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Separate Analyticity and Related Subjects

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Abstract. After recalling important historic steps, we give a survey of recent results on separately analytic functions (and mappings) and related subjects.

1. Historical Preliminaries

Let D and G be open sets in $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively. Given a complex function $f(z, w)$ on $D \times G$, we note by f_z the function $w \to f(z, w)$ for every fixed $z \in D$, and by f^w the function $z \to f(z, w)$ for every fixed $w \in G$. If f is continuous in $D \times G$ and if f_z and f^w are analytic in G and D, respectively, for every fixed $(z, w) \in D \times G$, then by Fubini's theorem and Cauchy's integral formula for polydiscs, one sees easily that f is analytic in $D \times G$. Osgood [33, 34] remarked in 1899 that the continuity of f is superfluous; it suffices that f is locally bounded in $D \times G$.

The first important step is Hartogs' 1906 paper [16], where local boundedness is dropped. The proof is based on the all important, so-called Hartogs lemma. Another important result was performed by Bernstein.

Theorem 1. *If f is a complex function on $[-a, a] \times [-b, b]$ such that*

(i) $\forall x \in [-a, a]$, *f_x has an analytic continuation to the open ellipse $E(b, S)$ of focis $\pm b$ and mean axis bS,*

(ii) $\forall y \in [-b, b]$, *f^y has an analytic continuation to the open ellipse $E(a, R)$ of focis $\pm a$ and mean axis aR,*

(iii) *these separated continuations are uniformly bounded, then f has an analytic continuation to the open set*

$$\bigcup_{0<\theta<1} E(a, R^\theta) \times E(b, S^{1-\theta}).$$

This is the first result related to global analyticity of separately analytic functions of real variables. Unfortunately, it was practically unknown for the past fifty years. Now we know of it thanks to Akhiezer and Ronkin [1]. It has been rediscovered by Cameron and Storvick [13] in a weaker form.

1961 marked the appearance of Lelong's work [24] on separate real analyticity. Introducing new tools, in particular the Real Hartogs lemma, he proved the analyticity of separately real analytic functions of the classes L_D. This enabled him to prove the harmonic analog of the Hartogs theorem. At the same time, motivated by analyticity of some distribution kernels, Browder [10] also considered functions of the same type and gave a weaker result which can be deduced from the Bernstein theorem. Naturally, he was unaware of this.

Another generalization of the Hartogs theorem was performed by Shimoda [41] (1957) and Terada [45] (1967). Continuing Shimoda's work [41], Terada has considerably weakened an assumption in the Hartogs theorem: Instead of "f_z is analytic in G for every fixed $z \in D$", it suffices to assume this for every fixed $z \in E$, where E is nonpluripolar in each connected component of D. He showed also that this assumption is optimal if E is a F_σ-set.

We now refer to the crucial works of Siciak and Zaharjuta. Unaware of the Bernstein theorem, Siciak [42] gave in 1969 a more general version of this theorem without any boundedness assumption. One year later he put his result in a general context [43]; this is the well-known Siciak theorem. In 1976, Zahajuta [49] generalized the Siciak theorem, resulting in the Siciak–Zaharjuta theorem. We need some preliminaries for the statement of this theorem and other results.

For an open set $D \subset \mathbf{C}^m$ and E arbitrary subset of $\mathbf{C}^m$, we denote by $\omega(\cdot, E, D)$ the upper regularized of

$$\sup\{u \in PSH(D),\, u \leq 1,\, u \leq 0 \text{ on } E \cap D\}.$$

E is called locally pluriregular at a point z if $\omega(z, E, V) = 0$ for every open neighborhood V of z. This property is equivalent to the local polynomial condition (L) of Leja. We pose

$$\tilde{\omega}(\cdot, E, D) = \lim \omega(\cdot, E, D_s),$$

where (D_s) is an increasing sequence of relatively compact open subsets of D such that $\cup D_s = D$ (the second member is independent of the choice of (D_s)).

