Vietnam Journal of Mathematics **25**:2 (1997) 91–98

Vietnam Journal
of
MATHEMATICS

Characterization of Rings Using Weakly Projective Modules

Dingguo Wang
Department of Mathematics, Qufu Normal University,
Qufu, Shandong, 273165, China

Received July 25, 1995

Abstract. Some rings characterized using projective modules are given. Our results generalize several well-known results by Golan [3], Huynh and Smith [5], and Rangaswamy and Vanaja [9].

1. Introduction

In their groundbreaking papers [8, 13], Miyashita, Wu and Jans introduced the notions of a self-projective or quasi-projective cover. These generalized the classical concepts of a projective module and a projective cover. Since its introduction, the study of quasi-projective modules and its generalization has been pursued with some success (see [6, 9, 10, 11, 14] etc.). Those notions turned out to be important tools in the ring and module theory nowadays. However, except for [3, 4], which describes some rings in terms of quasi-projective modules and quasi-projective covers, little results on characterizations of rings using quasi-projective modules have been obtained. Now many well-known theorems about projective and self-projective modules have been generalized by using weaker properties and an interest has grown for those "projective properties", and there are a number of well-known theorems which characterize rings in terms of those projectives (cf. [9, 10, 11, 14]). Our results concern the characterization of rings using the notion of weakly projective modules which was introduced by Zoschinger [14].

Throughout, all rings considered have an identity and modules are unital left modules. We will freely make use of the notation, terminology and results of [3, 4, 12].

Let M be an R-module. Recall that an R-module Q is called M-projective if, given an epimorphism ϕ of M onto another R-module N, every homomorphism $f: Q \to N$ can be lifted to a homomorphism $g: Q \to M$ relative to ϕ. Thus, an R-module is projective if and only if it is M-projective for all R-module M, and M is quasi-projective if and only if it is M-projective. M is called pseudo-projective [10] if, for any epimorphism $g: M \to A$ and $f: M \to A$, there exists $h \in \mathrm{End}(M)$ such that $f = gh$.

Following [11], we call a module underprojective if, for homomorphism $g: M/N \to M/N$, where N is a submodule of M, there exists a homomorphism

$g: M \to M$ such that $fh = hg$, where $h: M \to M/N$ is the canonical map. We call a module M weakly projective[a] if, for every pair (A, B) of submodules of M with $M = A + B$, there exists an endomorphism $f: M \to M$ such that $\mathrm{Im}(f) \subseteq A$ and $\mathrm{Im}(1 - f) \subseteq B$. Obviously, quasi-projective modules are underprojective. Tuganbaev [11, Lemma 2.1] proved that underprojective modules are weakly projective. Clearly, direct summand of weakly projective modules is also weakly projective.

The following lemma is very useful in this paper.

Lemma 1.1. *Let P be projective and $P \oplus M$ weakly projective. If there is an epimorphism $h: P \to M$, then M is projective.*

Proof. It is clear by [14, Lemma 1.2]. ■

2. Characterizing Rings by Weakly Projective Modules

A ring R is left PP if each principal left ideal is projective. We denote by R_n the ring of $n \times n$ matrices over R. If M is an R-module, then M^n is the product of n copies of M.

First, we give two characterizations of left PP-rings by means of weakly projective modules.

Proposition 2.1. *The following are equivalent:*
(1) *R is a left PP-ring.*
(2) *For any $r \in R$, $R \oplus Rr$ is weakly projective.*

Proof. (1) ⇒ (2) is trivial.

(2) ⇒ (1). Since there exists an epimorphism $R \to Rr$, by Lemma 1.1, Rr is projective. ■

A ring R is left (semi-)hereditary in case each (finitely generated) left ideal of R is projective. It is well known that R is left (semi-)hereditary if and only if each (finitely generated) submodule of a projective left R-module is projective. Golan [4] proved that a ring is left (semi-) hereditary if and only if (finitely generated) submodules of a projective left R-module are quasi-projective, if and only if principal left ideals of $\mathrm{End}(F)$ are quasi-projective for any (finitely generated) free R-module F. Here, we have the following.

Theorem 2.2. *Let R be a ring. The following conditions are equivalent:*
(1) *R is left hereditary.*
(2) *Every submodule of a projective R-module is weakly projective.*
(3) *Every principal left ideal of* End(F) *is weakly projective for any free R-module F.*

Proof. The implication (1) ⇒ (2) is trivial.

[a] Zoschinger [14] called it Kostetig.

(2) ⇒ (1). Let N be a submodule of a projective R-module P. Let F be a free R-module with an epimorphism $F \to M$. Then $F \oplus N$ is a submodule of the projective R-module $F \oplus M$, so $F \oplus N$ is weakly projective. Hence, N is projective and R is hereditary.

