Vietnam Journal of Mathematics **25**:2 (1997) 157–169

Vietnam Journal
of
MATHEMATICS

Explicit Cascade Decompositions of Unitary Systems

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Received January 29, 1996

Abstract. We consider the unitary linear dynamic system $\alpha = (X, U, V, A, B, C, D)$ of the form

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$
$$v(t) = Cx(t) + Du(t),$$

where $x(t) \in X$, $u(t) \in U$, $v(t) \in V$. The operators A, B, C, D are linear bounded and the operator

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : X \oplus U \to X \oplus V$$

is unitary. The purpose of this paper is to decompose explicitly a unitary system by factorization of its transfer function and by its invariant subspace.

1. Introduction

1.1. Livsis originated and has developed the theory of linear dynamic systems in infinite-dimensional spaces and the theory of unitary, dissipative systems [10, 11]. In different mathematical languages, this theory has been studied by Nagy–Foias [12], Brodskii [6], Arov [1, 2], De Branges [5], and Khanh [8, 9].

The problem of cascade coupling or cascade decomposition of systems was posed naturally. In the case of infinite-dimensional systems, the problem was studied by Bart, Gohberg, Kaashoek [4], Wang, Davison [13], Chen Chi Tong, Doeser [7], and several mathematicians. On the other hand, Livsis [11], Arov [1, 2], Khanh [8, 9], Ball and Kriete [3], and De Branges [5], have studied this problem for unitary or passive systems.

It is indicated in [6, 12] that there is a one-to-one correspondence between the existence of an invariant subspace of the main operator and the regular factorization of the transfer function. Then, the problem of decomposing a unitary system by factorization of its transfer function and by its invariant subspace is determined

[6, 10, 11, 12]. However, an explicit construction of these cascade decompositions is an interesting subject to consider. This is the main purpose of the paper.

1.2. Let X, U, V be separable Hilbert spaces. Consider a linear discrete stationary dynamic system $\alpha = (X, U, V, A, B, C, D)$ of the form:

$$x_{n+1} = Ax_n + Bu_n,$$

$$v_n = Cx_n + Du_n,$$

where $x_n \in X$, $u_n \in U$, $v_n \in V$. The operators $A : X \to X$, $B : U \to X$, $C : X \to V$, $D : U \to V$ are linear bounded.

The spaces X, U, V are called the state space, the input space, and the output space, respectively.

The operator function of the complex variable

$$\theta(z) = D + zC(I - zA)^{-1}B$$

is called the transfer function of system.

The subspaces $X_\alpha^C = \bigcup_0^\infty A^k BU$, $X_\alpha^0 = \bigcup_0^\infty A^{*k} C^* V$ stand for controllable and observable subspaces of α, respectively. The system is said to be controllable if $X = X_\alpha^C$, observable if $X = X_\alpha^0$, and simple if $X = X_\alpha^C \cup X_\alpha^0$.

Definition 1. *Let $\alpha_k = (X_k, U, V, A_k, B_k, C_k, D_k)$, $k = 1, 2$ be two linear systems. α_1, a_2 are said to be similar if there exists a linear continuously invertible operator $W : X_1 \to X_2$ such that*

$$A_2 = WA_1W^{-1},$$

$$B_2 = WB_1,$$

$$C_2 = C_1W^{-1},$$

$$D_2 = D_1.$$

If, moreover, the operator W is unitary, then the systems 1 and 2 are said to be unitarily equivalent.

Definition 2. *Let two linear systems $\alpha_k = (X_k, U, V, A_k, B_k, C_k, D_k)$, $k = 1, 2$, be such that $U_2 = V_1$.*

The linear system $\alpha = (X, U, V, A, B, C, D)$ is called a cascade coupling of α_1, α_2 and is written as $\alpha = \alpha_2\alpha_1$ if:

$$U = V_1,\ V = V_2,\ X = X_1 \oplus X_2;$$

$$A = A_1P_1 + A_2P_2 + B_3C_1P_1;$$

$$B = B_1 + B_2D_1;$$

$$C = C_2P_2 + D_2C_1P_1;$$

$$D = D_2D_1;$$

where P_k is the orthoprojection from X onto X_k $(k = 1, 2)$.

We have the following result. If $\alpha = \alpha_2\alpha_1$, then $\theta_\alpha(z) = \theta_{\alpha_2}(z)\theta_{\alpha_1}(z)$.

Definition 3. *The linear system is called an unitary system if the operator*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \oplus U \to X \oplus V$$

is unitary.

We have another equivalent definition.

Definition 4. *The linear system is said to be unitary if the following equalities hold:*

$$I - A^*A = C^*C, \tag{1.1}$$

$$I - AA^* = BB^*, \tag{1.2}$$

$$I - DD^* = CC^*, \tag{1.3}$$

$$I - D^*D = B^*B, \tag{1.4}$$

$$-A^*B = C^*D. \tag{1.5}$$

An unitary system is also called an unitary colligation [6] *or a conservative scattering system* [1, 2].

