniform.*

Proof. Let M be an indecomposable f-extending module. For every $x \in M$, $x \neq 0$,

Rx is essential in a direct summand A of M. Thus, $A = M$ and then Rx is essential in M. Hence, M is uniform.

The other implications are obvious. ■

Recall that a direct sum of submodules of M, $N = \bigoplus_\Lambda N_\lambda \subset M$, is said to be a local direct summand of M if $\bigoplus_A N_\alpha$ is a direct summand of M for every finite subset $A \subset \Lambda$.

Corollary 2.3. *Let M be an ef-extending R-module. Assume that*
(a) *every local direct summand of M is a direct summand or*
(b) $\mathrm{End}_R(M)$ *does not contain an infinite set of orthogonal idempotents.*

Then M is a direct sum of uniform modules.

Proof. In case (a), by [8, Theorem 2.17], and in case (b), by [5, 10.4], M is a direct sum of indecomposable modules. By Lemmas 2.1 and 2.2, M is a direct sum of uniform modules (in particular, in case (b), M is a finite direct sum of uniform modules). ■

A module $L \in \sigma[M]$ is called *M-singular* if $L \simeq N/K$ for some $N \in \sigma[M]$ and K is an essential submodule of N. As is well known, every module $L \in \sigma[M]$ contains a *largest M-singular submodule* which we denote by $Z_M(N)$. If $Z_M(N) = 0$, N is called *nonsingular in $\sigma[M]$* or *non-M-singular*. Note that, if M is non-M-singular, then every submodule of M has a unique maximal essential extension (see [10] for more details).

Proposition 2.4. *Let M be a module such that every submodule of M has a unique maximal essential extension. Then M is ef-extending if and only if M is f-extending.*

Proof. Let M be an f-extending module. Let L be a maximal essential extension of a finitely generated submodule K. Then K is essential in a direct summand H of M. By assumption, $H \subset L$. From this, L is a direct summand of M. Hence, M is ef-extending. ■

For any $m \in M$, we denote

$$l(m) = \{r \in R | rm = 0\}.$$

The following observation is fundamental.

Proposition 2.5. *Let M be an ef-extending R-module. If R satisfies ACC on left ideals of the form $l(m), m \in M$, then M contains a maximal local direct summand $N = \bigoplus_{i \in I} N_i$, with N_i uniform for each $i \in I$.*

Proof. Since R satisfies ACC on left ideals of the form $l(m), m \in M$, we can choose a nonzero element $m \in M$ such that $l(m)$ is maximal in $\{l(x) | 0 \neq x \in M\}$. By the ef-extending property of M, there exists a direct summand K of M such that Rm is essential in K. Suppose K is decomposable. Then there exist nonzero submodules K_1 and K_2 of K such that $K = K_1 \oplus K_2$. So we write $m = m_1 + m_2$ for some $m_1 \in K_1$, $m_2 \in K_2$. If $m_1 = 0$, then $m = m_2 \in K_2$, and $Rm \cap K_1 = 0$ giving $K_1 = 0$, a contradiction. Thus, $m_1 \neq 0$. It is easy to see that $l(m) \subset l(m_1)$. Hence, by the maximality of m, $l(m) = l(m_1)$. Similarly, $m_2 \neq 0$ and $l(m) = l(m_2)$. Because $m_1 \neq 0$ and Rm is essential in K, there exist $r_1, r_2 \in R$ such that

$$0 \neq r_1 m_1 = r_2 m = r_2(m_1 + m_2) = r_2 m_1 + r_2 m_2.$$

From this, $r_2 m_2 = 0$, and hence $r_2 \in l(m_2) \setminus l(m)$, a contradiction. So K is indecomposable.

By Lemma 2.1, K is ef-extending and by Lemma 2.2, K is uniform. It follows that any direct summand of M contains a uniform direct summand.