(1) ⇒ (3). If R is left hereditary, then S is left PP by [1, Theorem 2.3].

(3) ⇒ (1). If F is a free R-module with endomorphism ring S, then F^2 is a free R-module with endomorphism ring S_2. By (3), each principal left ideal of S_2 is weakly projective, so S is left PP by Proposition 2.1 and R is left hereditary by [1, Theorem 2.3]. ■

An analogous result for semihereditary rings is the following.

Theorem 2.3. *Let R be a ring. The following are equivalent:*

(1) *R is left semihereditary.*

(2) *Every finitely generated submodule of a (finitely generated) projective R-module is weakly projective.*

(3) *Every finitely generated (principal) left ideal of R_n is weakly projective for all $n \geq 1$.*

Using ideas of Huynh and Smith [5] and Liu [7], we can prove the following.

Theorem 2.4. *Let R be a ring. The following statements are equivalent:*

(1) *R is left hereditary.*

(2) *There exists a cardinal c such that every submodule of a projective left R-module is the direct sum of a weakly projective module and a c-limited ES-module.*

Proof. The implication (1) ⇒ (2) is trivial.

(2) ⇒ (1). Let M be a projective left R-module and N a submodule of M. There exists an exact sequence as follows:

$$0 \to K \to P \to N \to 0,$$

where P is projective. Set $L = N \oplus P$. Then L is a submodule of the projective left R-module $M \oplus P$.

Let $\{S_\omega\colon \omega \in \Omega\}$ denote a collection of representatives of the isomorphism classes of simple left R-modules and let $S = \oplus_{\omega\in\Omega} S_\omega$. Let K be an index set with $|K| \geq c$, and for each $\alpha \in K$, let $T_\alpha = S$, then define $T = \oplus_{\alpha\in K} T_\alpha$. Let I be an index set with $|I| > |E(T)|$. For each x in I, let $L_x = L$, and $F = \oplus_{x\in I} L_x$. Since L_x is a submodule of the projective left R-module $M \oplus P$, we obtain that F is a submodule of the projective left R-module $\oplus_{x\in I} M \oplus P$. By assumption, there exists a weakly projective module A and a c-limited ES-module B such that $F = A \oplus B$. Note that $\mathrm{Soc}(B)$ is a direct sum of at most c simple submodules of B; it is clear that there exists a monomorphism $f\colon \mathrm{Soc}(B) \to T$. Thus, we obtain a homomorphism $g\colon B \to E(T)$ such that $g|_{\mathrm{Soc}(B)} = f$. Since B is an ES-module, $\mathrm{Soc}(B)$ is an essential submodule of B, which implies that g is a monomorphism. Thus, we have $|B| \leq |E(T)|$. For each $b \in B$, there exists a finite subset $I(b)$ of I such that $b \in \oplus_{x\in I(b)} L_x$. Let $I' = \cup_{b\in B} I(b)$. If $|B|$ is finite, then I' is finite. Thus, $|I'| \leq |E(T)|$.

Now suppose $|B|$ is an infinite cardinal, then $|I'| \geq |B| \leq |E(T)|$. Set $I'' = I - I'$. From the construction of I, it follows that $|I| > |E(T)|$, and thus, $I'' \neq \emptyset$. Now, let $G = \oplus_{x \in I'} L_x$ and $H = \oplus_{x \in I''} L_x$. Then we have $F = G \oplus H = A \oplus B$ and $B \leq G$. Thus, it follows by modularity that $G = (A \cap G) \oplus B$. So $F = A \oplus B = (A \cap G) \oplus B \oplus H$, which implies that $A \cong (A \cap G) \oplus H$. Since A is weakly projective, it follows that H is weakly projective, too. Thus, $L = N \oplus P = L_x$, a direct summand of H, is weakly projective. Hence, N is projective and thus, R is a left hereditary ring. ■

Let R be a domain. R is called a Dedekind domain if R is a hereditary ring. We have the following.

Proposition 2.5. *Let R be a domain. The following statements are equivalent:*
(1) *R is a Dedekind domain.*
(2) *There exists a cardinal c such that every submodule of a projective left R-module is the direct sum of a weakly projective module and a c-limited ES-module.*

Koehler [6] and Golan [4] characterized semisimple rings using quasi-projective modules, and Tiwary and Pandeya [10] did so using pseudo-projectives. We can use Lemma 1.1 to generalize some of their results.

Theorem 2.6. *Let R be a ring. The following are equivalent:*
(1) *R is semisimple.*
(2) *Every (finitely generated) R-module is weakly projective.*
(3) *Every 2-generated R-module is weakly projective.*
(4) *The class of all weakly projective modules is closed under finite direct sums.*
(5) *The direct sum of two quasi-projective R-modules is weakly projective.*
(6) *The class of all weakly projective modules is coincidental with the class of all projective modules.*
(7) *There exists a cardinal c such that every R-module is the direct sum of a weakly projective module and a c-limited ES-module.*

Proof. The implication $(1) \Rightarrow (2) \Rightarrow (3)$, $(1) \Rightarrow (4) \Rightarrow (5)$, $(1) \Rightarrow (6)$, and $(1) \Rightarrow (7)$ are trivial.