It is known that if α_1 and α_2 are unitary systems, then $\alpha_2\alpha_1$ is also unitary, and according to the Livsis–Brodskii theorem, two simple unitary systems having the same transfer function are unitarily equivalent.

Let us consider the following function model of Nagy and Foias for a simple unitary system constructed by the given transfer function $\theta(z) \in \mathscr{A}(U, V)$.

$$X = [L_2^+(V) \oplus \overline{\Delta L_2(U)}] \ominus \{(\theta\omega, \Delta\omega) | \omega \in L_2^+(U)\},$$

$$A(\varphi \oplus \psi) = e^{-it}(\varphi(e^{it}) - \varphi(0)) \oplus e^{-it}\psi(e^{it}),$$

$$Bu = e^{-it}(\theta(e^{it}) - \theta(0))u \oplus e^{-it}\Delta(e^{it})u,$$

$$C(\varphi \oplus \psi) = \varphi(0),$$

$$Du = \theta(0)u,$$

where $\Delta(e^{it}) = (I - \theta^*(e^{it})\theta(e^{it}))^{1/2}$; $\mathscr{A}(U, V)$ denotes the class of all analytic functions in the unit disk $\{z : |z| < 1\}$, having values as contractive operators from U to V; $L_2^+(U)$ stands for the Hardy space of elements $f \in L_2(U)$ whose kth Fourier coefficient $\hat{f}(k) = 0$ for all $k < 0$.

Definition 5. *Let* $\theta(z) = \theta_2(z)\theta_1(z)$*;* $\theta(z) \in \mathscr{A}(U, V)$*;* $\theta_k(z) \in \mathscr{A}(U_k, V_k)$*;* $k = 1, 2$*,* $U = U_1$*;* $V_1 = U_2$*,* $V_2 = V$*.*

The factorization $\theta(z) = \theta_2(z)\theta_1(z)$ *is said to be regular if*

$$\overline{\{\Delta_2\theta_1 h \oplus \Delta_1 h : h \in L_2(U)\}} = \overline{\Delta_2 L_2(U_2)} \oplus \overline{\Delta_1 L_2(U_1)}.$$

This definition is equivalent to the following: the operator $\Delta h \mapsto \Delta_2\theta_1 h \oplus \Delta_1 h$ can be continuously extended to a unitary operator from $\overline{\Delta L_2(U)}$ onto $\overline{\Delta_2 L_2(U_2)} \oplus \overline{\Delta_1 L_2(U_1)}$.

From Definition 5, we have the following theorem.

Theorem 1. [12] *Suppose the factorization* $\theta(z) = \theta_2(z)\theta_1(z)$ *is regular. Then the space*

$$\hat{X} = [L_2^+(V) \oplus \overline{\Delta_2 L_2(U_2)} \oplus \overline{\Delta_1 L_2(U_1)}] \ominus \{(\theta\omega, \Delta_2\theta_1\omega, \Delta_1\omega) | \omega \in L_2^+(U)\}$$

contains the subspace

$$\hat{X}_2 = [L_2^+(V) \oplus \overline{\Delta_2 L_2(U_2)} \oplus \{0\}] \ominus \{(\theta_2 u, \Delta_2 u, 0) | u \in L_2^+(U_2)\}$$

invariant for the operator $\hat{A}$ *and it is the orthogonal complement in* $\hat{X}$ *of the subspace*

$$\hat{X}_1 = \{(\theta_2 u, \Delta_2 u, v) | u \in L_2^+(U_2), v \in \overline{\Delta_1 L_2(U_1)}\}$$
$$\ominus \{(\theta_2\theta_1\omega, \Delta_2\theta_1\omega, \Delta_1\omega) | \omega \in L_2^+(U)\}.$$

2. Explicit Cascade Decomposition of a Unitary System According to the Regular Factorization of its Transfer Function

Let a simple unitary system $\alpha = (X, U, V, A, B, C, D)$ be such that its transfer function $\theta(z)$ has a regular factorization $\theta(z) = \theta_2(z)\theta_1(z)$. We construct explicitly two simple unitary systems α_1 and α_2, whose transfer functions are $\theta_{\alpha_1}(z) = \theta_1(z)$, $\theta_{\alpha_2}(z) = \theta_2(z)$ and $\alpha = \alpha_2\alpha_1$, respectively.

