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Lower Estimations for the Lyapunov Exponents of Linear Systems of Differential Equations Perturbed by White Noise

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Abstract. The theory of Lyapunov exponents is a powerful tool in the qualitative theory of differential equations. Investigation of the behavior of the Lyapunov exponents is an effective way to study the problem of conditional stability of linear systems of differential equations. This paper deals with the behavior of the Lyapunov exponents of an arbitrary linear non-autonomous system of differential equations under small non-degenerate random perturbation. We obtain a lower estimation for Lyapunov exponents of a perturbed stochastic system by a kind of central exponents of the initial deterministic system.

1. Introduction

Given a linear system of differential equations

$$\dot{x} = A(t)x, \tag{1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $\sup_{t\in\mathbb{R}} \|A(t)\| < \text{const} < +\infty$, we shall consider its random perturbation

$$dy = A(t)y\,dt + \sigma \sum_{k=1}^{m} B_k y\, d\xi_k(t), \tag{2}$$

where $\xi_k(t)$ are mutually independent standard Wiener processes, $d\xi_k(t)$ are white noises on a probability space $(\Omega, \mathbb{P})$ and σ is a positive parameter. System (2) is a system of Ito differential equations. In case the matrix B_k has only one non-vanishing entry which is equal to 1 and is in (i,j) position we can interpret $\sigma B_k y\, d\xi_k(t)$ as a perturbation of coefficient $a_{ij}(t)$ of the system (1) by the white noise $d\xi_k(t)$ with intensity σ.

Throughout this paper we will assume that the perturbation satisfies the following non-degeneracy condition.

There exist positive numbers μ_1 and μ_2 such that for any vectors $y, z \in \mathbb{R}^n$

$$\mu_1 \|y\|^2 \|z\|^2 \le \sum_{k=1}^{m} (B_k y, z)^2 \le \mu_2 \|y\|^2 \|z\|^2. \tag{E}$$

The condition (E) means that (2) satisfies an elliptic condition.

Denote by $X(t,\tau)$ and $Y_\sigma(t,\tau;\omega)$ the Cauchy matrices of the systems (1) and (2), respectively. We give here the definition of the Lyapunov exponents λ_k, central exponents Ω_k and Θ_k ($k = 1,\ldots,n$) of (1) (cf. [5, 8, 10]).

Definition 1.1. *The numbers $\lambda_k, \Omega_k, \Theta_k$, $k = 1,\ldots,n$, defined by*

$$\lambda_k := \min_{\mathbf{R}^{n-k+1}\subset\mathbf{R}^n} \max_{\xi\in\mathbf{R}^{n-k+1}_*} \limsup_{t\to+\infty} \frac{1}{t}\ln\|X(t,0)\xi\|,$$

$$\Omega_k := \inf_{\mathbf{R}^{n-k+1}\subset\mathbf{R}^n} \int_{T\in\mathbf{R}^+} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln\|X((i+1)T,iT)_{|_{X(iT,0)\mathbf{R}^{n-k+1}}}\|,$$

$$\Theta_k := \sup_{\mathbf{R}^k\subset\mathbf{R}^n} \sup_{T\in\mathbf{R}^+} \limsup_{m\to+\infty} \frac{1}{mT}\sum_{i=0}^{m-1} \ln\|X(iT,(i+1)T)_{|_{X((i+1)T,0)\mathbf{R}^k}}\|^{-1},$$

where $\mathbf{R}^r$ denotes r-dimensional linear subspace in $\mathbb{R}^n$, $\mathbf{R}^r_* = \mathbf{R}^r\setminus\{0\}$, $X_{|_{\mathbf{R}^r}}$ denotes the restriction of X to $\mathbf{R}^r$, and are called Lyapunov exponents and central exponents of (1).

If in Definition 1.1 we replace $X(t,\tau)$ by $Y(t,\tau;\omega)$, we shall get Lyapunov exponents $\lambda_k(\sigma,\omega)$ and central exponents $\Omega_k(\sigma,\omega)$, $\Theta_k(\sigma,\omega)$ of the system (2). We note that the Lyapunov exponents and the central exponents of (2) actually do not depend on ω (see [12]). So we will drop ω and denote by $\lambda_k(\sigma)$ and $\Omega_k(\sigma)$, $\Theta_k(\sigma)$ the *Lyapunov exponents* and *central exponents* of (2), respectively.