$(3) \Rightarrow (1)$. Let I be a left ideal of R. Since $R \oplus (R/I)$ is weakly projective, R/I is projective. Therefore, I is a direct summand of R, proving (1).

$(5) \Rightarrow (1)$. If T is a simple R-module, then $R \oplus T$ is weakly projective by (5) and whence T is projective, hence R is semisimple by [13, 20.7].

$(6) \Rightarrow (1)$. Since every simple R-module S is quasi-projective, S is weakly projective. Then S is projective by (6). Thus R is semisimple.

$(7) \Rightarrow (1)$. Suppose M is a simple left R-module. There exists an exact sequence as follows:

$$0 \to K \to R \to M \to 0.$$

Set $L = M \oplus R$. Let $\{S_\omega : \omega \in \Omega\}$ denote a collection of representatives of the isomorphism classes of simple right R-modules and let $S = \oplus_{\omega \in \Omega} S_\omega$. Let K be an index set with $|K| \geq c$, and for each $\alpha \in K$, let $T_\alpha = S$ and define $T = \oplus_{\alpha \in K} T_\alpha$. Let I be an index set with $|I| > |E(T)|$. For each x in I, let $L_x = L$, and $F = \oplus_{x \in I} L_x$. Analogous to the proof of Theorem 2.4, we can prove that $L = M \oplus R$ is weakly projective. Hence, M is projective, and thus, R is semisimple.

3. Characterizing Rings by Weakly Direct Projective Covers

Golan [3] proved that R is left (semi-)perfect if and only if every (finitely generated) module has quasi-projective cover. Tiwary and Pandeya [10] introduced the concept of pseudo-projective covers and characterizations of semisimple rings and perfect rings using pseudo-projective modules and pseudo-projective covers.

For weak projectivity, we introduce the following concept.

Definition. *We call an epimorphism $f\colon Q \to M$ a weakly projective cover of M if Q is weakly projective and* $\ker f$ *is small in Q.*

We can prove the following.

Lemma 3.1. *Let P be a projective module and $P \oplus M$ has a weakly projective cover. If there is an epimorphism $f\colon P \to M$, then M has a projective cover.*

Proof. Suppose $f\colon P \oplus M$ is an epimorphism and $g\colon Q \to P \oplus M$ is the weakly projective cover of $P \oplus M$, *and $\phi\colon P \oplus M \to P$ is the projection map. Then there is an exact sequence*

$$0 \longrightarrow g^{-1}(M) \longrightarrow Q \overset{\phi g}{\longrightarrow} P \longrightarrow 0,$$

hence, $Q \cong P \oplus g^{-1}(M)$. Let $g' = g|_{g^{-1}}(M)$. Then we have the exact sequence

$$0 \to \ker(g) \to g^{-1}(M) \overset{g'}{\to} M \to 0.$$

Since $\ker(g)$ is small in Q, $\ker(g') = \ker(g)$ is small in $g^{-1}(M)$. In order to prove that $g^{-1}\colon g^{-1}(M) \to M$ is a projective cover of M, it only needs to prove that $g^{-1}(M)$ is a projective module.

Since P is projective, there is a homomorphism $h\colon P \to g^{-1}(M)$ such that $f = g'h$. Let $x \in g^{-1}(M)$, since f is an epimorphism, there exists $p \in P$ such that $g(x) = g'(x) = f(p)$, hence $g(h(p)) = g'(h(p)) = f(p) = g(x)$. Therefore, $x \in \mathrm{Im}(h) + \ker(g)$. Thus, $g^{-1}(M) = \mathrm{Im}(h) + \ker(g)$. By the fact that $\ker(g)$ is small in $g^{-1}(M)$, $g^{-1}(M) = \mathrm{Im}(h)$, i.e., h is an epimorphism. Because $P \oplus g^{-1}(M) \cong Q$ is weakly projective, $g^{-1}(M)$ is projective. ■

Proposition 3.2. *The following conditions are equivalent for any ring R:*

(1) *R is semisimple.*

(2) *Every R-module with a projective cover is precisely weakly projective module.*

Proof. Trivially, (1) implies (2).

Assume (2). Let M be a weakly projective module. By assumption, M possesses a projective cover P. Then $P \oplus M$ will have a projective cover and hence is weakly projective by hypothesis. Hence, M is projective. Since any simple R-module is quasi-projective, it becomes projective. Thus, R is semisimple. ∎

It is well known that a ring R is left perfect if and only if every flat left R-module is projective. The following two theorems characterize left (semi-)perfect rings by means of weakly projective modules and weakly projective covers. These results generalize several well-known results by Golan [3] and Rangaswamy and Vanaja [9].