According to the Livsis–Brodskii theorem, two simple unitary systems having the same transfer function are unitarily equivalent, so we can use the model of Nagy–Foias for a simple unitary system and still keep the generality of the problem. Besides, the factorization $\theta(z) = \theta_2(z)\theta_1(z)$ is regular, so instead of the Nagy–Foias function model $\alpha = (X, U, V, A, B, C, D)$, we can decompose the function model $\hat{\alpha} = (\hat{X}, U, V, \hat{A}, \hat{B}, \hat{C}, \hat{D})$, where

$$\hat{X} = [L_2^+(V) \oplus \overline{\Delta_2 L_2(U_2)} \oplus \overline{\Delta_1 L_2(U_1)}]$$
$$\ominus \{(\theta\omega, \Delta_2\theta_1\omega, \Delta_1\omega) | \omega \in L_2^+(U)\}, \tag{2.1}$$

$$\hat{A}(\varphi \oplus \psi \oplus \phi) = (e^{-it}(\varphi(e^{it}) - \varphi(0)) \oplus e^{-it}\psi(e^{it}) \oplus e^{-it}\phi(e^{it})), \tag{2.2}$$

$$\hat{B}u = \left(\frac{\theta(e^{it}) - \theta(0)}{e^{it}} u \oplus \frac{\Delta_2(e^{it})\theta_1(e^{it})}{e^{it}} u \oplus \frac{\Delta_1(e^{it})}{e^{it}} u\right), \tag{2.3}$$

$$\hat{C}(\varphi \oplus \psi \oplus \phi) = \varphi(0), \tag{2.4}$$

$$\hat{D}u = \theta(0)u. \tag{2.5}$$

By Theorem 1, the subspace $\hat{X}_2$ (of $\hat{X}$) is invariant for $\hat{A}$ and it is the orthogonal complement in $\hat{X}$ of $\hat{X}_1$.

For the spaces $\hat{X}_1$ and $\hat{X}_2$, we construct two systems $\hat{\alpha}_1 = (\hat{X}_1, U_1, V_1, \hat{A}_1, \hat{B}_1, \hat{C}_1, \hat{D}_1)$ and $\hat{\alpha}_2 = (\hat{X}_2, U_2, V_2, \hat{A}_2, \hat{B}_2, \hat{C}_2, \hat{D}_2)$ as follows:

$$\begin{aligned}\hat{X}_1 = \{(\theta_2 u \oplus \Delta_2 u \oplus v) | u \in L_2^+(U_2), v \in \overline{\Delta_1 L_2(U_1)}\} \\ \ominus \{(\theta_2\theta_1\omega \oplus \Delta_2\theta_1\omega \oplus \Delta_1\omega) | \omega \in L_2^+(U)\},\end{aligned} \tag{2.6}$$

$$\hat{A}_1(\theta_2 u \oplus \Delta_2 u \oplus \Phi) = \left(\theta_2(e^{it})\left[\frac{u(e^{it}) - u(0)}{e^{it}}\right] \oplus \Delta_2(e^{it})\left[\frac{u(e^{it}) - u(0)}{e^{it}}\right] \oplus \frac{\Phi(e^{it})}{e^{it}}\right), \tag{2.7}$$

$$\hat{B}_1 u = \left(\theta_2(e^{it})\left(\frac{\theta_1(e^{it}) - \theta_1(0)}{e^{it}} u\right) \oplus \Delta_2(e^{it})\left(\frac{\theta_1(e^{it}) - \theta_1(0)}{e^{it}} u\right) \oplus \frac{\Delta_1(e^{it})}{e^{it}} u\right), \tag{2.8}$$

$$\hat{C}_1(\theta_2 u \oplus \Delta_2 u \oplus \Phi) = u(0), \tag{2.9}$$

$$\hat{D}_1 u = \theta_1(0) u, \tag{2.10}$$

$$\begin{aligned}\hat{X}_2 = [L_2^+(V) \oplus \overline{\Delta_2 L_2(U)} + \{0\}] \\ \ominus \{(\theta_2 u \oplus \Delta_2\omega, 0 | u \in L_2^+(U_2)\},\end{aligned} \tag{2.11}$$

$$\hat{A}_2(f \oplus g \oplus 0) = (e^{-it}(f(e^{it}) - f(0)) \oplus e^{-it} g(e^{it}) \oplus 0), \tag{2.12}$$

$$\hat{B}_2 u = (e^{-it}(\theta_2(e^{it}) - \theta_2(0)) u \oplus e^{-it}\Delta_2(e^{it}) u \oplus 0), \tag{2.13}$$

$$\hat{C}_2(f \oplus g \oplus 0) = f(0), \tag{2.14}$$

$$\hat{D}_2 u = \theta_2(0) u. \tag{2.15}$$

Then we have the following.

Theorem 2. *The simple unitary system of the form* (2.1)–(2.5) *has an explicit decomposition into two simple unitary systems* $\hat{\alpha}_1$ *and* $\hat{\alpha}_2$, $\hat{\alpha} = \hat{\alpha}_2\hat{\alpha}_1$, *by the formulas* (2.6)–(2.15).

Proof. Theorem 1 leads to $\hat{X} = \hat{X}_1 \oplus \hat{X}_2$.

Now, we have to prove all operators of the systems $\hat{\alpha}_1, \hat{\alpha}_2$ are correctly determined.

