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On the One Point Lyapunov Spectrum of Ergodic Stationary Difference Systems Perturbed by Random Noise

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Abstract. In this paper, we consider the Lyapunov spectrum of perturbed random difference systems of the form

$$x(n+1) = A(n)x(n) + \xi(n),$$
$$x(0) = x_0 \in R^d,\ n = 0, 1, 2, \ldots,$$

where $A = (A(n))_{n\geq 0}$ is an ergodic stationary matrix-valued sequence and $\xi(n)$ is a random perturbation. It is well known by the famous multiplicative ergodic theorem due to Oseledec and Millionshchikov [3, 4, 5] that the Lyapunov spectrum of the unperturbed system corresponding to (1), i.e., $x(n+1) = A(n)x(n)$ consists of r real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_r$, $r \leq d$. A wide class of random noise $\xi(n)$ will be characterized so that this perturbed system has the unique Lyapunov exponent $\lambda = \max(0, \lambda_r)$.

1. Introduction

In this paper, we are concerned with the Lyapunov spectrum (i.e., the set of the Lyapunov exponents of all solutions) of an ergodic stationary difference system perturbed by a general random noise of the form

$$x(n+1) = A(n)x(n) + \xi(n),$$
$$x(0) = x_0 \in R^d, n = 0, 1, 2, \ldots. \tag{1}$$

A continuous time version of this model has been investigated in [2], where we have proved that all solutions of an ergodic stationary differential system perturbed by Gaussian white noise have a unique Lyapunov exponent $\lambda = \max(0, \lambda_d)$, where λ_d denotes the top exponent of the corresponding unperturbed system. Unlike in [2] the random noise is now not necessarily Gaussian. An assertion similar to that in [2] remains true for the discrete model (1) under a fairly general noise $\xi(n)$ as we will see in the present paper.

It is necessary to emphasize that some steps of the proof in [2] now must be changed. Indeed, since the noise is now not assumed to be Gaussian, the law of the iterated logarithm applied to the Wiener process in [2] is no longer used and some conditional distributions cannot be explicitly computed. However, the diagonalization technique applied to the homogenous system

$$x(n+1) = A(n)x(n),$$
$$x(0) = x_0 \in R^d,\ n = 0, 1, 2, \dots \tag{2}$$

by using a discrete version of Floquet-type representation of Wihstutz for the fundamental solution matrix of (2) (cf. [1, 7]) is still available.

2. The Results

As usual, the Lyapunov exponent of a sequence of d-dimensional random variables $(\eta(n))_{n\geq 0}$ is defined as

$$\lambda[\eta] = \limsup_{n\to\infty} \frac{1}{n} \log|\eta(n)|,$$

where $|\cdot|$ means the Euclidean norm in R^d. Thus, $\lambda[\eta]$ is in general a random variable.

Let us consider the perturbed difference system (1) and suppose that the system matrix $A(n)$ satisfies the following condition:

(i) $A = (A(n))_{n\geq 0}$ is an ergodic stationary $Gl(d, R)$-valued sequence defined on some probability space $(\Omega, \mathscr{F}, P)$ such that

$$E\log^+|A^{-1}(0)| < \infty,\ E|A(0)| < \infty.$$

Then note that Oseledec's integrability condition $E\log^+|A^{\pm 1}(0)| < \infty$ is satisfied. Therefore, the deep multiplicative ergodic theorem of Oseledec and Millionshchikov [3, 4, 5] tells us that there are r $(1 \leq r \leq d)$ real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ and subspaces $E_1, E_2, \dots, E_r$ of R^d with $\dim E_i = d_i$, $\sum_{i=1}^r d_i = d$, such that

$$R^d = E_1 \oplus E_2 \oplus \cdots \oplus E_r$$

and the exact Lyapunov exponent of the solutions of (2) starting in E_i is λ_i, i.e.,

$$\lim_{n\to\infty} \frac{1}{n} \log|x(n, x_0)| = \lambda_i$$

uniformly for all $x_0 \in E_i$, $|x_0| = 1$; $i = 1, 2, \dots, r$. It should be noted that r, λ_i, and E_i are, in general, random, i.e., $r = r(\omega)$, $\lambda_i = \lambda_i(\omega)$, and $E_i = E_i(\omega)$, $\omega \in \Omega$.

