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Some Characterizations of *SC*-Modules

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Abstract. A ring R is called a right SC-ring if every singular right R-module is continuous. Similarly, a module M is said to be an SC-module if every M-singular module is continuous. In this paper, several characterizations of right SC-rings and SC-modules are given.

1. Introduction

Throughout this paper, R is an associative ring with identity and Mod-R denotes the category of all unital right R-modules. For amodule $M \in$ Mod-R, by $\sigma[M]$, we denote the subcategory of Mod-R whose objects are submodules of M-generated modules. The socle, the singular submodule and the injective hull of M are denoted by Soc(M), $Z(M)$, and $E(M)$, respectively. A module $N \in$ Mod-R is said to be semisimple if Soc$(N) = N$, nonsingular if $Z(N) = 0$, and singular if $Z(N) = N$. For a given module $M \in$ Mod-R, a module $N \in$ Mod-R is called M-singular if it is singular in $\sigma[M]$, i.e., there exists a module $K \in \sigma[M]$ with an essential submodule L such that $N \simeq K/L$. By [1, 2], the class of all M-singular modules and the class of all singular modules are closed under taking direct sums, submodules and homomorphic images. For a module $N \in$ Mod-R, the maximal M-singular submodule of N is called its M-singular submodule and is denoted by $Z_M(N)$.

For a module $M \in$ Mod-R we consider the following conditions:

(C_0) If $M = M_1 \oplus M_2$, then M_1 and M_2 are relatively injective.

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule of M isomorphic to a direct summand of M is itself a direct summand of M.

(C_3) If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

A module M is called *continuous* (resp., *quasi-continuous*) if it satisfies (C_1) and (C_2) (resp., (C_1) and (C_3)). By [6], every quasi-continuous module satisfies (C_0) (see [6, 2.10]).

It is easy to see that the following hierarchy holds:

semisimple $\Rightarrow$ quasi-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ (C_0).

A module $M \in$ Mod-R is defined to be an SI-module (resp., SC-module) if every M-singular right R-module is M-injective (resp., continuous). If $M = R_R$, we have notions of right SI-rings and right SC-rings, respectively.

Let $\mathscr{A}$ be a class of modules in Mod-R. We say that a module $N \in$ Mod-R is $\mathscr{A}$-injective if it is A-injective for every $A \in \mathscr{A}$.

The class of right SI-rings was introduced by Goodear [2]. Later, Yousif and Huynh developed this idea in relation to modules. Right SC-ring and SC-modules are weaker forms of right SI-rings and SI-modules, respectively. In this paper, we generalize some results in [5, 8] and develop the idea used in [4] to give some characterizations of SC-modules and right SC-rings.

2. Results

In the first part of this paper, we give some necessary and sufficient conditions for the fact that every module in a given class to be continuous. Then we apply these to obtain some characterizations of SC-modules, SC-rings, locally noetherian modules, right noetherian rings. In the second part, we give characterizations of SC-modules and SC-rings by requiring singular continuous modules satisfying the *restricted semisimple condition* (RSSC) to be injective relative to all singular modules.

Theorem 1. *Let $\mathscr{A}$ be a class of modules in* Mod-R *such that $\mathscr{A}$ is closed under taking direct sums and submodules. Then the following statements are equivalent:*

(a) *Every module in $\mathscr{A}$ is semisimple.*
(b) *Every cyclic module in $\mathscr{A}$ is semisimple.*
(c) *Every module in $\mathscr{A}$ is quasi-injective.*
(d) *Every module in $\mathscr{A}$ is continuous.*
(e) *Every module in $\mathscr{A}$ is quasi-continuous.*
(f) *Every module in $\mathscr{A}$ satisfies* (C_0).
(g) *Every module in $\mathscr{A}$ is $\mathscr{A}$-injective.*

Proof. (b) $\Leftrightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) are obvious.

(f) $\Rightarrow$ (g). Let A, B be modules in $\mathscr{A}$. Since $\mathscr{A}$ is closed under taking direct sums, we have $A \oplus B \in \mathscr{A}$. By (f), $A \oplus B$ satisfies (C_0), therefore, A is B-injective. Hence, A is $\mathscr{A}$-injective.