1.1. The Siciak–Zaharjuta Theorem

Let D and G be pseudoconvex domains in $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively, and E and F compact subsets of D and G, respectively, each of them is locally pluriregular at every of its points. Let f be a complex function on the crossed set

$$X = (E \times G) \cup (D \times F).$$

If f is separately analytic on X, i.e. f_z (resp. f^w) is analytic in G (resp. D) for every $z \in E$ (resp. $w \in F$), then f has an analytic continuation to

$$\hat{X} = \{(z, w) \in D \times G\colon \omega(z, E, D) + \omega(w, F, G) < 1\}.$$

Remark. The theorem is also true for connected Stein manifolds D and G. Siciak gave the p-separate analyticity version with $X = (D_1 \times E_2 \times \cdots \times E_p) \cup \cdots \cup (E_1 \times \cdots \times E_{p-1} \times D_p)$, where $E_j \subset D_j \subset \mathbf{C}$.

The present paper is divided into three parts:

- A general version of the Siciak–Zaharjuta theorem (and its direct consequences).
- Separate harmonicity and separate subharmonicity.
- Miscellaneous.

2. A General Version of the Siciak–Zaharjuta Theorem

Theorem 2. *Let $E \subset D \subset \mathbf{C}^m$, $F \subset G \subset \mathbf{C}^n$, where E and F are nonpluripolar, and D and G are open. Let f be a separately analytic function on $X = (E \times G) \cup (D \times F)$. Denote by E' the set of locally pluriregular points of E.*

(i) *If G is connected, then there exists a function $\hat{f}$ analytic in an open neighborhood Ω of $E' \times G$ such that $\hat{f} = f$ on $\Omega \cap X$ where E' is the set of locally pluriregular points for E in D.*

(ii) *If D is pseudoconvex, then there exists a function $\hat{f}$ analytic in*

$$\hat{X} = \{(z, w) \in D \times G\colon \tilde{\omega}(z, E', D) + \tilde{\omega}(w, F, G) < 1\}$$

such that $\hat{f} = f$ on $\hat{X} \cap X$.

Indications for the Proof.

(i) Schiffman [38] proved this part by Siciak's interpolation method.

(ii) We observe that without loss of generality, one can suppose the boundedness of D and G (that implies $\omega(\cdot, E, D) = \tilde{\omega}(\cdot, E, D)$ and $\omega(\cdot, F, G) = \tilde{\omega}(\cdot\, F, G)$). We need the following lemma: (ii) is true when E is $\mathscr{K}$-analytic with $\omega(z, E, D)$ instead of $\omega(z, E', D)$. This result is proved in [29] (see also [30]) by series expansion with respect to a Bergman's doubly orthogonal system. Now we indicate how to do without the hypothesis "E is $\mathscr{K}$-analytic": it suffices to use (i) and to observe that E' is a G_δ-set, so it is $\mathscr{K}$-analytic and non-pluripolar (Bedford and Taylor [8]), so $\omega(\cdot, E, D) = \omega(\cdot, E', D)$.

Remark. Sadullaev [36] has sketched a proof of (ii) using tensor product of two doubly orthogonal systems, with the assumption that D and G are pseudoconvex, and that E and F are Borel sets.

Very recently, Alehyane [4] gave a proof of the theorem using only the method of [29].

The theorem is probably true without the assumption "D (or G) is pseudoconvex".

2.1. Direct Consequences

- *Consequence of Part (i)*, [38, Corollary 3]. With the notations and hypothesis of the theorem, if $D \backslash E$ is of Lebesgue measure 0, then there exists $\hat{f}$ analytic in $D \times G$ such that $\hat{f} = f$ almost everywhere.
- *Consequence of Part (i)*, [38, Theorem 1]. Let Ω be an open subset of $\mathbf{R}^m$, G a domain in $\mathbf{C}^n$, and $f(x, w)$ a complex function defined on $\Omega \times G$ such that f_x is analytic in G for every $x \in \Omega$ and f^w is (real) analytic in Ω for every $z \in F$, where F is a nonpluripolar subset of G. Then f is analytic in $\Omega \times G$.

- *Consequence of Part (ii).* With the notations and hypothesis of the theorem, if we suppose D is connected and $\tilde{\omega}(\cdot, F, G) = 0$, then there exists $\hat{f}$ analytic in $D \times G$ such that $\hat{f} = f$ on X.

