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Short Communication

An Algebraic Characterization of Complex Hyperbolic Hypersurfaces

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In 1970, S. Kobayashi conjectured that a generic projective hypersurface of large degree is hyperbolic. More precisely, let P^N, where $N = \left[{n+d \atop n} \right] - 1$, be the projective space whose points parametrize hypersurfaces of degree d in P^n (not necessarily reduced). Let $H(n,d) \subset P^N$ be the subset corresponding to hyperbolic hypersurfaces. Then one could ask *whether $H(n,d)$ contains a Zariski open subset of P^N for d large enough with respect to n.*

Due to Brody, the property of being Kobayashi hyperbolic for a compact variety X is equivalent to the property that every holomorphic map from C into X is constant. From now on, by hyperbolicity we mean the Brody hyperbolicity.

The Kobayashi conjecture attracts attention because of the conjectural relation of hyperbolicity and the finiteness of rational points on a projective variety (Lang's conjecture).

Lang's conjecture. *Let X be a projective variety defined over a number field k. Then X has only finitely many rational points on every finite extension of k if and only if the corresponding complex variety $X(C)$ is Brody hyperbolic.*

Due to Brody and Zaidenberg [3], the set $H_{n,d}$ of hyperbolic hypersurfaces is open in the usual topology. Thus, if Lang's and Kobayashi's conjectures are true, then the property of being hyperbolic would be an *algebraic property*.

The aim of this paper is to point out the part, already known to be "algebraic", of the property being hyperbolic. More precisely, we represent the set of hyperbolic hypersurfaces as an intersection of two sets, and show that one of them is Zariski dense and the other contains "sufficiently" many hypersurfaces.

Note that the Borel lemma is one of the most effective tools in the study of hyperbolic spaces. First, let us recall it.

Borel's Lemma. *Let $f_1, f_2, \ldots, f_n$ be nonzero holomorphic functions on C such that*

$$f_1 + f_2 + \ldots + f_n \equiv 0.$$

Then there exists a decomposition of the set of indices $\{1, 2, \ldots, n\} = \cup_\xi I_\xi$ such that

(i) $\# I_\xi \geq 2$ *for every* ξ,

(ii) *for* $\alpha, \beta \in I_\xi$, *the ratios of* f_α *and* f_β *are constants,*

(iii) *for any* $\xi, \sum_{i \in I_\xi} f_i \equiv 0$.

Borel's Lemma suggests us to go to the following.

Definition 1. *Let X be a projective hypersurface of degree d in P^n defined by the equation*

$$\sum_{\alpha_1 + \cdots + \alpha_{n+1} = d} c_{\alpha_1 \ldots \alpha_{n+1}} z_1^{\alpha_1} \ldots z_{n+1}^{\alpha_{n+1}} = 0,$$

where $c_\alpha \neq 0$ with α in a set $I(X)$ of indices. For simplicity, we write

$$X : \sum_{\alpha \in I(X)} c_\alpha M_\alpha = 0.$$

Let $f = (f_1, \ldots, f_{n+1})$ be a holomorphic curve in X. We call f a Borel curve *in X if there is a decomposition of the set of indices $I(X) = I_0 \cup_\xi I_\xi$ such that*

(i) $M_i \circ f \equiv 0$ *if and only if* $i \in I_0$;

(ii) $\# I_\xi \geq 2$ *for every* ξ, *and for* $\alpha, \beta \in I_\xi$, *the ratios of* $M_\alpha \circ f$ *and* $M_\beta \circ f$ *are constants;*

(iii) for any $\xi, \sum_{i \in I_\xi} c_i M_i f \equiv 0$.

Definition 2. *A hypersurface X is said to be* good *if every holomorphic curve in X is a Borel curve.*

Examples. (1) *X is the Fermat hypersurface*

$$z_0^d + z_1^d + \cdots + z_n^d = 0,$$

where $d \geq n^2$ [1].

(2) *X is a hypersurface defined by the equation as in Definition* 1, *and* $d > s(s-2)$ [2].