Firstly, $\hat{A}_1$ is an operator from $\hat{X}_1$ to $\hat{X}_1$. Indeed, since $u \in L_2^+(U_2)$, then we have $\frac{u(e^{it}) - u(0)}{e^{it}} \in L_2^+(U_2)$. Moreover, note that for $u \in L_2^+(U_2)$, and $\Phi \in \overline{\Delta_1 L_2(U_1)}$, $(\theta_2 u, \Delta_2 u, \Phi)$ belongs to the space $\hat{X}_1$ if and only if $\theta_1^* u + \Delta_1\Phi$ belongs to $L_2^-(U_1)$. Hence, for $(\theta_2 u, \Delta_2 u, \Phi) \in \hat{X}_1$, we have that

$$\theta_1^*\left(\frac{u(e^{it}) - u(0)}{e^{it}}\right) + \Delta_1\frac{\Phi(e^{it})}{e^{it}} = \frac{\theta_1^*(e^{it})u(e^{it}) + \Delta_1(e^{it})\Phi(e^{it})}{e^{it}} - \frac{\theta_1^*(e^{it})u(0)}{e^{it}}$$

belongs to $L_2^-(U_1)$. Therefore, $\hat{A}_1(\theta_2 u, \Delta_2 u, \Phi) \in \hat{X}_1$.

Secondly, $\hat{B}_1$ is an operator from U_1 to $\hat{X}_1$ because we have

$$\frac{\theta_1(e^{it}) - \theta_1(0)}{e^{it}} u \in L_2^+(U_2)$$

and

$$\theta_1^*(e^{it})\left[\frac{\theta_1(e^{it})-\theta_1(0)}{e^{it}}u\right]+\Delta_1(e^{it})\frac{\Delta_1(e^{it})}{e^{it}}u=\frac{u}{e^{it}}-\theta_1^*(e^{it})\frac{\theta_1(0)}{e^{it}}\in L_2^-(U_1).$$

The remainer operators are obviously determined.

Now, we prove $\hat{\alpha}=\hat{\alpha}_2\hat{\alpha}_1$. It is easy to verify that

$$\begin{aligned}
&(\hat{A}_1P_1+\hat{A}_2P_2+\hat{B}_2\hat{C}_1P_1)(\varphi,\psi,\phi)\\
&\quad=\hat{A}_1(\theta_2u,\Delta_2u,\Phi)+\hat{A}_2(\varphi-\theta_2u,\psi-\Delta_2u,0)+\hat{B}_2\hat{C}_1(\theta_2u,\Delta_2u,\Phi)\\
&\quad=\left[\theta_2(e^{it})\left(\frac{u(e^{it})-u(0)}{e^{it}}u\right),\Delta_2(e^{it})\left(\frac{u(e^{it})-u(0)}{e^{it}}u\right),\frac{\phi(e^{it})}{e^{it}}u\right]\\
&\qquad+\left(\frac{\varphi(e^{it})-\theta_2(e^{it})u(e^{it})-\varphi(0)+\theta_2(0)u(0)}{e^{it}},\frac{\psi(e^{it})-\Delta_2(e^{it})u(e^{it})}{e^{it}},0\right)\\
&\qquad+\left(\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}u(0),\frac{\Delta_2(e^{it})}{e^{it}}u(0),0\right)\\
&\quad=\left(\frac{\varphi(e^{it})-\varphi(0)}{e^{it}}u,\frac{\psi(e^{it})}{e^{it}},\frac{\Phi(e^{it})}{e^{it}}\right)=\hat{A}(\varphi,\psi,\Phi),\\
&(\hat{B}_1+\hat{B}_2\hat{D}_1)u=\hat{B}_1u+\hat{B}_2(\theta_1(0)u)\\
&\quad=\left[\theta_2(e^{it})\left(\frac{\theta_1(e^{it})-\theta_1(0)}{e^{it}}u\right),\Delta_2(e^{it})\left(\frac{\theta_1(e^{it})-\theta_1(0)}{e^{it}}u\right),\frac{\Delta_1(e^{it})}{e^{it}}u\right]\\
&\qquad+\left(\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}\theta_1(0)u,\frac{\Delta_2(e^{it})}{e^{it}}\theta_1(0)u,0\right)\\
&\quad=\left(\frac{\theta_2(e^{it})\theta_1(e^{it})-\theta_2(0)\theta_1(0)}{e^{it}}u,\frac{\Delta_2(e^{it})\theta_1(e^{it})}{e^{it}}u,\frac{\Delta_1(e^{it})}{e^{it}}u\right)\\
&\quad=\left(\frac{\theta(e^{it})-\theta(0)}{e^{it}}u,\frac{\Delta_2(e^{it})\theta_1(e^{it})}{e^{it}}u,\frac{\Delta_1(e^{it})}{e^{it}}u\right)=\hat{B}u,\\
&(\hat{C}_2P_2+\hat{D}_2\hat{C}_1P_1)(\varphi,\psi,\phi)\\
&\quad=\hat{C}_2(\varphi-\theta_2u,\psi-\Delta_2u,0)+\hat{D}_2\hat{C}_1(\theta_2u,\Delta_2u,\Phi)\\
&\quad=\varphi(0)-\theta_2(0)u(0)+\theta_2(0)u(0)\\
&\quad=\varphi(0)-\theta_2(0)u(0)+\hat{D}_2(u(0))=\varphi(0)=\hat{C}(\varphi,\psi,\phi),\\
&\hat{D}_2\hat{D}_1u=\theta_2(0)\theta_1(0)u=\theta(0)u=\hat{D}u.
\end{aligned}$$

Since $\hat{X}=\hat{X}_1\oplus\hat{X}_2$ and the operators of the systems $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{\alpha}$ satisfy the equalities of cascade coupling, we have $\hat{\alpha}=\hat{\alpha}_2\hat{\alpha}_1$.