Theorem 3.3. *The following conditions are equivalent for a ring R:*
(1) *R is left perfect.*
(2) *Every left R-module has a weakly projective cover.*
(3) *Every flat left R-module has a weakly projective cover.*
(4) *Every flat left R-module is weakly projective.*
(5) *A direct limit of quasi-projective R-modules is weakly projective.*
(6) *A direct limit of finitely generated quasi-projective R-module is weakly projective.*
(7) *There exists a cardinal c such that every flat left R-module is the direct sum of a weakly projective module and a c-limited ES-module.*

Proof. The implications (1) ⇒ (2) ⇒ (3), (4) ⇒ (3), (1) ⇒ (7) and (5) ⇒ (6) are trivial. The implication (1) ⇒ (5) is clear by [9, Theorem 6.1]. The implication (7) ⇒ (1) is analogous to the proof of Theorem 2.4.

(3) ⇒ (1). By [2, Theorem 2.1], it is only to prove that every flat left R-module M has a projective cover. Take a projective module P and an epimorphism $P \to M$. Since $P \oplus M$ is flat, $P \oplus M$ has a weakly projective cover, by hypothesis. Then Lemma 3.1 implies that M has a projective cover.

(6) ⇒ (4). Since, by [12, 36.2], every flat R-module is a direct limit of finitely generated projective R-modules, the implication is immediate. ∎

Corollary 3.4. *If the direct limit of (finitely generated) left weakly projective R-modules is also weakly projective, then R is left perfect.*

Theorem 3.5. *The following conditions are equivalent for a ring R:*
(1) *R is left semiperfect.*
(2) *Every finitely generated left R-module has a weakly projective cover.*
(3) *Every 2-generated left R-module has a weakly projective cover.*

Proof. The implications (1) ⇒ (2) ⇒ (3) are trivial.

(3) ⇒ (1). Let I be a left ideal of R, by hypothesis, $R \oplus (R/I)$ has a weakly projective cover. By Lemma 3.1, every cyclic R-module R/I has a projective cover, thus R is left semiperfect. ∎

A ring R is called quasi-perfect, if every finitely generated flat R-module is projective. It is well known that rings and left Noetherian rings are all quasi-perfect rings.

Proposition 3.6. *The following conditions are equivalent for a ring R:*
(1) *R is quasi-perfect.*
(2) *Every finitely generated flat R-module is weakly projective.*
(3) *Every finitely generated flat R-module has a weakly projective cover.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are trivial.

$(3) \Rightarrow (1)$. Let M be a finitely generated flat R-module. Take a finitely generated projective R-module P and an epimorphism $P \to M$. Since $P \oplus M$ is also a finitely generated R-module, by hypothesis, $P \oplus M$ has a weakly projective cover, hence, M has a projective cover by Lemma 3.1. Thus, R is quasi-perfect. ■

Recall that a ring R is called semilocal in case $R/J(R)$ is semisimple. An R-module M is called J-semisimple if the Jacobson radical of M is zero.

Proposition 3.7. *The following conditions are equivalent for a ring R:*
(1) *R is semilocal.*
(2) *Every J-semisimple R-module is weakly projective.*
(3) *Every finitely generated J-semisimple R-module is weakly projective.*
(4) *Every 2-generated, J-semisimple R-module is weakly projective.*

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$ are trivial.

$(4) \Rightarrow (1)$. Let $R' = R/J(R)$. To prove that R' is a semisimple ring, it needs to prove that every simple R'-module S is projective. Since R' and S are all J-semisimple as R-modules and $R' \oplus S$ is 2-generated, by hypothesis, $R' \oplus S$ is weakly projective as a R'-module. By Lemma 1.1, the simple R'-module S is projective. Thus, R' is a semisimple ring, i.e., R is semilocal. ■

We conclude this paper with the following remark.

Remark. It is of interest to ask, whether Morita equivalence preserves weakly projective modules. If this is true, using ideas of Golan [3] and [4], we can prove the following results:

(1) R is a left PP-ring if and only if every principal left ideal of R_2 generated by a diagonal matrix is weakly projective.
(2) R is a semisimple ring if and only if, for all $n \geq 1$, every cyclic R_n-module is weakly projective, if and only if there exists some $n > 1$ such that every cyclic R_n-module is weakly projective.
(3) R is left semiperfect if and only if for all natural numbers n, every cyclic left R_n-module has a weakly projective cover, where R_n denotes the ring of all $n \times n$ matrices over R, if and only if there exists a natural number $n > 1$ such that every cyclic left R_n-module has a weakly projective cover.

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