Remark. The last result is proved in [52] for the case $G = \mathbf{C}^n$. The general case is given by Zahajuta [49], however, the proof indicated by Zaharjuta cannot work; it uses the *local pluriregularity* of *compact* sets E and F.

3. Separate Harmonicity and Separate Subharmonicity

3.1. Harmonic Analogs of the Siciak and Terada Theorems

Theorem 3. [50] *For $j = 1, 2, \ldots, p$, let E_j be a compact subset of $\mathbf{R}^2$ satisfying the local Harmonic polynomial condition (H) at every of its points, and D_j be a domain in $\mathbf{R}^2$ containing E_j. If f is a separately harmonic function on*

$$X = (D_1 \times E_2 \times \cdots \times E_p) \cup \cdots \cup (E_1 \times \cdots \times E_{p-1} \times D_p),$$

then f has a harmonic continuation to

$$\hat{X} = \left\{(x_1, \ldots, x_p) \in D_1 \times \cdots \times D_p;\ \sum_{j=1}^{p} \omega(x_j, E_j, D_j) < 1\right\}.$$

This result has been proved earlier [26] under strong assumptions. We recall the following.

Definition of (H). *$E \subset \mathbf{R}^m$ satisfies (H) at a point x_0 if, for every neighborhood V of x_0, every family $\mathscr{F}$ of harmonic polynomials of m real variables, verifying*

$$\sup\{|f(x)|: f \in \mathscr{F}\} < \infty, \ \forall x \in V \cap E,$$

and every $b > 1$, there exists a neighborhood W of x_0 and a positive constant M such that

$$|f(x)| \leq M.b^{d^\circ f}, \ \forall x \in W, \forall f \in \mathscr{F}.$$

Very recently the author has given the following analog of the Terada theorem.

Theorem 4. [28] *Let D be a domain in $\mathbf{R}^m$, E a subset of D satisfying (H) at some point of D, and G an open subset of $\mathbf{R}^n$. If $f(x, y)$ is a complex function in $D \times G$ such that f_x is harmonic in G for every $x \in E$, and f^y is harmonic in D for every $y \in G$, then f is harmonic in $D \times G$.*

We now give a result similar to Cor. 3 of Shiffman [37] (see I.2. above).

Proposition 5. *In the preceding theorem if we suppose that f^y is harmonic in D for almost all $y \in G$, and f_x is harmonic in G for every $x \in E \subset D$, where E is non-pluripolar as a subset of $\mathbf{C}^m = \mathbf{R}^m + i\mathbf{R}^m$, then there exists $\hat{f}$ harmonic in $D \times G$ such that $\hat{f} = f$ almost everywhere.*

Proof. Let

$$\hat{D} = \bigcup_{x \in D} \{z \in \mathbf{C}^m \colon \|z - x\| < 2^{-1/2} \operatorname{dist}(x, \partial D)\},$$

$$\hat{G} = \bigcup_{y \in G} \{w \in \mathbf{C}^n \colon \|w - y\| < 2^{-1/2} \operatorname{dist}(y, \partial G)\}.$$

Let $F \subset G$ such that $\operatorname{mes}(G \backslash F) = 0$ and f^y is harmonic in D for every $y \in F$. Let (D_s) be a sequence of subdomains of D such that $D_s \subset D_{s+1}$ and $\cup D_s = D$. For $s \geq s_0$, $E_s = E \cap D_s$ is nonpluripolar in $\mathbf{C}^m$. We remark that it suffices to prove that the conclusion is true for D_s instead of D ($s \geq s_0$). Let $\tilde{D}_s$ be a pseudoconvex open connected neighborhood of $\overline{D}_s$ (which is a connected polynomially convex compact of $\mathbf{C}^m$) in $\hat{D}$. For $y \in F$ (resp. $x \in E_s$), f^y (resp. f_x) is analytically continuable to $\tilde{D}_s$ (resp. $\hat{G}$), so we can define a function $\tilde{f}$ separately analytic on $X_s = (\tilde{D}_s \times F) \cup (E \times \hat{G})$ and equal to f on $(D_s \times F) \cup (E \times G)$. By Theorem 2(ii), there exists $\hat{f}$ analytic in