By Zorn's lemma, M contains a maximal local direct summand $N = \bigoplus_{i \in I} N_i$, where N_i is a uniform submodule of M for each $i \in I$. ∎

Corollary 2.6. *Let M be an ef-extending R-module. If R satisfies ACC on left ideals of the form $l(m)$, $m \in M$, and every maximal local direct summand is essentially finite, then M is a finite direct sum of uniform modules. Consequently, M is extending.*

Proof. By Proposition 2.5, M contains a maximal local direct summand $N = \bigoplus_{i \in I} N_i$. By [5, 8.1(1)], N is closed in M. Hence, N is a direct summand of M, say

$$M = N \oplus N',$$

for some submodule N' of M. If $N' \neq 0$, then by the same argument as in the proof of Proposition 2.5, $N' = U \oplus U'$ for some submodule U, U', with U uniform. Then $N \oplus U$ is a local direct summand, contradicting the maximality of N. Then $N' = 0$ and $M = \bigoplus_{i \in I} N_i$ is a direct sum of uniform submodules of M. By assumption, M contains an essential finitely generated submodule V. It follows that there exists a finite subset J of I such that $V \subset \bigoplus_J N_j$. Since V is essential in M, $M = \bigoplus_J N_j$, proving our claim. ∎

We also have some kind of decomposition of an ef-extending module.

Proposition 2.7. *Let M be an ef-extending R-module which is projective in $\sigma[M]$. Then $M = \bigoplus_{i \in I} M_i$, where each M_i is essentially finite.*

Proof. By Kaplansky's theorem (e.g. [13, 8.10]), the module M is a direct sum of countably generated submodules. Thus, without loss of generality, we may assume that M is countably generated, i.e., there exists a countable set of elements $m_1, m_2, m_3, \ldots$ in M such that

$$M = \sigma_{i=1}^{\infty} Rm_i.$$

By hypothesis, there exist submodules M_1, N_1 of M such that $M = M_1 \oplus N_1$ and Rm_1 is essential in M_1. Then $m_2 = rm_1 + n_2$. Suppose that $n_2 \neq 0$. By Lemma 2.1, N_1 is again an ef-extending module, hence, there exists a direct summand M_2 of N_1 which contains Rn_2 as an essential submodule; moreover,

$$Rm_1 + Rm_2 \subseteq M_1 \oplus M_2.$$

Continuing in this manner we obtain a direct sum $M_1 \oplus M_2 \oplus \cdots$ of submodules in the modules M such that

$$Rm_1 + Rm_2 + \cdots + Rm_k \subseteq M_1 \oplus M_2 \oplus \cdots \oplus M_k,$$

for all $k \in \mathbb{N}$. It follows that $M = \bigoplus_{i \in \mathbb{N}} M_i$.

Moreover, by construction, each submodule M_i is essentially finite.

Corollary 2.8. *Let M be an ef-extending R-module which is projective in $\sigma[m]$ and non-M-singular. Then M is a sum of finitely generated modules.*

Proof. By Proposition 2.7, M is a direct sum of essentially finite ef-extending modules. Since a non-N-singular module N which is projective in $\sigma[N]$ and has an essentially finite submodule is finitely generated (see [5, 4.7]), hence the desired proof follows. ■

A module M is called *π-injective* if for any $L_1, L_2 \in M$ with $L_1 \cap L_2 = 0$, there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$ and $L_i \subset M_i$ $(i = 1, 2)$.

Now we have a characterization of an ef-extending module via a property that is close to the property of a π-injective module.

Theorem 2.9. *For an R-module M, the following conditions are equivalent:*
(a) *For any $L_1, L_2 \subset M$ with $L_1 \cap L_2 = 0$ and L_1 essentially finite, there exist submodules M_1, M_2 of M such that $M = M_1 \oplus M_2$, L_1 is essential in M_1 and $L_2 \subset M_2$,*
(b) *M is an ef-extending module and whenever $M = M_1 \oplus M_2$ with M_1 essentially finite, then M_1 is M_2-injective.*

Proof. (a) ⇒ (b). First, we prove that M is an ef-extending module. Let A be a closed essentially finite submodule in M and L_2 a complement of A in M; then, by assumption (a), there exist submodules M_1, M_2 of M such that A is essential in M_1, $L_2 \subset M_2$ and $M = M_1 \oplus M_2$. It follows that $L_2 = M_2$ and $A = M_1$. So A is a direct summand of M. Hence, M is an ef-extending module.