By a discrete version of the Floquet type theorem of Wihstutz (cf. [1, 7]) we know that Eq. (2) has a fundamental solution matrix of the form

$$\Psi(n) = S(n)\exp(n\Lambda + o(n)),\quad \frac{o(n)}{n} \to 0 \text{ a.s. } (n \to \infty), \tag{3}$$

with $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$, where $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_d$ is the complete Lyapunov spectrum of (2) and $(S(n))_{n\ge 0}$ is a sequence of nonsingular random $d \times d$-matrices having the columns $S_k(n)$, $k = 1, 2, \ldots, d$, with $|S_k(n)| = 1$ for any $n \ge 0$ and $k = 1, 2, \ldots, d$.

We now characterize a class of random noises $\xi(n)$ so that the perturbed system (1) has the one-point Lyapunov spectrum (see theorem below).

For (1), besides condition (i) on the matrix $A(n)$, we assume further that the random perturbation $\xi(n)$ satisfies the following conditions:

(ii) $(\xi(n))_{n\ge 0}$ is a sequence of d-dimensional independent random variables with $E|\xi(n)|^2 < \infty$ for any $n \ge 0$ and $(\xi(n))_{n\ge 0}$ is independent of $(A(n))_{n\ge 0}$.

(iii) The second order moment of $\xi(n)$ increases not very fast in the sense that

$$\limsup_{n\to\infty} \frac{1}{n} \log E|\xi(n)|^2 \le 0. \tag{4}$$

(iv) For any $\varepsilon > 0$ we have

$$\sum_{n=0}^{\infty} P(|\xi(n)| > \exp(-\varepsilon n)) = \infty \tag{5}$$

which means that the noise $\xi(n)$ is not too weak.

(v) There exists a number $n_0 > 0$ such that the distribution function of $\langle \alpha, \xi(n_0) \rangle$ is continuous for any $0 \ne \alpha \in R^d$, where $\langle \cdot\, , \cdot \rangle$ denotes the scalar product in R^d.

Some examples given at the end of this work show that conditions (ii)–(v) on the noise $\xi(n)$ in (1) are also necessary for the validity of the following.

Theorem. *If conditions* (i)–(v) *are satisfied, then the Lyapunov spectrum of the perturbed system* (1) *consists of only one element, namely, the number* $\lambda = \max(0, \lambda_d)$, *where* λ_d *is the top exponent of the unperturbed part* (2).

Proof. Using the random transformation

$$x(n) = S(n)y(n), \tag{6}$$

where $S(n)$ is given in (3), the system (1) becomes

$$\begin{aligned} y(n+1) &= \exp(Q(n))y(n) + S^{-1}(n+1)\xi(n), \\ y(0) &= S^{-1}(0)x_0 =: y_0 \in R^d, \end{aligned} \tag{7}$$

where $Q(n) = \mathrm{diag}(Q_1(n), Q_2(n), \ldots, Q_d(n)) = \Lambda + o(n+1) - o(n)$. If we denote

$$T_k(n) = \sum_{i=0}^{n} Q_k(i),$$

then

$$\lim_{n\to\infty} \frac{T_k(n)}{n} = \lambda_k \quad \text{a.s.} \quad (k = 1, 2, \ldots, d) \tag{8}$$

(cf. [1, Remark 4.6]). It is easy to check that the solution of (7) is given by

$$y_k(n+1) = \exp(T_k(n)) \left[y_k(0) + \sum_{i=0}^{n} \exp(-T_k(i)) \langle S_k^{-1}(i+1), \xi(i) \rangle \right], \tag{9}$$

where $S_k^{-1}(n)$ is the k-th row vector of the inverse matrix $S^{-1}(n)$ $(k = 1, 2, \ldots, d)$.