$$\hat{X}_s = \{(x, w) \in \tilde{D}_s \times \hat{G} \colon \omega(z, E_s, \tilde{D}_s) + \omega(w, F, \hat{G}) < 1\}$$

and is equal to f on $\hat{X}_s \cap X_s$. Following [15], F is locally pluriregular at every point of G, thus, $\omega(\cdot, F, G) = 0$ in G. On the other side, $\omega(\cdot, E_s, \tilde{D}_s) < 1$ in $\tilde{D}_s$, because E_s is nonpluripolar in the domain $\tilde{D}_s$. Thus, $D_s \times G \subset \hat{X}$, $\hat{f}$ is real-analytic in $D_s \times G$, and $\hat{f} = \tilde{f} = f$ on $D_s \times F$ (so $\hat{f} = f$ a.e. in $D_s \times G$). $\Delta\hat{f}$ is real-analytic in $D_s \times G$ and for every $(x_0, y_0) \in D_s \times G$, $\Delta_{x,y}\hat{f}(x_0, y_0) = \Delta_x f(\cdot, y_0)(x_0) + \Delta_y \hat{f}(x_0, \cdot) = \Delta_x f(\cdot, y_0) + \Delta_y f(y_0, \cdot)(y_0) = 0$. Because $E_s \times F$ is a uniqueness set for real-analytic functions in $D_s \times G$, we have $\Delta\hat{f} \equiv 0$.

Remark. Following the proof we have f harmonic in $D \times G$ if $F = G$. This result can be considered as an immediate consequence of the preceding theorem. In fact, following [8], E is locally pluriregular at a point $x_0 \in E$, so E verifies (H) at x_0. The converse is not true. One can find in [44] an example of a set $E \subset \mathbf{R}^2$, which is pluripolar in $\mathbf{C}^2$ and verifies (H).

3.2. Separate Subharmonicity

The following problem: "Let D and G be open sets in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Is every separately subharmonic function in $D \times G$ subharmonic?" has been open until 1988, when Wiegerinck [47] gave a very simple counter-example.

Subharmonicity of separately subharmonic functions was first studied by Lelong [24] and his student Avanissian [7]. They have given a positive answer under the hypothesis that f is locally upper bounded. Arsove [6] assumed only that f has a L^1_{loc} majorant. More recent results of this kind are due to Riihentauss [53] and Armitage and Gardiner [5]. Assumptions on the partial functions f_x and f^y are also considered:

(i) $f(x, y)$ is real-analytic and subharmonic in x, and harmonic in y [17] (this result can be considered as an immediate consequence of [38, Theorem 1] cited above in Sec. 2.1).

(ii) $f(x, y)$ is C^2 and subharmonic in x and harmonic in y [22].

We also cite a result of Cegrell and Sadullaev [14] used in [22].

Let B_1 and B_2 be open balls in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively and f a real function defined in a neighborhood of $\overline{B_1 \times B_2}$, subharmonic in x and harmonic in y. Then there exist two closed sets with empty interiors $E_1 \subset B_1$ and $E_2 \subset B_2$ such that f is subharmonic in $(B_1 \times B_2)\setminus(E_1 \times E_2)$.

We end this paragraph by a result of Wiegerinck and Zeinstra [48].

Every separately $(1, p)$-subharmonic function in $\mathbf{R}^n$ is subharmonic $(0 \leq p < n)$.

We recall that a function $f(x_1, \ldots, x_n)$ defined in an open set $\Omega \in \mathbf{R}^n$ is called separately $(1, p)$-subharmonic iff, for every fixed $x^0_{i_1}, \ldots, x^0_{i_{n-p}}$, the function $f(x_1, \ldots, x_n)$ restricted to $\Omega \cap \{x_j = x^0_j, j = i_1, \ldots, i_{n-p}\}$ is subharmonic.