Lyapunov exponents are introduced by Lyapunov [8] to investigate the stability of the origin of linear systems of differential equations. They play an important role in qualitative theory of differential equations. The central exponent Ω_1 has been introduced by Vinograd (see [4]). It is greater than the top Lyapunov exponent λ_1 and is an indicator for stability of all systems in a neighborhood of (1). In analogy with Vinograd's central exponent Ω_1 Millionshchikov introduced central exponents Ω_k, $k = 1,\ldots,n$, for investigating Lyapunov exponents. Ω_k makes an upper estimation for the Lyapunov exponent λ_k, whereas Θ_k makes a lower estimation. For more references on Lyapunov exponents and central exponents, we refer to [11, 12].

This work deals with the problem of parameter dependence of Lyapunov exponents of linear systems of stochastic differential equations. Although this problem attracts attention of many researchers, not much progress has been made. Results on the problem are concerned with particular classes of perturbations (see, e.g., [1–3, 13–15]). In a general set up, using his turning solution method Millionshchikov [9] proved the continuity of $\lambda(\sigma)$ provided (1) is absolutely regular. This paper is based on the method of Millionshchikov [9].

For a non-degenerate $n\times n$ matrix X, we denote by $d_1(X) \geq \cdots \geq d_n(X)$ its singular numbers, i.e., the positive square roots of the eigenvalues of the matrix

X^*X. Clearly, for any $k \in \{1, \ldots, n\}$, we have

$$d_k(X) = \inf_{\mathbf{R}^{n-k+1} \subset \mathbf{R}^n} \sup_{x \in \mathbf{R}_*^{n-k+1}} \frac{\|Xx\|}{\|x\|} = \sup_{\mathbf{R}^k \subset \mathbf{R}^n} \inf_{x \in \mathbf{R}_*^k} \frac{\|Xx\|}{\|x\|}. \tag{3}$$

For $k = 1, \ldots, n$, we put

$$e_k(X) = d_1(X) \cdots d_k(X).$$

It is easily seen that

$$e_k(X) = \sup_{\substack{x_1,\ldots,x_k \in \mathbb{R}^n \\ \dim \operatorname{span}\{x_1,\ldots,x_k\} = k}} \frac{G_{Xx_1 \cdots Xx_k}}{G_{x_1 \cdots x_k}}, \tag{4}$$

where $G_{x_1,\ldots x_k}$ denotes the Gram's volume of the vectors $x_1, \ldots, x_k$, i.e.,

$$G_{x_1 \ldots x_k} = \det \begin{pmatrix} (x_1, x_1) & (x_1, x_2) & \cdots & (x_1, x_k) \\ (x_2, x_1) & (x_2, x_2) & \cdots & (x_2, x_k) \\ \cdots & \cdots & \cdots & \cdots \\ (x_k, x_1) & (x_k, x_2) & \cdots & (x_k, x_k) \end{pmatrix}$$

Furthermore, for any $k \in \{i, \ldots, n\}$ and matrices X, Y,

$$e_k(XY) \leq e_k(X) e_k(Y). \tag{5}$$

Definition 1.2. *The numbers v_k defined by*

$$v_k := \limsup_{T \to +\infty} \limsup_{s \to +\infty} \frac{1}{sT} \sum_{i=0}^{s-1} \ln d_k(X((i+1)T, iT)), \quad k = 1, \ldots, n, \tag{6}$$

are called auxiliary exponents of the system (1).

These exponents are introduced by Millionshchikov [9] for the investigation of Lyapunov spectrum of linear systems of differential equations perturbed by random noises.

Definition 1.3. *The functions $v_k(\sigma, T)$ defined by*

$$v_k(\sigma, T) := \limsup_{m \to +\infty} \mathbf{E} \ln d_k(Y_\sigma((i+1)T, iT; \omega)), \quad k = 1, \ldots, n,$$

where $\mathbf{E}\xi(\omega)$ denotes the expectation of the random variable $\xi(\omega)$, are called auxiliary functions of the systems (2).