Definition 3. *A hypersurface X is said to be* Borel hyperbolic *if every Borel curve in X is constant.*

Examples. *We can verify that the surfaces in P^3 defined by the following equations are Borel hyperbolic.*

(1) $z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_0 z_1 + z_0 z_2 = 0.$

(2) $z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_0 z_1 z_2 = 0.$

Denote by $H(n,d)$, $G(n,d)$, $B(n,d)$, resp., the set of hyperbolic, good, Borel hyperbolic hypersurfaces. Then we have from the definitions

$$H(n,d) = G(n,d) \cap B(n,d).$$

The main result is the following.

Theorem 4. *For every n, there exits $d(n)$ such that, for $d \geq d(n)$, $B(n,d)$ is Zariski dense.*

Before giving the proof of Theorem 4 let us make some remarks.

Remark 1. Let X be a hypersurface of degree d in P^n defined by the equation

$$c_1 M_1 + \cdots + c_s M_s = 0.$$

Denote by X_l the hypersurface defined by

$$c_1 M_1^l + \cdots + c_s M_s^l = 0.$$

If $X \in B(n,d)$, then $X_l \in B(n,dl)$.

Remark 2. We can give explicit examples of hyperbolic hypersurfaces by the following method. First, choose a Borel hyperbolic hypersurface X (the set of them is dense by Theorem 4). Then X_l is hyperbolic if l is large enough. For example, assume X is of degree d, and the defining equation of X contains s monomials. Then it suffices to take $l > s(s-2)$. By using this method, we can prove that the hypersurfaces constructed by Brody–Green and Masuda–Noguchi are hyperbolic for smaller degrees.

From Theorem 4 and Remark 2, we have the following.

Corollary 5. *For every n, there exists $d(n)$ such that, for $d > d(n)$ and for every hypersurface X of degree d in a Zariski dense set Z in $P^{N(n,d)}$, X_l is hyperbolic if $l > N(N-2)$.*

Proof of Theorem 4 (An outline). The proof uses some techniques developed in [2]. Only the difference is that we will show when the set of hypersurfaces which are not Borel hyperbolic is a *proper* algebraic subset.

Let X be hypersurface defined by the equation as in Definition 1 and suppose X is not Borel hyperbolic. There exists a Borel curve in $X: f = (f_0, \dots, f_n)$: $C \to X$. For simplicity, we may assume that any $f_i \not\equiv 0$. Then we have the decomposition as in Definition 1 such that, for every ξ and $i, j \in I_\xi$, we have

$$f_0^{\alpha_{io}} \dots f_n^{\alpha_{in}} = b_{ij} f_0^{\alpha_{jo}} \dots f_n^{\alpha_{jn}},$$

where b_{ij} are nonzero constants.

Hence, the linear system of equations $AY = B$, where $A = \{\alpha_{il} - \alpha_{jl}\}$, $Y = \{y_0, \dots, y_n\}$, $B = \{\log b_{ij}\}$, has the solution $\{\log f_0, \dots, \log f_n\}$. Thus, the matrix A satisfies a certain condition on the rank (this is an algebraic condition).

On the other hand, by (iii) in Definition 1, there exist $(A_0, \ldots, A_n) \in P^n$ such that $\{c_i\} \in (C^*)^s$ satisfies the following equation:

$$\sum_{i \in \xi} c_i A_0^{\alpha_{i0}} \ldots A_n^{\alpha_{in}} = 0.$$

Hence, $\{c_i\} \in (C^*)^s$ belongs to the projection Σ of an algebraic subset in $(C^*)^s \times P^n$. It remains to prove that when $s = N(n, d)$ and d is large enough with respect to n, Σ is proper. It suffices to show the existence of a Borel hyperbolic hypersurface X corresponding to a point in $P^{N(n,d)}$ with all nonzero coordinates. Indeed, by Masuda–Noguchi's results [2], for every n, there exists $d(n)$ such that, for $n > d(n)$, there exists a hyperbolic hypersurface. Since the set of hyperbolic hypersurfaces is open (in the usual topology), we can take a hyperbolic hypersurface as required.

References

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