(g) $\Rightarrow$ (a). Let $A \in \mathscr{A}$ and B be a submodule of M. We must prove that B is a direct summand of A. Since $\mathscr{A}$ is closed under submodules, we have $B \in \mathscr{A}$. By (g), B is $\mathscr{A}$-injective. Then B is A-injective. Hence, B is a direct summand of A. ∎

As a consequence of Theorem 1 we obtain the following results.

Corollary 2. [8, Theorem 3] *The following statements are equivalent for a module* $M \in$ Mod-R:

(a) *M is a SC-module.*
(b) *Every M-singular right R-module is semisimple.*

(c) *Every M-singular cyclic right R-module is semisimple.*
(d) *Every M-singular right R-module is quasi-injective.*
(e) *Every M-singular right R-module is quasi-continuous.*
(f) *Every M-singular right R-module satisfies* (C_0).
(g) *Every M-singular right R-module is A-injective for each M-singular right R-module A.*

Corollary 3. *The following statements are equivalent for a ring R:*
(a) *R is a right SC-ring.*
(b) *Every singular right R-module is semisimple.*
(c) *Every singular cyclic right R-module is semisimple.*
(d) *Every singular right R-module is quasi-injective.*
(e) *Every singular right R-module is quasi-continuous.*
(f) *Every singular right R-module satisfies* (C_0).
(g) *Every singular right R-module is A-injective for each singular right R-module A.*

The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) in Corollary 3 have been obtained in [7].

Let c be a cardinal. Following [3], we call a module $M \in$ Mod-R c-limited if the cardinality of every family of independent submodules in M is at most c. In terms of c-limited modules, several characterizations of rings with chain conditions have been obtained in [3]. A module M with an essential socle is called an *ES*-module. A module is locally noetherian if every finitely generated submodule is noetherian. As is well known, a ring R is right noetherian if and only if the class of injective right R-modules is closed under taking direct sums (cf. [9]). By [5], this is equivalent to the fact that every direct sum of injective right R-modules is (quasi-)continuous. By [9], a module M is locally noetherian if and only if the class of M-injective right modules is closed under taking direct sums. On the other hand, we easily see that the condition (C_0) is inherited by direct summands. The following easy proposition extends some results obtained in [5].

Proposition 4. *The following conditions are equivalent for a module $M \in$ Mod-R:*
(a) *Every direct sum of M-injective right modules is quasi-continuous.*
(b) *Every direct sum of M-injective right R-modules satisfies the condition* (C_0).
(c) *There exists a cardinal c such that every direct sum of M-injective modules is a direct sum of a module satisfying* (C_0) *and a c-limited ES-module.*

In this case M is locally noetherian.

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a). Let A_i, $i \in I$ be M-injective modules and $B = \bigoplus_{i \in I} A_i$. Then $B \oplus B$ is a direct sum of M-injective modules. By (b), $B \oplus B$ satisfies (C_0). Thus, B is quasi-injective. Hence, B is quasi-continuous.

(b) $\Leftrightarrow$ (c) follows from [5, Theorem 1].

In this case, let $B = \bigoplus_{i \in I} A_i$ be a direct sum of M-injective modules A_i and $E(M)$ the injective hull of M. Then by (b), $B \oplus E(M)$ satisfies (C_0). Hence, B is $E(M)$-injective. Thus, B is M-injective as desired. ■

Corollary 5. *The following conditions are equivalent for a ring R:*
(a) *R is right noetherian.*
(b) *Every direct sum of injective right R-modules is quasi-continuous.*
(c) *Every direct sum of injective right R-modules satisfies the condition* (C_0).
(d) *There exists a cardinal c such that every direct sum of injective right R-modules is a direct sum of a module satisfying* (C_0) *and a c-limited ES-module.*

Similarly, we get some characterizations of semisimple modules.

Proposition 6. *The following conditions are equivalent for a module $M \in$ Mod-R:*
(a) *M is semisimple.*
(b) *Every module in $\sigma[M]$ is (quasi-)continuous.*
(c) *Every module in $\sigma[M]$ satisfies the condition* (C_0).
(d) *There exists a cardinal c such that every module in $\sigma[M]$ is a direct sum of a module satisfying* (C_0) *and a c-limited ES-module.*

Proof. (a) $\Leftrightarrow$ (b) follows from Theorem 1 and the fact that a module M is semisimple if and only if every module in $\sigma[M]$ is semisimple.
(b) $\Leftrightarrow$ (c) by Theorem 1.
(c) $\Leftrightarrow$ (d) follows from [5, Theorem 1].