Lemma 1. *There exist the following limits*

$$\lim_{n\to\infty} \frac{1}{n} \log|S_k(n)| = 0, \tag{10}$$

$$\lim_{n\to\infty} \frac{1}{n} \log|S_k^{-1}(n)| = 0, \; k = 1, 2, \ldots, d. \tag{11}$$

Proof. Of course, only (11) has to be proved because $|S_k(n)| = 1$. It follows from $|S_k(n)| = 1$ and $\langle S_k(n), S_k^{-1}(n) \rangle = 1$ that $|S_k^{-1}(n)| \geq 1$. Hence,

$$\liminf_{n\to\infty} \frac{1}{n} \log|S_k^{-1}(n)| \geq 0. \tag{12}$$

By (3), we have

$$\det S(n) = \det \Psi(n) \exp\left(-n \sum_{i=1}^{d} \lambda_i + o(n) \right). \tag{13}$$

Taking into account

$$\lim_{n\to\infty} \frac{1}{n} \log|\det \Psi(n)| = \sum_{i=1}^{d} \lambda_i \quad \text{a.s.}$$

(cf. [2]), we obtain $\lambda[\det S] = 0$ a.s. By definition, $S^{-1}(n) = (S_{ij}(n)/\det S(n))$, where $S_{ij}(n)$ is the algebraic complement of the element $s_{ji}(n)$ of the matrix $S(n)$, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log|S_k^{-1}(n)| \leq 0 \quad \text{a.s.} \tag{14}$$

Thus, (11) is proved by (12) and (14). ■

Corollary. *The random transformation* (6) *leaves the Lyapunov spectrum of the system* (1) *invariant, or in other words, the systems* (1) *and* (7) *have the same Lyapunov spectrum.*

Lemma 2. *Under the conditions above, we have*

$$\lambda[x(\,\cdot\,, x_0)] \geq 0 \quad \text{a.s.} \tag{15}$$

for any $x_0 \in R^d$, *where* $x(\,\cdot\,, x_0)$ *denotes the solution of the Cauchy problem* (1).

Proof. Suppose there exists an $x_0 \in R^d$ such that $p(\lambda[x(\,\cdot\,, x_0)] < 0) > 0$. Hence, there is a $\delta > 0$ such that, if $B := \{\omega \in \Omega \colon \lambda[x(\,\cdot\,, x_0)] \leq -3\delta\}$, then $P(B) > 0$.

By the definition of the Lyapunov exponent, for any $\omega \in B$, we have

$$\lim_{n\to\infty} |x(n, x_0)| \exp(2\delta n) = 0. \tag{16}$$

Multiplying both sides of (1) by $\exp(\delta n)$, we get

$$\exp(\delta n)x(n+1) = A(n)\exp(\delta n)x(n) + \exp(\delta n)\xi(n). \tag{17}$$

Since $(A(n))_{n\geq 0}$ is an ergodic stationary sequence and $E|A(0)| < \infty$ (see (i)), we have

$$\lim_{n\to\infty} A(n)\exp(\delta n)x(n) = \lim_{n\to\infty} A(n)\exp(-\delta n)\exp(2\delta n)x(n) = 0 \tag{18}$$

for any $\omega \in B$.

From (16), (17), and (18), it follows that

$$\lim_{n\to\infty} \exp(\delta n)\xi(n) = 0 \tag{19}$$

for any $\omega \in B$. By condition (iv), we have

$$\sum_{n=0}^{\infty} P(\exp(\delta n)|\xi(n)| \geq 1) = \infty.$$

Using the Borel–Cantelli lemma, we obtain

$$\limsup_{n\to\infty} \exp(\delta n)|\xi(n)| \geq 1 \quad \text{a.s.}$$

This contradicts (19). Lemma 2 is proved. ■

By the corollary of Lemma 1, it suffices to prove that the Lyapunov spectrum of (7) contains a unique number $\lambda = \max(0, \lambda_d)$. Lemma 2 said that $\lambda[y(\,\cdot\,, y_0)] = \lambda[x(\,\cdot\,, x_0)] \geq 0$ a.s. for any $y_0 \in R^d$, where $y(\,\cdot\,, y_0)$ denotes the solution of the Cauchy problem (7). Let us consider the following two cases (a) and (b).

