Vietnam Journal of Mathematics **25**:1 (1997) 75–79

Vietnam Journal
of
MATHEMATICS

Short Communication

Spherical Classes Detected by the Algebraic Transfer

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Received August 28, 1996

We are interested in the following classical conjecture on spherical classes in Q_0S^0, i.e., elements belonging to the image of the Hurewicz homomorphism

$$H\colon \pi_*^s(S^0) \cong \pi_*(Q_0S^0) \to H_*(Q_0S^0).$$

Here and throughout the note, the coefficient ring for homology and cohomology is always $\mathbf{F}_2$, the field of two elements.

Conjecture 1 (conjecture on spherical classes). *There are no spherical classes in Q_0S^0, except the Hopf invariant one and the Kervaire invariant one elements.*

Some topologists believe that the conjecture is due to Madsen, while others say it is due to Curtis. (See Curtis [5] and Wellington [20] for a discussion.)

Let E^k be an elementary Abelian 2-group of rank k. The general linear group $GL_k = GL(k, \mathbf{F}_2)$ acts on E^k and therefore on $H^*(BE^k)$ in the usual way. Let D_k be the Dickson algebra of k variables, $D_k := H^*(BE^k)^{GL_k} \cong \mathbf{F}_2[x_1, \ldots, x_k]^{GL_k}$, where $P_k = \mathbf{F}_2[x_1, \ldots, x_k]$ is the polynomial algebra on k generators $x_1, \ldots, x_k$, each of dimension 1. As the action of the (mod 2) Steenrod algebra, $\mathscr{A}$, and that of GL_k on P_k commute with each other, D_k is an algebra over $\mathscr{A}$.

Let $\varphi_k\colon \mathrm{Ext}_{\mathscr{A}}^{k,k+i}(\mathbf{F}_2, \mathbf{F}_2) \to (\mathbf{F}_2 \otimes_{\mathscr{A}} D_k)_i^*$ be the Lannes–Zarati homomorphism, which is compatible with the Hurewicz homomorphism (see [8], [9, p. 46]). The domain of φ_k is the E_2-term of the Adams spectral sequence converging to $\pi_*^s(S^0) \cong \pi_*(Q_0S^0)$. According to Madsen's theorem [10], which asserts that D_k is dual to the coalgebra of Dyer–Lashof operations of length k, the range of φ_k is a submodule of $H_*(Q_0S^0)$. By compatibility of φ_k and the Hurewicz

This work was partially supported by the National Basic Research Programme in Natural Science, No. 1.4.3.

homomorphism, we mean φ_k is a "lifting" of the latter from the "E_∞-level" to the "E_2-level".

The Hopf invariant one and the Kervaire invariant one elements are respectively represented by certain permanent cycles in $\mathrm{Ext}_{\mathscr{A}}^{1,*}(\mathbf{F}_2, \mathbf{F}_2)$ and $\mathrm{Ext}_{\mathscr{A}}^{2,*}(\mathbf{F}_2, \mathbf{F}_2)$, on which φ_1 and φ_2 are nonzero (see Adams [1], Browder [4], Lannes–Zarati [9]).

Therefore, Conjecture 1 is a consequence of the following one.

Conjecture 2. *$\varphi_k = 0$ in any positive stem i for $k > 2$.*

It is well known that the Ext group has intensively been studied, but remains very mysterious. In order to avoid the shortage of our knowledge of the Ext group, we have combined the above data with Singer's algebraic transfer (see [12]).

Singer defined in [19] the algebraic transfer

$$Tr_k \colon \mathbf{F}_2 \bigotimes_{GL_k} PH_i(BE^k) \to \mathrm{Ext}_{\mathscr{A}}^{k,k+i}(\mathbf{F}_2, \mathbf{F}_2),$$

where $PH_*(BE^k)$ denotes the submodule consisting of all $\mathscr{A}$-annihilated elements in $H_*(BE^k)$. It is shown to be an isomorphism for $k \le 2$ by Singer [19] and for $k = 3$ by Boardman [2]. Singer also proved in [19] that it is not an isomorphism for $k = 5$, and conjectured that Tr_k is a monomorphism for any k.