Corollary 7. *The following conditions are equivalent for a ring R:*
(a) *R is semisimple.*
(b) *Every right (left) R-module is (quasi-)continuous.*
(c) *Every right (left) R-module satisfies the condition* (C_0).
(d) *There exists a cardinal c such that every right (left) R-module is a direct sum of a module satisfying* (C_0) *and c-limited ES-module.*

A module M is said to satisfy the RSSC if, for each essential submodule E of M, M/E is semisimple. It is clear that every semisimple module satisfies the RSSC, but the converse is not true in general. The following theorem characterizes SC-modules by M-singular continuous right R-modules satisfying the RSSC. The following lemma of [4] is important for the proof of the next theorem.

Lemma 8. [4, Lemma 1] *Let M be an indecomposable quasi-injective right R-module. If H is a submodule of H such that M/H is noetherian, then H is a continuous module.*

Theorem 9. *Let M be a right R-module and S_M the class of all M-singular right R-modules. Then the following conditions are equivalent:*
(a) *M is a SC-module.*
(b) *Every M-singular continuous right R-module is S_M-injective.*
(c) *Every M-singular continuous right R-module satisfying RSSC is S_M-injective.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.
(c) $\Rightarrow$ (a). By Theorem 1, it is enough to prove that every M-singular cyclic right R-module is semisimple.

Let N be an M-singular cyclic right R-module. First, we claim that N has a finite uniform dimension. Assume on the contrary that N has an infinite uniform dimension. Then N contains an infinite direct sum of nonzero submodules $\oplus_{i\in I}N_i$. For each $i \in I$, let x_i be a nonzero element in N_i. Then N contains the infinite direct sum $\oplus_{i\in I}x_iR$. Each x_iR contains a maximal submodule L_i. Then $\oplus_{i\in I}(x_iR/L_i)$ is an M-singular semisimple module. In particular, it is an M-singular continuous module satisfying the RSSC. Put $L = \oplus_{i\in I}L_i$. Then N/L contains a submodule K such that K is isomorphic to the direct sum $\oplus_{i\in I}(x_iR/L_i)$. Thus, K is an M-singular continuous module satisfying the RSSC. By (c), K is M-injective and hence N/L-injective. Thus, K is a direct summand of the cyclic module N/L, a contradiction.

To continue, we assume that $U_1, U_2, \ldots, U_n$ are finite many independent uniform submodules of N such that the sum

$$U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

is essential in N. We wish to show that every submodule U_i is simple. First, we show that each U_i is noetherian. Assume on the contrary that there exists a module U_i which is not noetherian. Then U_i contains an infinitely strictly ascending chain of the form:

$$y_1R \subset y_1R + y_2R \subset \cdots . \qquad (*)$$

Then y_1R contains a maximal submodule H_1. The factor module y_1R/H_1 is an M-singular simple module. By (c), it is M-injective, and hence, $(y_1R+y_2R)/H_1$-injective. This means that $(y_1R+y_2R)/H_1 = y_1R/H_1 \oplus N_1/H_1$, where N_1 is a suitable submodule of y_1R+y_2R containing H_1. For a maximal submodule H_2 of N_1 containing H_1, $(y_1R+y_2R)/H_2 \simeq ((y_1R+y_2R)/H_1)/(H_2/H_1) \simeq y_1R/H_1 \oplus N_1/H_2$ is semisimple. By induction we easily see that, for each integer number $m \geq 1$, $y_1R + y_2R + \cdots + y_mR$ contains a submodule H_m containing H_{m-1} such that,

$$(y_1R + y_2R + \cdots + y_mR)/H_m \simeq E_1 \oplus E_2 \oplus \cdots \oplus E_m,$$

where E_i is simple, for all $i = 1, 2, \ldots, m$. Since each M-singular simple right R-module is M-injective, we can find a submodule H contained in the union of the chain $(*)$ such that N/H has the infinitely generated socle. Then $\mathrm{Soc}(N/H)$ is S_M-injective, and hence, N/H-injective. Thus, it is a direct summand of the cyclic module N/H, a contradiction.