(a) There is an index $1 \leq k \leq d$ such that $\lambda_k \leq 0$. For each given $\varepsilon > 0$, we put

$$b_n^\varepsilon = \sum_{i=0}^{n} \exp(i(-\lambda_k + \varepsilon)).$$

It is easy to see that $b_n^\varepsilon \uparrow \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} \frac{1}{n} \log b_n^\varepsilon = -\lambda_k + \varepsilon. \tag{20}$$

By the assumption (iii), there exists a constant $M > 0$ such that

$$E|\xi(n)|^2 \leq M.\exp(\varepsilon n/4),\ n = 0, 1, \ldots \tag{21}$$

Hence, by using the Chebyshev inequality and Borel–Cantelli lemma, we get

$$\lim_{n\to\infty} \exp(-\varepsilon n/4)|\xi(n)| = 0 \quad \text{a.s.} \tag{22}$$

On the other hand, Lemma 1 implies that

$$\lim_{n\to\infty} \exp(-\varepsilon n/4)|S_k^{-1}(n+1)| = 0 \quad \text{a.s.} \tag{23}$$

From (22) and (23), by the Cauchy–Schwarz–Bunyakovskii inequality, we conclude that

$$\lim_{n\to\infty} \exp(-\varepsilon n/2)\langle S_k^{-1}(n+1), \xi(n)\rangle = 0 \quad \text{a.s.} \tag{24}$$

From (8) it follows that

$$\lim_{n\to\infty} \exp\Big(-T_k(n) + \Big(\lambda_k - \frac{\varepsilon}{2}\Big)n\Big) = 0 \quad \text{a.s.} \tag{25}$$

It follows from (24), (25), and the Toeplitz theorem [6, p. 377] that

$$\begin{aligned}
&\frac{1}{b_n^\varepsilon}\sum_{i=0}^{n} \exp(-T_k(i))\langle S_k^{-1}(i+1), \xi(i)\rangle \\
&= \frac{1}{b_n^\varepsilon}\sum_{i=0}^{n} \exp((\varepsilon-\lambda_k)i)\exp(-T_k(i) + (\lambda_k - \varepsilon/2)i) \\
&\quad \times \exp(-\varepsilon i/2)\langle S_k^{-1}(i+1), \xi(i)\rangle \to 0
\end{aligned}$$

as $n\to\infty$. Therefore, by (8), (9), and (20), we get

$$\begin{aligned}
\lambda[y_k] &= \limsup_{n\to\infty} \frac{1}{n}\log|y_k(n+1)| \\
&\le \limsup_{n\to\infty} \frac{1}{n} T_k(n) + \limsup_{n\to\infty}\frac{1}{n}\log b_n^\varepsilon \\
&\quad + \limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{b_n^\varepsilon}\Big|y_k(0) + \sum_{i=0}^{n}\exp(-T_k(i))\langle S_k^{-1}(i+1), \xi(i)\rangle\Big| \\
&\le \lim_{n\to\infty}\frac{1}{n} T_k(n) + \lim_{n\to\infty}\frac{1}{n}\log b_n^\varepsilon = \lambda_k - \lambda_k + \varepsilon = \varepsilon.
\end{aligned}$$

This means $\lambda[y_k] \le 0$ a.s. Thus, if $\lambda_d \le 0$, then $\lambda[y_k] \le 0$ a.s. for any $k = 1, 2, \ldots, d$, so $\lambda[y] = \max_k \lambda[y_k] \le 0$. Lemma 2 said that $\lambda[y] = \lambda[x] \ge 0$, hence, $\lambda[y] = 0 = \max(0, \lambda_d)$.

(b) There exists an index $1 \le k \le d$ such that $\lambda_k > 0$. Based on (4), it is easy to see that

$$\sum_{n=0}^{\infty} \exp(-\lambda_k n/3)E|\xi(n)| < \infty,$$

$$\sum_{n=0}^{\infty} \exp(-2\lambda_k n/3)D|\xi(n)| < \infty.$$

Hence, by the Two-Series Theorem of Kolmogorov [6, p. 373], we conclude that

$$\sum_{n=0}^{\infty} \exp(-\lambda_k n/3)|\xi(n)| < \infty \quad \text{a.s.} \tag{26}$$

It follows from (8), (11), and (26) that

$$P\left(\sum_{n=0}^{\infty}\exp(-T_k(n))\langle S_k^{-1}(n+1),\xi(n)\rangle<\infty\right)=1.$$

In the sequel we need the following.