The system $\hat{\alpha}_2$ is constructed according to the function model of Nagy–Foias so $\hat{\alpha}_2$ is simple, unitary, and has $\theta_2(z)$ as the transfer function. To prove that the system $\hat{\alpha}_1$ is simple, unitary, and has $\theta_1(z)$ as the transfer function, we consider the system $\alpha_1=(X_1,U_1,V_1,A_1,B_1,C_1,D_1)$ constructed by the Nagy–Foias model corresponding to the transfer function $\theta_1(z)$. For the state space X_1 of α_1,

$$X_1 = [L_2^+(V_1) \oplus \overline{\Delta_1 L_2(U_1)}] \ominus \{(\theta_1\omega, \Delta_1\omega) | \omega \in L_2^+(U_1)\},$$

we consider the operator

$$\Gamma : X_1 \to \hat{X}_1,$$
$$(u \oplus v) \mapsto (\theta_2 u \oplus \Delta_2 u \oplus v).$$

The operator is determined because we have that $(u \oplus v)$ belongs to X if and only if $\theta_1^* u + \Delta_1 v$ belongs to $L_2^-(U_1)$, and the latter happens if and only if $(\theta_2 u \oplus \Delta_2 u \oplus v)$ belongs to $\hat{X}_1$.

Obviously, the operator Γ is surjective. Moreover, for any two elements $(u_1 \oplus v_1)$, $(u_2 \oplus v_2)$ of X_1, we have

$$\begin{aligned}\langle \Gamma(u_1 \oplus v_1), \Gamma(u_2 \oplus v2)\rangle_{\hat{X}_1} &= \langle \theta_2 u_1, \theta_2 u_2\rangle + \langle \Delta_2 u_1, \Delta_2 u_2\rangle + \langle v_1, v_2\rangle \\ &= \langle \theta_2^*\theta_2 u_1 + \Delta_2^2 u_1, u_2\rangle + \langle v_1, v_2\rangle \\ &= \langle u_1, u_2\rangle + \langle v_1, v_2\rangle \\ &= \langle (u_1 \oplus v_1), (u_2 \oplus v_2)\rangle_{X_1}.\end{aligned}$$

Thus, Γ is unitary. Besides, it is easy to check that

$$\Gamma A_1 = \hat{A}_1\Gamma, \Gamma B_1 = \hat{B}_1, \; C_1 = \hat{C}_1\Gamma, \; D_1 = \hat{D}_1.$$

Hence, the system $\hat{\alpha}_1$ is unitary equivalent to α_1. Then $\hat{\alpha}_1$ is simple, unitary, and has $\theta_1(z)$ as the transfer function.

This completes the proof of the theorem. ■

3. Explicit Cascade Decomposition of a Unitary System by its Variant Subspace

Consider a unitary system $\alpha = (X, U, V, A, B, C, D)$ having X_2 as a subspace invariant for A. We construct explicitly two unitary systems α_1 and α_2, whose state spaces are $X_1 = X \ominus X_2$, respectively, and X_2 such that $\alpha = \alpha_2\alpha_1$.

The idea in this section is based on [6].

By hypothesis, since the subspace X_2 is invariant for A, then $T(X_2)$ is included in $X_2 \oplus V$ where T is defined in Definition 3. Let R be the orthogonal complement of $T(X_2)$ in the space $X_2 \oplus V$, i.e., $R = [X_2 \oplus V] \ominus T(X_2)$. Now we construct two systems $\alpha_1 = (X_1, U_1, V_1, A_1, B_1, C_1, D_1)$ and $\alpha_2 = (X_2, U_2, V_2, A_2, B_2, C_2, D_2)$ with $U_2 = V_1 = R$, as follows:

$$U_1 = U, \; V_2 = V, \; X_1 = X \ominus X_2, \tag{3.1}$$

$$A_1 = P_1 A|_{X_1}, \tag{3.2}$$

$$B_1 = P_1 B, \tag{3.3}$$

$$C_1 = P_2 A|_{X_1} \oplus C|_{X_1}, \tag{3.4}$$

$$D_1 = P_2 B \oplus D, \tag{3.5}$$

$$A_2 = A|_{X_2}, \tag{3.6}$$

$$C_2 = C|_{X_2}, \tag{3.7}$$

$$B_2 : R \to X_2,\ B_2(x_2 \oplus v) = x_2, \tag{3.8}$$

$$D_2 : R \to V,\ D_2(x_2 \oplus v) = v. \tag{3.9}$$

Then we have the following.