We now show that each module U_i satisfies the RSSC. Suppose there is a module U_i which does not satisfy the RSSC. Then U_i is not artinian. Since U_i is noetherian, it contains a submodule D maximal with respect to the property that the module $K = U_i/D$ is not artinian. Then U_i/D' is artinian for every submodule D' of U_i properly containing D. Hence, each factor module of K by its nonzero submodule is artinian, hence, semisimple. This means that K is a uniform module satisfying the RSSC. Let $V = E(K)$ be the injective hull of K and $Z_M(V)$ the M-singular submodule of V. Then V is uniform and $Z_M(V)$ is a nonzero fully invariant submodule of V containing K. The sum W of all cyclic submodules of V satisfying the RSSC is also a nonzero fully invariant submodule of V. Then $F = Z_M(V) \cap W$ is a nonzero fully invariant submodule of V. Hence, F is a quasi-injective submodule of V. Moreover, F is an M-singular module satisfying the RSSC. By (c), F is S_M-injective. Then it is $Z_M(V)$-injective. Hence, F is a direct summand of $Z_M(V)$. This means that $Z_M(V) = F$, and hence, $Z_M(V)$ satisfies the RSSC. Let x be a nonzero element in $Z_M(V)$. Then xR contains a maximal submodule X. Since xR/X is an M-singular simple module, xR/X is $Z_M(V)/X$-injective. Thus, xR/X is a direct summand of $Z_M(V)/X$, say, $Z_M(V)/X = xR/R \oplus Y/X$, for some submodule Y of $Z_M(V)$ containing X. Thus, $Z_M(V)/Y \simeq (Z_M(V)/X)/(Y/X) \simeq xR/X$ is simple. Since $Z_M(V)$ is quasi-injective, Y is then a continuous module by Lemma 8. On the other hand, Y is an M-singular submodule of $Z_M(V)$ satisfying the RSSC. By (c), Y is S_M-injective, and hence, $Z_M(V)$-injective. Thus, Y is a direct summand of $Z_M(V)$. Since $Z_M(V)$ is indecomposable, we have $Y = 0$. This means that $Z_M(V)$ is simple. Thus, K is simple, a contradiction. Hence, each module U_i satisfies the RSSC.

To end the proof, we may use the previous argument to show that each module U_i is simple. Then the sum

$$U = U_1 \oplus U_2 \oplus \cdots \oplus U_n$$

is a M-singular semisimple submodule of N. Again by (c), U is a direct summand of N. Thus, $N = U$, i.e., N is semisimple as desired. ■

Corollary 10. *Let R be a ring and S the class of all singular right R-modules. Then the following statements are equivalent:*

(a) *R is a right SC-ring.*

(b) *Every singular continuous right R-module is S-injective.*

(c) *Every singular continuous right R-module satisfying the RSSC is S-injective.*

We note that the following results have been obtained in [4]:

(i) *A ring R is semisimple if and only if every continuous module is injective.*

(ii) *A ring is right SI if and only if every singular continuous right R-module satisfying the RSSC is injective.*

One can obtain from (i) an interesting fact that, whenever a ring R is not semisimple, Mod-R contains two continuous right R-modules N and M such that $N \oplus M$ is not (quasi-)continuous.

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References

1. N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending Modules, Research Notes in Mathematics Series*, vol. 313, Pitman, London, 1994.
2. K. R. Goodearl, Singular torsion and splitting properties, *Memoirs Amer. Math. Soc.* **124** (1972).
3. D. V. Huynh and P. F. Smith, Some rings characterised by their modules, *Comm. Algebra* **18** (6) (1990) 1971–1988.

4. D. V. Huynh and S. T. Rizvi, An approach to Boyle's conjecture, preprint, January, 1995.
5. Liu Zhongkui, Characterizations of rings by their modules, *Comm. Algebra* **21** (10) (1993) 3663–3671.
6. S. H. Mohamed and B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Notes, vol. 147, Cambridge University Press, 1990.
7. S. T. Rizvi and M. F. Yousif, On continuous and singular modules, *Proc. Conf. Athens Ohio*, Lecture Notes in Mathematics, vol. 1448, Springer-Verlag, 1990, pp. 116–124.
8. N. V. Sanh, On *SC*-modules, *Bull. Austral. Math. Soc.* **48** (1993) 251–255.
9. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.