Lemma 3. *Suppose $F^A=\sigma(A(n))$, $n=0,1,\dots$ is the algebra generated by the variables $A(n)$ $(n=0,1,\dots)$ and*

(α) *$(g(n))_{n\geq 0}$ is a sequence of d-dimensional random vectors independent of the sequence $(\xi(n))_{n\geq}0$ such that $g(n)\neq 0$ a.s. for each n and there exists an $\varepsilon>0$ such that*

$$P\left(\sum_{n=0}^{\infty}\exp(\varepsilon n/2)|g(n)|<\infty\right)=1. \tag{27}$$

(β) *The sequence $(\xi(n))_{n\geq 0}$ satisfies conditions* (ii), (iii) *and* (v) *above.*
Then the random variable

$$\eta(\omega)=g(0)+\sum_{n=0}^{\infty}\langle g(n+1),\xi(n)\rangle$$

is defined almost everywhere and it has a continuous distribution function.

Proof. By the assumption (α) and (22), it is easy to check that the series (28) converges almost surely, i.e., η is defined a.s. Using Bayes' formula, we have

$$P(\eta<x)=\int_{\Omega}P(\eta<x|g(0)=\alpha_0,g(1)=\alpha_1,\dots).P(g(0)\in d\alpha_0,g(1)\in d\alpha_1,\dots).$$

Hence,

$$P(\eta<x)=\int_{\Omega}F_{\alpha_0,\alpha_1,\dots}(x).P(g(0)\in d\alpha_0,g(1)\in d\alpha_1,\dots),$$

where

$$\begin{aligned}F_{\alpha_0,\alpha_1,\dots}(x)&:=P(\eta<x|g(0)=\alpha_0,g(1)=\alpha_1,\dots)\\&=P\left(\alpha_0+\sum_{n=0}^{\infty}\langle\alpha_{n+1},\xi(n)\rangle<x\right),\end{aligned}$$

because $(g(n))_{n\geq 0}$ is independent of $(\xi(n))_{n\geq 0}$ (see (α)).

On the other hand, taking into account that $(\xi(n))_{n\geq 0}$ is the sequence of independent random variables (see (ii)) and the distribution function of $\langle\alpha_{n_0+1},\xi(n_0)\rangle$ is continuous, we can use the property of the convolution operator to conclude that the function $F_{\alpha_0,\alpha_1,\dots}(x)$ is continuous in x for any $(\alpha_0,\alpha_1,\dots)$. From this, applying the theorem of Lebesgue, we see that the mapping $x\to P\ (\eta<x)$ is continuous. Lemma 3 is proved.

Applying Lemma 3 for $g(0)=y_0$ and $g(n)=\exp(-T_k(n-1)).S_k^{-1}(n)$, $n=1,2,\dots$, we obtain

$$P\left(y_0+\sum_{n=0}^{\infty}\exp(-T_k(n))\langle S_k^{-1}(n+1),\xi(n)\rangle\neq 0\right)=1.$$

This equality and (9) imply

$$\begin{aligned}\lambda[y_k]&=\limsup_{n\to\infty}\frac{1}{n}\log|y_k(n+1)|\\&=\lim_{n\to\infty}\frac{1}{n}T_k(n)+\lim_{n\to\infty}\frac{1}{n}\log\left|y(0)+\sum_{i=0}^{n}\exp(-T_k(i))\langle S_k^{-1}(i+1),\xi(i)\rangle\right|\\&=\lambda_k+0=\lambda_k\quad\text{a.s.}\end{aligned}$$

Hence, $\lambda[y]=\max_k\lambda[y_k]=\lambda_d=\max(0,\lambda_d)$.

Thus our theorem is proved in both cases (a) and (b). ∎

3. Examples and Remarks

First, let us give two examples of noise sequence $\xi(n)$ satisfying the conditions (ii)–(v) stated in Sec. 2.

Example 1. Suppose $(\xi(n))_{n\geq 0}$ is a sequence of i.i.d. random variables such that $E|\xi(0)|^2\leq\infty$, independent of $(A(n))_{n\geq 0}$, and for any $\alpha\neq 0$, the distribution function of the variable $\langle\alpha,\xi(0)\rangle$ is continuous (for example, $\xi(0)=(\xi_1(0),\xi_2(0),\dots,\xi_d(0))$, where $\xi_1(0),\xi_2(0),\dots,\xi_d(0)$ are n independent random variables having the continuous distribution function). Such a noise then clearly fulfills all the conditions (ii)–(v).