Let $M = M_1 \oplus M_2$ with M_1 essentially finite and N a submodule of M such that $N \cap M_1 = 0$. Then by assumption (a), there exist submodules M' and M_1' of M such that

$$M = M' \oplus M_1'$$

and $N \subset M'$, $M_1 \subset M_1'$. Hence, by the Modular Law,

$$M_1' = M_1' \cap M = M_1' \cap (M_1 \oplus M_2) = M_1 \oplus (M_1' \cap M_2)$$

and

$$M = M' \oplus M_1 \oplus (M_1' \cap M_2) = M' \oplus (M_1' \cap M_2) \oplus M_1.$$

By [5, 7.5], M_1 is M_2-injective.

(b) ⇒ (a). Let L_1 be essentially finite with a finitely generated essential submodule B and $L_2 \subset M$ such that $L_1 \cap L_2 = 0$. We take the complement C of B in M. This is to say that $L_1 \cap C = 0$. Then by Zorn's lemma, there exists maximal submodule $D \subset M$ such that $D \cap C = 0$ and $D \supset B$. Moreover, B is essential in D.

So D is a closed essentially finite submodule of M. Since M is ef-extending, D is a direct summand of M, say

$$M = D \oplus F.$$

Since L_1 is also essential in D, $D \cap L_2 = 0$. By (b), D is F-injective. Hence, by [5, 7.5], there exists a submodule P' of M such that $M = D \oplus P'$ and $L_2 \subset P'$. So (a) follows. ■

3. Finitely Σ-F-(ef)-Extending Modules

For any R-module M, denote by Add M (add M, resp.) the full subcategory of $\sigma[M]$ whose objects are direct summands of (finite) coproducts of copies of M. Of course, Add R is just the class of all projective R-modules and it does not contain (R)-singular modules. It is clear that if M is projective in $\sigma[M]$, or M has no M-singular submodule, then Add M contains no M-singular modules (see [3, Lemma 1.10]).

An R-module M is said to be *direct-projective* if, for every direct summand X of M, every epimorphism $M \to X$ splits. M is *Σ-direct-projective* if any coproduct of copies of M is direct-projective. Of course, if M is projective in $\sigma[M]$, then it is Σ-direct-projective.

A module N is called finitely presented in M if it is finitely generated and in every exact sequence

$$0 \to K \to L \to N \to 0$$

in $\sigma[M]$, with L finitely generated, K is also finitely generated.

If $M = R$, then the class of all finitely presented modules is the class of all modules of the form R^n/H where $H \subset R^n$ finitely generated in [13, 2.5].

Definition. *We call an R-module M (finitely)* Σ-f-extending *if any (finite) coproduct of copies of M is f-extending.*

M is called (finitely) Σ-ef-extending *if any (finite) coproduct of copies of M is ef-extending.*

Recall the following results about regular rings and left and right hereditary serial Artinian rings.

Proposition 3.1. *For a ring R, the following conditions are equivalent.*
(a) *R is von Neumann regular and*
(b) *every finitely presented left (and right) R-module is projective.*

Proof. See [13, 37.6]. ■

Proposition 3.2. *For a ring R, the following conditions are equivalent:*
(a) *R is a left (and right) hereditary serial Artinian ring and*
(b) *R is fight nonsingular and every nonsingular left R-module is projective.*

Proof. See [7, Theorem 5.23]. ■

Now we consider properties of finitely Σ-f-extending which are related to Propositions 3.1 and 3.2.

Lemma 3.3. *Let M be ef-extending (f-extending) and K an essentially finite (a finitely generated, resp.) submodule of M such that M/K is non-M-singular. Then K is a direct summand of M.*

Proof. Since M is f-extending, there exists an essential extension $\overline{K}$ of K which is a direct summand of M, say

$$M = \overline{K} \oplus V.$$

From this

$$M/K \simeq (\overline{K}/K) \oplus V.$$

Since M/K is non-M-singular, it follows that $K = \overline{K}$ and then K is a direct summand of M.

The proof for ef-extending is similar. ■

Corollary 3.4. *If R is a left finitely Σ-f-extending ring, then every finitely presented nonsingular left R-module is projective.*

Theorem 3.5. *Let M be a module. We consider the following conditions:*
(a) *M is finitely ef-extending,*
(b) *every module M,* (a) (b) (c), *and*
(c) *every factor of M^n by any closed essentially finite submodule of M^n is in Add M.*

Then for every module M, (a) ⇔ (b) ⇔ (c).
If M is Σ-direct projective, then (c) ⇒ (a).

Proof. (a) ⇔ (b). By Lemma 3.1, any direct summand of an ef-extending module is also ef-extending.