Theorem 3. *The unitary system has an explicit decomposition into two unitary systems* α_1, α_2 *defined by the formulas* (3.1)–(3.9), *and* $\alpha = \alpha_2\alpha_1$.

Proof. Obviously, the operators $A_1, B_1, A_2, B_2, C_2, D_2$ are determined. To prove C_1 and D_1 are determined, first we observe that, for any $x_2 \in X_2$, $v \in V$, $x_2 \oplus v$ belongs to R if and only if $A_2^* x_2 + C_2^* v = 0$. Indeed,

$$\begin{aligned} x_2 \oplus v \in R &\Leftrightarrow \langle x_2 \oplus v, ux_2' \rangle_{X_2 \oplus V} = 0, \quad \forall x_2' \in X_2 \\ &\Leftrightarrow \langle x_2 \oplus v, Ax_2' \oplus Cx_2' \rangle_{X_2 \oplus V} = 0, \forall x_2' \in X_2 \\ &\Leftrightarrow \langle x_2, A_2 x_2' \rangle_{X_2} + \langle v, C_2 x_2' \rangle_V = 0, \forall x_2' \in X_2 \\ &\Leftrightarrow \langle A_2^* x_2 + C_2^* v, x_2' \rangle_{X_2} = 0, \ \forall x_2' \in X_2 \\ &\Leftrightarrow A_2^* x_2 + C_2^* v = 0. \end{aligned}$$

With the operator C_1 defined by $C_1 x_1 = P_2 A(x_1 \oplus 0) \oplus C(x_1 \oplus 0)$, we have

$$\begin{aligned} A_2^* P_2 A(x_1 \oplus 0) + C_2^* C(x_1 \oplus 0) &= P_2 A^* A(x_1 \oplus 0) + P_2 C^* C(x_1 \oplus 0) \\ &= P_2(A^*A + C^*C)(x_1 \oplus 0), \end{aligned}$$

by virtue of α being unitary, so the equality (1.1) in Definition 4 leads to

$$A_2^* P_2 A(x_1 \oplus 0) + C_2^* C(x_1 \oplus 0) = P_2 I_X(x_1 \oplus 0) = 0.$$

Thus, $C_1 x_1 \in R$ and C_1 is an operator from X_1 to R. With the operator D_1 defined by $D_1 u = P_2 Bu \oplus Du$, it results that D_1 is an operator from U to R. Indeed, from relation (1.5) in Definition 4, we can deduce, for every $u \in U$,

$$\begin{aligned} &- A^* Bu = C^* Du \\ &\Rightarrow - P_2 A^* Bu = P_2 C^* Du \\ &\Rightarrow - A_2^* P_2 Bu = C_2^* Du \\ &\Rightarrow A_2^* P_2 Bu + C_2^* Du = 0. \end{aligned}$$

Hence, $P_2 Bu \oplus Du \in R$.

Now, we prove that the operators of the systems α, α_1, α_2 satisfy the equalities of cascade coupling:

$$\begin{aligned}
A_1P_1 + A_2P_2 + B_2C_1P_1 &= P_1A|_{X_1} + A|_{X_2} + B_2[P_2A|_{X_1} \oplus C|_{X_1}] \\
&= P_1A|_{X_1} + A|_{X_2} + P_2A|_{X_1} \\
&= A|_{X_1} + A|_{X_2} = A, \\
B_1 + B_2D_1 &= P_1B + B_2[P_2B \oplus D] \\
&= P_1B + P_2B = B, \\
C_2P_2 + D_2C_1P_1 &= C|_{X_2} + D_2[P_2A|_{X_1} \oplus C|_{X_1}] \\
&= C|_{X_2} + C|_{X_1} = C, \\
D_2D_1 &= D_2[P_2B \oplus D] = D.
\end{aligned}$$

Finally, we prove that the systems α_1, α_2 defined above are unitary. Consider the system α_2 with the operator

$$T_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} : X_2 \oplus R \to X_2 \oplus V = T(X_2) \oplus R.$$

For every $x_2 \in X_2$, we have

$$T_2(x_2 \oplus 0) = A(x_2 \oplus 0) \oplus C(x_2 \oplus 0) = T_2(x_2 \oplus 0) \tag{3.10}$$

and, for every $r = x_2 \oplus v$ belonging to R, we have

$$T_2(0 \oplus r) = B_2r \oplus D_2r = x_2 \oplus v = r. \tag{3.11}$$

From (3.10) and (3.11), it follows that $T_2 = T|_{X_2} \oplus I_R$. Since T is unitary, then so is T_2.