Restricting φ_k to the image of Tr_k, we stated in [12] the following conjecture.

Conjecture 3 (weak conjecture on spherical classes).

$$\varphi_k \cdot Tr_k \colon \mathbf{F}_2 \bigotimes_{GL_k} PH_*(BE^k) \to P\left(\mathbf{F}_2 \bigotimes_{GL_k} H_*(BE^k)\right) := \left(\mathbf{F}_2 \bigotimes_{\mathscr{A}} D_k\right)^*$$

is zero in positive dimensions for $k > 2$.

In other words, there are no spherical classes in Q_0S^0, which can be detected by the algebraic transfer, except the Hopf invariant one and the Kervaire invariant one elements.

In [12], we have proved that the inclusion of D_k into P_k is a lifting of $Tr_k^* \cdot \varphi_k^* \colon \mathbf{F}_2 \otimes_{\mathscr{A}} D_k \to (\mathbf{F}_2 \otimes_{\mathscr{A}} P_k)^{GL_k}$. So, we get the following result.

Theorem 4. *The weak conjecture on spherical classes is equivalent to the conjecture that the homomorphism*

$$j_k \colon \mathbf{F}_2 \bigotimes_{\mathscr{A}} (P_k^{GL_k}) \to \left(\mathbf{F}_2 \bigotimes_{\mathscr{A}} P_k\right)^{GL_k}$$

induced by the identity map on P_k is zero in positive dimensions for $k > 2$.

Let D_k^+, $\mathscr{A}^+$ be respectively the submodules of D_k and $\mathscr{A}$ consisting of all elements of positive dimensions. Then the above conjecture on j_k can equivalently be stated as follows.

Conjecture 5. *If $k > 2$, then $D_k^+ \subset \mathscr{A}^+ \cdot P_k$.*

In [12], we have proof of this conjecture for $k=3$.

To introduce a new approach, we need to summarize Singer's invariant-theoretic description of the lambda algebra [18]. By Dickson [6], $D_k \cong \mathbb{F}_2[Q_{k,k-1}, \dots, Q_{k,0}]$, where $Q_{k,i}$ denotes the Dickson invariant of dimension $2^k - 2^i$. Singer set $\Gamma_k = D_k[Q_{k,0}^{-1}]$, the localization of D_k given by inverting $Q_{k,0}$, and defined $\Gamma_k^\wedge$ to be a certain "not too large" submodule of Γ_k. He also equipped $\Gamma^\wedge = \oplus_k \Gamma_k^\wedge$ with a differential $\partial\colon \Gamma_k^\wedge \to \Gamma_{k-1}^\wedge$ and a coproduct. Then, he showed that the differential coalgebra $\Gamma^\wedge$ is dual to the lambda algebra of the six authors of [3]. Thus, $H_k(\Gamma^\wedge) \cong Tor_k^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$. (Originally, Singer used the notation Γ_k^+ to denote $\Gamma_k^\wedge$. However, by $D_k^+, \mathscr{A}^+$ we always mean the submodules of D_k and $\mathscr{A}$ respectively consisting of all elements of positive dimensions, so Singer's notation would make a confusion in this note. Therefore, we prefer the notation $\Gamma_k^\wedge$ to Γ_k^+).

One of the main results of this note is the following theorem.

Theorem 6. *There exists a lifting* $L_k\colon \Gamma_k^\wedge \to \mathbb{F}_2[x_1^{\pm 1}, \dots, x_k^{\pm 1}]$ *of* $Tr_k^*\colon Tor_k^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \mathbb{F}_2 \otimes_{\mathscr{A}} P_k$, *whose restriction to* $D_k \subset \Gamma_k^\wedge$ *is the inclusion of* D_k *into* P_k.

Note that every $q \in D_k^+$ is a cycle in the differential module $\Gamma_k^\wedge$. By means of this theorem, the weak conjecture on spherical classes is an immediate consequence of the following conjecture.