(a) ⇒ (c). Let H be a finitely generated submodule of M^n and $\overline{H}$ any maximal essential extension of H. Then $\overline{H}$ is a direct summand of M^n, say $M^n = \overline{H} \oplus K$. It is easy to see that $M^n/\overline{H} \simeq K$ is in add M.

(c) ⇒ (a). Let M be Σ-direct projective. Assume (c). Let H be a finitely generated submodule of M^n and $\overline{H}$ any maximal essential extension of H. Then by (c), $M^n/\overline{H}$ is in Add M. Since M is Σ-direct projective, $\overline{H}$ is a direct summand of M^n. It follows that M^n is ef-extending.

For the f-extending property, we obtain the following:

Theorem 3.6. *Let M be an R-module. We consider the following conditions:*
(a) *M is Σ-f-extending,*
(b) *M is finitely Σ-f-extending,*

(c) *every factor of M^n by some closed essentially finite submodule of M^n is in Add M,*
(d) *every factor of M^n by a finitely generated submodule is a direct sum of a module in Add M and an M-singular module,*
(e) *every non-M-singular factor of M^n by any finitely generated submodule is in Add M.*
Then we have the following implications:
(1) *For every module M,* (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (e).
(2) *If M is Σ-direct projective, then* (c) $\Rightarrow$ (a), (c) $\Rightarrow$ (d).

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (a). Let H be a finitely generated submodule of $M^{(\Lambda)}$, where Λ any index set. Then there exists $n \in \mathbb{N}$ such that $H \subset M^n$. By (b), H is essential in a direct summand $\overline{H}$ of M^n. It is easy to see that $\overline{H}$ is also a direct summand of $M^{(\Lambda)}$. So (a) follows.

The other implications are proved similarly to Theorem 3.5. ∎

Corollary 3.7.
(1) *For a ring R, the following conditions are equivalent:*
(a) *R is left finitely Σ-ef-extending;*
(b) *every finitely generated projective is left R-module ef-extending;*
(c) *every factor of R^n by any closed essentially finite submodule of R^n is projective.*
(2) *For a ring R, the following conditions are equivalent:*
(a′) *R is left Σ-f-extending;*
(b′) *R is left finitely Σ-f-extending;*
(c′) *every factor of R^n by some closed essentially finite submodule of R^n is projective.*
(3) *If ${}_RR$ is nonsingular, then all conditions in* (1) *and* (2) are equivalent.
(4) *One of the above conditions in* (1) *and* (2) $\Rightarrow$ (d) $\Rightarrow$ (e) are as follows:
(d′) *every finitely presented left R-module is a direct sum of a projective module and a singular module;*
(e′) *every nonsingular finitely presented left R-module is projective.*

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References

1. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, New York, 1974.
2. P. Ara and J. K. Park, On continuous semiprimary rings, *Comm. Algebra* **17** (7) (1991) 1945–1957.

3. J. Clark and R. Wisbauer, Σ-extending modules, *J. Pure Applied Algebra* 1995 (to appear).
4. F. Dischinger and W. Müller, Left PF is not right PF, *Comm. Algebra* **14** (7) (1986) 1223–1227.
5. N. V. Dung, D. V. Huynh, P. Smith, and R. Wisbauer, *Extending Modules*, Pitman, London, 1994.
6. N. V. Dung and P. F. Smith, Σ-CS modules, *Comm. Algebra* **22** (1994) 83–93.
7. K. R. Goodearl, *Ring Theory*, Marcel Dekker, New York, 1976.
8. S. H. Mohamed and B. J. Müller, Continuous and discrete modules, *London Math. Soc. Lecture Notes Ser.* **147**, Cambridge University Press, Cambridge, 1990.
9. K. Oshiro, Continuous modules and quasi-continuous modules, *Osaka J. Math.* **20** (1983) 681–694.
10. P. F. Smith, Modules for which every submodule has a unique closure, *Ring Theory, Proc. of the Biennial Ohio-State Denison Conference*, World Scientific, 1992, pp. 302–312.
11. Y. Utumi, Self-injective rings, *J. Algebra* **6** (1967) 56–64.
12. N. Vanaja, Characterization of rings using extending and lifting modules, *Ring Theory* (Denison Conf.), World Scientific, Singapore, 1993.
13. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.