2. Main Result

Theorem 2.1. *For any $\varepsilon > 0$, there exists a positive number σ_1 such that for all $\sigma \in (0, 1)$ and $k = 1, \ldots, n$, the following inequalities hold*

$$\lambda_k(\sigma) \geq \Theta_k - \varepsilon.$$

For the proof of this theorem we need the following two lemmas which are proved in [5] and [6].

Lemma 2.2. *There is a positive constant c such that for any* $\varepsilon \in (0,1)$, *there exists* $\delta(\varepsilon) > 0$ *such that for all* $\sigma \in (0,1)$, $T \in \mathbf{N}$ *and* $k \in \{1, \ldots, n\}$, *the following inequalities hold*

$$|\Theta_k(\sigma) - \nu_k(\sigma, T)| \leq c\sqrt{\varepsilon} - \frac{1}{T}\ln\frac{\delta\sigma^{n3}}{2},$$

$$|\Omega_k(\sigma) - \nu_k(\sigma, T)| \leq c\sqrt{\varepsilon} - \frac{1}{T}\ln\frac{\delta\sigma^{n3}}{2}.$$

Lemma 2.3. *For any* $\sigma \in (0,1)$ *and* $k \in \{1, \ldots, n\}$, *there exists the limit*

$$\nu_k(\sigma) := \lim_{\substack{T\to+\infty \\ T\in\mathbf{N}}} \nu_k(\sigma, T),$$

and the following equalities hold

$$\Omega_k(\sigma) = \lambda_k(\sigma) = \Theta_k(\sigma) = \nu_k(\sigma).$$

Proof of Theorem 2.1. We fix an arbitrary $\varepsilon \in (0,1)$. From the definition of Θ_k, it follows that there exists a k-dimensional subspace $\mathbf{R}_1^k$ of $\mathbb{R}^n$ such that

$$\Theta_k \geq \sup_{T\in\mathbf{R}^+} \limsup_{\substack{m\to+\infty \\ m\in\mathbf{N}}} \frac{1}{mT}\sum_{i=0}^{m-1} \ln\|X(iT,(i+1)T)|_{X((i+1)T,0)\mathbf{R}_1^k}\|^{-1} \geq \Theta_k - \varepsilon.$$

By virtue of the property of the norm of operators we have the following property of quasimonotonity:

For any $l \in \mathbf{N}$ and $T \in \mathbb{R}^+$

$$\limsup_{\substack{m\to+\infty \\ m\in\mathbf{N}}} \frac{1}{mT}\sum_{i=0}^{m-1} \ln\|X(iT,(i+1)T)|_{X((i+1)T,0)\mathbf{R}_1^k}\|^{-1}$$

$$\leq \limsup_{\substack{m\to+\infty \\ m\in\mathbf{N}}} \frac{1}{mlT}\sum_{i=0}^{m-1} \ln\|X(ilT,(i+1)lT)|_{X((i+1)lT,0)\mathbf{R}_1^k}\|^{-1}.$$

Consequently, there exists a natural number $T_1 \in \mathbf{N}$ such that

$$\Theta_k \geq \limsup_{\substack{m\to+\infty \\ m\in\mathbf{N}}} \frac{1}{mT_1}\sum_{i=0}^{m-1} \ln\|X(iT_1,(i+1)T_1)|_{X((i+1)T_1,0)\mathbf{R}_1^k}\|^{-1} \geq \Theta_k - 2\varepsilon.$$

Hence, there exists an unbounded set $\mathscr{A} \subset \mathbf{N}$ such that

$$\Theta_k \geq \lim_{\substack{m\to+\infty \\ m\in\mathscr{A}}} \frac{1}{mT_1}\sum_{i=0}^{m-1} \ln\|X(iT_1,(i+1)T_1)|_{X((i+1)T_1,0)\mathbf{R}_1^k}\|^{-1} \geq \Theta_k - 2\varepsilon.$$