Conjecture 7. *If* $q \in D_k^+$, *then* $[q] = 0$ *in* $Tor_k^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $k > 2$.

This is a corollary of the following conjecture.

Conjecture 8. *Let* $\operatorname{Ker} \partial_k$ *be the submodule of all cycles in* $\Gamma_k^\wedge$. *Then, for* $k > 2$,

$$D_k^+ \subset \mathscr{A}^+ \cdot \operatorname{Ker} \partial_k .$$

We get the following result, which is weaker than the above conjecture.

Theorem 9. *If* $k > 2$, *then* $D_k^+ \subset \mathscr{A}^+ . \Gamma_k^\wedge$.

We also have the following theorem.

Theorem 10. *Conjecture 7 is true for* $k = 3$.

Conjecture 5 is related to the difficult problem of the determination of $\mathbb{F}_2 \otimes_{\mathscr{A}} P_k$. This problem has first been studied by Peterson [15], Wood [21], Singer [19], and Priddy [16] who show its relationship to several classical problems in homotopy theory. $\mathbb{F}_2 \otimes_{\mathscr{A}} P_k$ has explicitly been computed for $k \leq 3$. The cases $k=1$ and 2 are not difficult, while the case $k=3$ is very complicated and was solved by Kameko [7]. There is also another approach, the qualitative one, to the problem. By this we mean giving conditions on elements of P_k to show that they go to zero in $\mathbb{F}_2 \otimes_{\mathscr{A}} P_k$, i.e., belong to $\mathscr{A}^+ \cdot P_k$. Peterson's conjecture, which was established by Wood [21], claims that $F_2 \otimes_{\mathscr{A}} P_k = 0$ in dimension d such that $\alpha(d+k) > k$. Here, $\alpha(n)$ denotes the number of ones in the dyadic expansion of n. Recently, Singer, Monks and Silverman have refined the method of Wood to show that many more monomials in P_k are in $\mathscr{A}^+.P_k$. (See Silverman [17] and its references). *Conjecture 5 presents a large family, whose elements are predicted to be in* $\mathscr{A}^+.P_k$.

Remark. Conjecture 7, and therefore Conjecture 8, is false when $k = 1$ or 2. Indeed, $[Q_{1,0}^{2^i-1}]$ and $[Q_{2,1}^{2^i-1}]$ are nonzero in $Tor_*^{\mathscr{A}}(F_2, F_2)$. They are respectively dual to the Adams element $h_i \in \mathrm{Ext}^1_{\mathscr{A}}(F_2, F_2)$ and its square $h_i^2 \in \mathrm{Ext}^2_{\mathscr{A}}(F_2, F_2)$. One easily verifies this assertion by combining Theorem 6 with Theorem 2.1 of [12] and Proposition 5.3 of [9]. The only Hopf invariant one elements are represented by h_i for $i = 1, 2, 3$ (see Adams [1]). Furthermore, the only Kervaire invariant one elements are represented by h_i^2, wherever h_i^2 is a permanent cycle in the Adams spectral sequence for spheres (see Browder [4]).

Let $\varphi_k^*: F_2 \otimes_{\mathscr{A}} D_k \to Tor_k^{\mathscr{A}}(F_2, F_2)$ be the dual of the Lannes–Zarati homomorphism, which is compatible with the Hurewicz one $H: \pi_*(Q_0S^0) \to H_*(Q_0S^0)$. In [12] we have proved that the inclusion $D_k \subset P_k$ is a lifting of $Tr_k^* \cdot \varphi_k^*: F_2 \otimes_{\mathscr{A}} D_k \to F_2 \otimes_{\mathscr{A}} P_k$. This together with Theorem 6 lead us to the following conjecture.

Conjecture 11. *The inclusion* $D_k \subset \Gamma_k^\wedge$ *is a lifting of* φ_k^*.

This conjecture and Conjecture 7 imply the classical conjecture on spherical classes.

The results of this note will be published in detail elsewhere.

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