Example 2. We consider a sequence of independent random variables $\xi(n)\sim N(0,G_n)$, $n=0,1,\dots$ which is independent of the sequence $(A(n))_{n\geq 0}$, where G_n is a positive definite $d\times d$-matrix. Denote $M_n=\mathrm{tr}(G_n)$ and the least eigenvalue of G_n by m_n and suppose

$$\limsup_{n\to\infty}\frac{1}{n}\log M_n=0,\ \liminf_{n\to\infty}\frac{1}{n}\log m_n=0. \tag{29}$$

This noise sequence satisfies all the conditions (ii)–(v).

Indeed, all the conditions except (iv) are obviously satisfied and only (iv) must be checked. To do this, let us first note that (29) implies

$$\lim_{n\to\infty}\frac{1}{n}\log M_n=\lim_{n\to\infty}\frac{1}{n}\log m_n=0. \tag{30}$$

By (30), for each $\varepsilon > 0$, there is a constant $C > 0$ such that

$$\frac{\exp(-2\varepsilon n)}{m_n} \leq C \tag{31}$$

for all $n \geq 0$. Using (31), we get the estimation

$$\begin{aligned} P(|\xi(n)| > \exp(-\varepsilon n)) &= [(2\pi)^d \det(G_n)]^{-1/2} \int_{|x|>\exp(-\varepsilon n)} \exp(-x^T G_n^{-1} x/2)\, dx \\ &= (2\pi)^{-d/2} \int_{y^T G_n y > \exp(-2\varepsilon n)} \exp(-|y|^2/2)\, dy \\ &\geq (2\pi)^{-d/2} \int_{|y|^2 > \exp(-2\varepsilon n)/m_n} \exp(-|y|^2/2)\, dy \\ &\geq (2\pi)^{-d/2} \int_{|y|^2 > C} \exp(-|y|^2/2)\, dy. \end{aligned}$$

Thus, the series (5) is divergent, i.e., assumption (iv) is satisfied.

It is interesting to remark that if one of conditions (iii)–(v) on the noise $\xi(n)$ is not satisfied, then the assertion of our theorem is no longer valid as shown by the following counterexamples.

Take $d = 1$, $A = 1$, and $\xi(n) \sim N(0, e^n)$. Then assumption (iii) is not satisfied and, in this case, $x(n) = x(n-1) + \xi(n-1)$. Hence, for all solutions $x(n)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log|x(n)| = \frac{1}{2} \neq \lambda_1 = 0 \quad \text{a.s.},$$

i.e., the assertion of our theorem is not valid in this case.

If we take $d = 1$, $A = 1/2$, and $\xi(n) \sim N(0, e^{-2n})$, then condition (iv) is not satisfied. The Lyapunov spectrum of the unperturbed system $x(n+1) = A(n)x(n)$ is $\{-\ln 2\}$. It is easy to check that the perturbed system $x(n+1) = A(n)x(n) + \xi(n)$ has the Lyapunov spectrum $\{-\ln 2\}$ too. Thus, in this case, the assertion of our theorem is not true.

Finally, if $d = 1$, $A_n = 2$, and $P\{\xi(n) = 1\} = 1 - (1/2^n)$; $P\{\xi(n) = -1\} = (1/2^n)$, then of course condition (v) is violated and the Lyapunov exponent of solutions of system $x(n+1) = A(n)x(n)$ equals to $\ln 2$ for any $x(0) \neq 0$. On the other hand, from

$$P\{\xi(1) = 1, \xi(2) = 1, \ldots\} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{2n}\right) = \alpha > 0,$$

it follows that the Lyapunov exponent of solutions of system $x(n+1) = A(n)\,x(n) + \xi(n)$ with $x(0) = -1$ equals to 0 with a probability $\geq \alpha$.

It is noted that condition (v) is not necessarily one. For example, if $d = 1$, $A(n) = 2$, and $P\{\xi(n) = 1\} = P\{\xi(n) = -1\} = (1/2)$, then it is easy to show that our conclusion is still true.

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