By virtue of the boundedness of $A(\cdot)$, it follows that for any $N \in \mathbf{N}$, the following inequalities hold

$$\Theta_k \geq \lim_{\substack{m\to+\infty \\ d(m,\mathscr{A})\leq N}} \frac{1}{mT_1}\sum_{i=0}^{m-1} \ln\|X(iT_1,(i+1)T_1)_{|_{X((i+1)T_1,0)\mathbf{R}_1^k}}\|^{-1} \geq \Theta_k - 2\varepsilon,$$

where $d(m,\mathscr{A})$ denotes the distance between the point m and the set $\mathscr{A}$ in the real axis. Consequently, for any $N \in \mathbf{N}$,

$$\begin{aligned}
&\liminf_{\substack{m\to+\infty \\ d(mN,\mathscr{A})\leq N}} \frac{1}{mNT_1}\sum_{i=0}^{m-1} \ln\|X(iNT_1,(i+1)NT_1)_{|_{X((i+1)NT_1,0)\mathbf{R}_1^k}}\|^{-1} \\
&\geq \liminf_{\substack{m\to+\infty \\ d(m,\mathscr{A})\leq N}} \frac{1}{mT_1}\sum_{i=0}^{m-1} \ln\|X(iT_1,(i+1)T_1)_{|_{X((i+1)T_1,0)\mathbf{R}_1^k}}\|^{-1} \geq \Theta_k - 2\varepsilon.
\end{aligned}$$

By the definition of Θ_k, this implies that for any $N \in \mathbf{N}$, the following inequalities hold

$$\begin{aligned}
\Theta_k &\geq \limsup_{\substack{m\to+\infty \\ d(mN,\mathscr{A})\leq N}} \frac{1}{mNT_1}\sum_{i=0}^{m-1} \ln\|X(iNT_1,(i+1)NT_1)_{|_{X((i+1)NT_1,0)\mathbf{R}_1^k}}\|^{-1} \\
&\geq \liminf_{\substack{m\to+\infty \\ d(mN,\mathscr{A})\leq N}} \frac{1}{mNT_1}\sum_{i=0}^{m-1} \ln\|X(iNT_1,(i+1)NT_1)_{|_{X((i+1)NT_1,0)\mathbf{R}_1^k}}\|^{-1} \\
&\geq \Theta_k - 2\varepsilon. \qquad (7)
\end{aligned}$$

We define a function $b(\cdot)\colon \mathbf{N} \to \mathbf{R}$ by

$$b(N) := \liminf_{\substack{m\to+\infty \\ d(mN,\mathscr{A})\leq N}} \frac{1}{mNT_1}\sum_{i=0}^{m-1} \ln e_k(X((i+1)NT_1, iNT_1)).$$

From (5) and the boundedness of $A(\cdot)$, it follows that $b(\cdot)$ satisfies the condition

$$b(lN) \leq b(N) \text{ for all } l,\ N \in \mathbf{N}. \qquad (8)$$

Set

$$b_0 := \inf_{N\in\mathbf{N}} b(N).$$

From (8), it follows that there exist a number $N_1 \in \mathbf{N}$ and an unbounded set $\mathscr{A}_1 \subset \mathscr{A}$ such that

$$b_0 \leq b(N_1) \leq b_0 + \frac{\varepsilon^2}{8n^3}$$

and

$$\begin{aligned}
&\lim_{\substack{m\to+\infty \\ d(mN_1,\mathscr{A}_1)\leq N_1}} \frac{1}{mN_1T_1}\sum_{i=0}^{m-1} \ln e_k(X((i+1)N_1T_1, iN_1T_1)) \\
&= \liminf_{\substack{m\to+\infty \\ d(mN_1,\mathscr{A})\leq N_1}} \frac{1}{mN_1T_1}\sum_{i=0}^{m-1} \ln e_k(X((i+1)N_1T_1, iN_1T_1)).
\end{aligned}$$

Consequently,

$$b_0 \leq \lim_{\substack{m\to+\infty \\ d(mN_1,\mathscr{A}_1)\leq N_1}} \frac{1}{mN_1T_1}\sum_{i=0}^{m-1}\ln e_k(X((i+1)N_1T_1, iN_1T_1)) \leq b_0 + \frac{\varepsilon^2}{8n^3}. \quad (9)$$