4. Miscellaneous

4.1. Singular Sets of Separately Analytic Functions of Real Variables

For a function $f(x, y)$ separately analytic in an open set Ω of $\mathbf{R}^m_x \times \mathbf{R}^n_y$, we pose

$$A(f) = \{(x, y) \in \Omega\colon f \text{ is analytic in a neighborhood of } (x, y)\},$$
$$S(f) = \Omega\setminus A(f).$$

$S(f)$ is called the singular set of f.

Theorem 6 (Saint Raymond–Siciak). *Let S be a closed subset of an open set Ω in $\mathbf{R}^m \times \mathbf{R}^n$, and S_1 and S_2 the projection of S on $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively. Then S_1 and S_2 are pluripolar in $\mathbf{C}^m = \mathbf{R}^m + i\mathbf{R}^m$ and $\mathbf{C}^n = \mathbf{R}^n + i\mathbf{R}^n$, respectively, if and only if S is the singular set of a separately analytic function $f(x, y)$ on Ω $(x \in \mathbf{R}^m, y \in \mathbf{R}^n)$.*

This theorem is from Siciak [55] who proved a more general version (see also [51]). It has been first proved by Saint Raymond [54] for $m = n = 1$.

4.2. Separate Analyticity in Infinite Dimension

The Hartogs theorem was extended to separately analytic functions on complete metrizable topological vector spaces (F-spaces) by Noverraz [32]. We give here an extension of the Terada theorem.

Theorem 7. *Let D and G be open subsets in F-space A and B, respectively, D connected. Let E be a subset of D satisfying the local Polynomial Condition of Leja at some point of D. If $f\colon D \times G \to \mathbf{C}$ is such that f_x (resp. f^y) is analytic in G (resp. D) for every $x \in E$ (resp. $y \in G$), then f is analytic in $D \times G$.*

In [27], the author proved this result with an additional assumption on f which was dropped recently by Bui and Nguyen [12].

4.3. Separately Analytic Mappings

Let X be a complex analytic space having the Hartogs Extension Property (HEP). If f is an analytic mapping from

$$H(r) = \{(z, w) \in \mathbf{C}^2 \colon |z| < r \text{ or } |w| > 1 - r\}, 0 < r < 1,$$

into X, then f is the restriction to $H(r)$ of an analytic mapping from the bidisc Δ^2 into X.

Analogs of the Terada theorem for analytic mappings into $X \in$ (HEP) are given by Shiffman [39]. Recently, Alehyane [2] has extended Theorem 2 to these mappings: same statement, with a complex analytic space $X \in$ (HEP) instead of **C**.

4.4. Separately Meromorphic Functions and Mappings

Firstly, the meromorphic analog of the Hartogs theorem was given in the 1950s by Rothstein [35] and Sakai [37]. In 1976, Kazarian [20] gave the analog of the Siciak theorem, ten years after [21], the analog of the Siciak–Zaharjuta theorem. Recently, Alehyane [3] has generalized Theorem 2 to meromorphic mappings into a complex analytic space having the p-Meromorphic Extension Property (p-MEP) with $p = m + n$: same statement with $X \in$ (p-MEP) instead of **C**.

We recall the definition: $X \in$ (p-MEP) signifies that every meromorphic mapping from

$$H_p(r) = \{(z', z_p) \in \mathbf{C}^{p-1} \times \mathbf{C} \colon |z'| < r \text{ or } |z_p| > 1 - r\}, 0 < r < 1$$

into X is the restriction of a meromorphic mapping from the polydisc Δ^p into X. Every compact Kähler manifold verifies (p-MEP) for $p \geq 2$ [18] (see also [31] for similar results). In 1994, Shiffman [40] gave at Dolbeault's colloquium various results on separately meromorphic mappings into compact Kähler manifolds.

4.5. Applications

Results on separately analytic functions have various applications in Partial Differential Equations and Theoretical Physics (Feynman Integrals, "Edge of the Wedge" theorem). For the 1960s, see the Introduction of [11]. Akhiezer and Ronkin [1] gave an application to the "Fine End of the Wedge". A more recent application to the study of Anosov and geodesic flows is in [19].

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