To show that α_1 is unitary, we first prove

$$T = (T_2 \oplus I_{X_1})(I_{X_2} \oplus T_1),$$

where

$$T_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}.$$

We have

$$I_{X_2} \oplus T_1 = \begin{pmatrix} I_{X_2} & 0 & 0 \\ 0 & A_1 & B_1 \\ 0 & C_1 & D_1 \end{pmatrix}$$

is an operator from $X_2 \oplus X_1 \oplus U$ to $X_2 \oplus X_1 \oplus R$ and

$$T_2 \oplus I_{X_1} = \begin{pmatrix} A_2 & 0 & B_2 \\ 0 & I_{X_1} & 0 \\ C_2 & 0 & D_2 \end{pmatrix}$$

is an operator from $X_2 \oplus X_1 \oplus R$ to $X_2 \oplus X_1 \oplus V$,

$$(T_2 \oplus I_{X_1})(I_{X_2} \oplus T_1) = \begin{pmatrix} A_2 & 0 & B_2 \\ 0 & I_{X_1} & 0 \\ C_2 & 0 & D_2 \end{pmatrix} \begin{pmatrix} I_{X_2} & 0 & 0 \\ 0 & P_1A|_{X_1} & P_1B \\ 0 & P_2A|_{X_1} \oplus C|_{X_1} & P_2B \oplus D \end{pmatrix}$$

$$= \begin{pmatrix} A_2 & B_2(P_2A|_{X_1} \oplus C|_{X_1}) & B_2(P_2B \oplus D) \\ 0 & P_1A|_{X_1} & P_1B \\ C_2 & D_2(P_2A|_{X_1} \oplus C|_{X_1}) & D_2(P_2B \oplus D) \end{pmatrix}$$

$$= \begin{pmatrix} A_2 & P_2A|_{X_1} & P_2B \\ 0 & P_1A|_{X_1} & P_1B \\ C_2 & C|_{X_1} & D \end{pmatrix}.$$

Since $P_1B + P_2B = B$, $C_2 + C|_{X_1} = C|_{X_2} + C|_{X_1} = C$, $A_2 + P_2A|_{X_1} + P_1A|_{X_1} = A|_{X_2} + A|_{X_1} = A$, then the matrix above is nothing else but the operator $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ from $X \oplus U$ to $X \oplus V$, where $X = X_2 \oplus X_1$. Thus, $(I_{X_2} \oplus T_1) = (T_2 \oplus I_{X_1})^{-1}T$. From the fact that T and T_2 are unitary, we conclude that T_1 is unitary and the proof is complete. ■

4. The Relation Between Cascade Decompositions of a Unitary System by its Transfer Function and by its Invariant Subspace

Note that in the first case of decomposition, given a system with its transfer function $\theta(z)$ having a regular factorization $\theta(z) = \theta_2(z)\theta_1(z)$, the intermediate space U_2 is also given. In the second case of decomposition, given a system α with its subspace X_2 being invariant for A, the intermediate space is not given, and we construct this space as $R = [X_2 \oplus V] \ominus T(X_2)$. However, we can prove that these two spaces U_2 and R are unitarily equivalent.

Theorem 4. *Let $\alpha = (X, U, V, A, B, C, D)$ be a simple, unitary system constructed according to the Nagy–Foias model. Assume the transfer function $\theta(z)$ of α has a regular factorization $\theta(z) = \theta_2(z)\theta_1(z)$, $\theta(z) \in \mathcal{A}(U, V)$, $\theta_i(z) \in \mathcal{A}(U_i, V_i)$, $i = 1, 2$, $U = U_1$, $V = V_2$, $U_2 = V_1$, and assume X_2 is a subspace invariant for A. Then the space U_2 is unitarily equivalent to the space $R = [X_2 \oplus V] \ominus T(X_2)$ through an operator defined as follows:*

$$\Gamma : U_2 \to R,$$
$$\Gamma u = (e^{-it}(\theta_2(e^{it}) - \theta_2(0))u, e^{-it}\Delta_2(e^{it})u) \oplus \theta_2(0)u.$$

Proof. First, Γ is determined since we have

$$A_2^*\left(\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}u, \frac{\Delta_2(e^{it})}{e^{it}}u\right)+C_2^*\theta_2(0)u$$
$$=((\theta_2(e^{it})-\theta_2(0))u-\theta_2\omega, \Delta_2(e^{it})u-\Delta_2\omega)$$
$$+(\theta_2(0)u-\theta_2(e^{it})\theta_2^*(0)\theta_2(0)u, -\Delta_2(e^{it})\theta_2^*(0)\theta_2(0)u),$$

where

$$\omega=\frac{1}{2\pi}\int_0^{2\pi}\left[e^{it}\theta_2^*(e^{it})\left(\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}\right)u+e^{it}\Delta_2(e^{it})\frac{\Delta_2(e^{it})}{e^{it}}u\right]dt$$
$$=\frac{1}{2\pi}\int_0^{2\pi}[u-\theta_2^*(e^{it})\theta_2(0)u]\,dt=u-\theta_2^*(0)\theta_2(0)u.$$

From this, it follows that

$$A_2^*\left(\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}u, \frac{\Delta_2(e^{it})}{e^{it}}u\right)+C_2^*\theta_2(0)u=0.$$

Hence, $\Gamma u\in R$.