Hence, by (5) and the definition of b_0, for any $M \in \mathbf{N}$,

$$\begin{aligned} b_0 &\leq \liminf_{\substack{m\to+\infty \\ d(mMN_1,\mathscr{A}_1)\leq MN_1}} \frac{1}{mMN_1T_1}\sum_{i=0}^{m-1}\ln e_k(X((i+1)MN_1T_1, iMN_1T_1)) \\ &\leq \limsup_{\substack{m\to+\infty \\ d(mMN_1,\mathscr{A}_1)\leq MN_1}} \frac{1}{mMN_1T_1}\sum_{i=0}^{m-1}\ln e_k(X((i+1)MN_1T_1, iMN_1T_1)) \\ &\leq b_0 + \frac{\varepsilon^2}{8n^3}. \end{aligned} \quad (10)$$

Now we define a function $b'(\cdot)\colon \mathbf{N} \to \mathbb{R}$ by

$$b'(N) := \liminf_{\substack{m\to+\infty \\ d(mN,\mathscr{A}_1)\leq N}} \frac{1}{mNT_1}\sum_{i=0}^{m-1}\ln e_{k-1}(X((i+1)NT_1, iNT_1)).$$

This function is an analogue of $b(\cdot)$. By (5) and the boundedness of $A(\cdot)$, we have

$$b'(lN) \leq b'(N) \text{ for all } l,\, N \in \mathbf{N}. \quad (11)$$

Put

$$b_0' := \inf_{N\in\mathbf{N}} b'(NN_1).$$

Then there exist a number $N_1' \in \mathbf{N}$ and an unbounded set $\mathscr{A}_2 \subset \mathscr{A}_1$ such that

$$b_0' \leq b'(N_1'N_1) \leq b_0' + \frac{\varepsilon^2}{8n^3}$$

and

$$\begin{aligned} &\lim_{\substack{m\to+\infty \\ d(mN_1'N_1,\mathscr{A}_2)\leq N_1'N_1}} \frac{1}{mN_1'N_1T_1}\sum_{i=0}^{m-1}\ln e_{k-1}(X((i+1)N_1'N_1T_1, iN_1'N_1T_1)) \\ &= \liminf_{\substack{m\to+\infty \\ d(mN_1'N_1,\mathscr{A}_1)\leq N_1'N_1}} \frac{1}{mN_1'N_1T_1}\sum_{i=0}^{m-1}\ln e_{k-1}(X((i+1)N_1'N_1T_1, iN_1'N_1T_1)). \end{aligned}$$

Consequently,

$$\begin{aligned} b_0' &\leq \lim_{\substack{m\to+\infty \\ d(mN_1'N_1,\mathscr{A}_2)\leq N_1'N_1}} \frac{1}{mN_1'N_1T_1}\sum_{i=0}^{m-1}\ln e_{k-1}(X((i+1)N_1'N_1T_1, iN_1'N_1T_1)) \\ &\leq b_0' + \frac{\varepsilon^2}{8n^3}. \end{aligned}$$

Hence, by (5) and the definition of b_0', for any $M \in \mathbf{N}$

$$\begin{aligned} b_0' \le & \liminf_{\substack{m\to+\infty \\ d(mN_1'N_1,\mathscr{A}_2)\le MN_1'N_1}} \frac{1}{mMN_1'N_1T_1}\sum_{i=0}^{m-1} \ln e_{k-1}(X((i+1)MN_1'N_1T_1, iMN_1'N_1T_1)) \\ \le & \limsup_{\substack{m\to+\infty \\ d(mN_1'N_1,\mathscr{A}_2)\le MN_1'N_1}} \frac{1}{mMN_1'N_1T_1}\sum_{i=0}^{m-1} \ln e_{k-1}(X((i+1)MN_1'N_1T_1, iMN_1'N_1T_1)) \\ \le & \, b_0' + \frac{\varepsilon^2}{8n^3}. \end{aligned} \tag{12}$$