We show that Γ preserves the scalar-product. Indeed

$$\langle\Gamma u_1,\Gamma u_2\rangle=\left\langle\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}u_1, \frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}u_2\right\rangle_{L_2(V)}$$
$$+\left\langle\frac{\Delta_2(e^{it})}{e^{it}}u_1, \frac{\Delta_2(e^{it})}{e^{it}}u_2\right\rangle_{L_2(U_2)}+\langle\theta_2(0)u_1,\theta_2(0)u_2\rangle_V$$
$$=-\langle\theta_2(e^{it})u_1,\theta_2(0)u_2\rangle+\langle\theta_2(0)u_1,\theta_2(0)u_2\rangle$$
$$-\langle\theta_2(0)u_1,\theta_2(e^{it})u_2\rangle+\langle\theta_2(0)u_1,\theta_2(0)u_2\rangle+\langle u_1,u_2\rangle$$
$$=\langle u_1,u_2\rangle.$$

Next, we prove that Γ is surjective.

Let $(\varphi,\psi,v)\in R\ominus\Gamma U_2$; we will prove $(\varphi,\psi,v)=0$.

Indeed, for any $(\varphi,\psi,v)\in R$, we have $\langle(\varphi,\psi,v),\Gamma u\rangle=0$ for every $u\in U_2$ if and only if the conditions below are satisfied:

$$\theta_2^*\varphi+\Delta_2\psi\in L_2^-(U_2),\ \varphi\in L_2^+(V),\ \psi\in\overline{\Delta_2L_2(U_2)},\tag{4.1}$$
$$A_2^*(\varphi,\psi)+C_2^*v=0,\tag{4.2}$$
$$\left\langle\varphi,\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}u\right\rangle+\left\langle\psi,\frac{\Delta_2(e^{it})}{e^{it}}u\right\rangle+\langle v,\theta_2(0)u\rangle=0.\tag{4.3}$$

Condition (4.3) is equivalent to

$$\langle e^{it}(\theta_2^*\varphi+\Delta_2\psi,u\rangle-\langle e^{it}\varphi,\theta_2(0)u\rangle+\langle v,\theta_2(0)u\rangle=0.$$

This implies

$$\langle\omega,u\rangle-\langle e^{it}\varphi,\theta_2(0)u\rangle+\langle v,\theta_2(0)u\rangle=0,$$

where

$$\omega = \frac{1}{2\pi}\int_0^{2\pi} e^{it}(\theta_2^*\varphi + \Delta_2\psi)\,dt \in U_2.$$

On the other hand, we have $\varphi \in L_2^+(V)$, $\theta_2(0)u \in U_2$. Hence

$$\langle e^{it}\varphi, \theta_2(0)u\rangle = 0.$$

Then condition (4.3) is equivalent to the condition $\langle \omega + \theta_2^*(0)v, u\rangle = 0$ for every u belonging to U_2. This implies $\omega + \theta_2^*(0)v = 0$.

Condition (4.2) is equivalent to

$$e^{it}\varphi(e^{it}) - \theta_2(e^{it})\omega + v - \theta_2(e^{it})\theta_2^*(0)v = 0, \tag{4.4}$$

$$e^{it}\psi(e^{it}) - \Delta_2(e^{it})\omega - \Delta_2(e^{it})\theta_2^*(0)v = 0. \tag{4.5}$$

From (4.4), it follows that

$$e^{it}\varphi(e^{it}) + v = \theta_2(e^{it})[\omega + \theta_2^*(0)v] = 0.$$

Therefore, $\varphi(e^{it}) = e^{-it}v \in L_2^-(V)$. Since $\varphi \in L_2^+(V)$, we must have $\varphi = 0$ and hence, $v = 0$.

From (4.5), it follows that

$$e^{it}\psi = \Delta_2[\omega + \theta_2^*(0)v] = 0.$$

Hence, $\psi = 0$.

We have already proved that $R \ominus \Gamma U_2 = \{0\}$, so we conclude Γ is surjective. Concretely, we can determine

$$\Gamma^{-1}(\varphi, \psi, v) = \omega + \theta_2^*(0)v, \text{ where } \omega = \frac{1}{2\pi}\int_0^{2\pi} e^{it}(\theta_2^*\varphi + \Delta_2\psi)\,dt.$$

Thus, Γ is unitary and this completes the proof. ■

Besides, U_2 and R are unitarily equivalent. We also observe that the operators A_2, B_2, C_2, D_2 can be considered as the same in both cases of decompositions. Indeed, since $\alpha = \alpha_2\alpha_1$, then $A_2 = A|_{X_2}$, $C_2 = C|_{X_2}$. In the first case of decomposition, we have

$$B_2u = P_{X_2}(\Gamma u),\ D_2u = P_{V_2}(\Gamma u),\ B_2u + D_2u = \Gamma u,$$

while in the second case, we have

$$B_2u = P_{X_2}u,\ D_2u = P_{V_2}u,\ B_2u + D_2u = u.$$

Once α_2 is constructed, α_1 is determined uniquely corresponding to α_2 from the equality $\alpha = \alpha_2\alpha_1$.

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