Now we set

$$N_2 := N_1'N_1.$$

Denote by Σ_M the set consisting of those half-intervals $[iMN_2T_1, (i+1)MN_2T_1)$ such that at least one of the following inequalities holds

$$\left|\frac{1}{MN_2T_1}\sum_{s=1}^{M} \ln e_k(X(iMN_2T_1 + sN_2T_1, iMN_2T_1 + (s-1)N_2T_1)) - \frac{1}{MN_2T_1}\ln e_k(X((i+1)MN_2T_1))\right| \le \frac{\varepsilon}{8n^3},$$

$$\left|\frac{1}{MN_2T_1}\sum_{s=1}^{M} \ln e_{k-1}(X(iMN_2T_1 + sN_2T_1, iMN_2T_1 + (s-1)N_2T_1)) - \frac{1}{MN_2T_1}\ln e_{k-1}(X((i+1)MN_2T_1))\right| \le \frac{\varepsilon}{8n^3}.$$

From (8), (10), (11), and (12) and the choice of $\mathscr{A}_2 \subset \mathscr{A}_1$, it follows that for any $M \in \mathbf{N}$ the following inequality holds

$$\limsup_{\substack{m\to+\infty \\ d(mMN_2,\mathscr{A}_2)\le MN_2}} \frac{meas(\Sigma_M \cap (0, mMN_2T_1))}{mMN_2T_1} \le 2\varepsilon,$$

where *meas* denotes the Lebesgue measure on the real axis.

Put

$$f_\sigma := \sup_{i\in\mathbf{N}} \mathbf{E}(\|I - X((i+1)N_2T_1, iN_2T_1)Y_\sigma(iN_2T_1, (i+1)N_2T_1;\omega)\|).$$

It is easily seen that there exists a positive constant b_1 which depends on N_2, T_1, $A(\cdot)$ but not on σ such that

$$0 < f_\sigma < b_1\sigma.$$

Take a number $\eta \in (0,1)$ such that the inequalities $|t-\tau| \le N_2T_1$ and $\sin\angle(x,y) < \eta$ imply

$$\|X(t,\tau)y\| \cdot \|x\| \le \exp\left(\frac{\varepsilon N_2T_1}{8n^3}\right)\|X(t,\tau)x\| \cdot \|y\|.$$

We choose a natural number $3 < M_1 \in \mathbf{N}$ such that

$$\delta(b_1^{-1}\varepsilon\eta)^{n^3} \exp\left(\frac{\varepsilon M_1 N_2 T_1}{4}\right) > 2, \tag{13}$$

where the number $\delta = \delta(\varepsilon)$ is defined as in Lemma 2.2, and the inequality $M \le M_1$ implies

$$M \le \exp\left(\frac{\varepsilon M N_2 T_1}{8n^3}\right).$$

Let $\sigma_1 \in (0, 1)$ be such that

$$\delta\sigma_1^{n^3} \exp(\varepsilon M_1 N_2 T_1) < 2. \tag{14}$$

The number σ_1 depends only on ε and on the system (1). Let $\sigma \in (0, \sigma_1)$ be arbitrary. We take a number $S > M_1 N_2 T_1$ such that

$$\delta\sigma^{n^3} \exp(\varepsilon S) = 2. \tag{15}$$

Let M_2 be the least natural number such that $M_2 N_2 T_1 \ge S$. From (13), (14) and (15), it follows that

$$\frac{f_\sigma}{\varepsilon} \exp\left(\frac{\varepsilon}{2n^3} M_2 N_2 T_1\right) < \eta, \tag{16}$$

$$\delta\sigma^{n^3} \exp(\varepsilon M_2 N_2 T_1) \ge 2. \tag{17}$$

We introduce the notation

$$X_i = X((i+1)M_2 N_2 T_1, iM_2 N_2 T_1),$$

$$X_{i,s} = X(iM_2 N_2 T_1 + sN_2 T_1, iM_2 N_2 T_1), \ X_{i,s+1,s} = X_{i,s+1} X_{i,s}^{-1},$$

and similarly for $Y_\sigma(t, \tau, \omega)$, for instance,

$$Y_{\sigma,i}(\omega) = Y_\sigma((i+1)M_2 N_2 T_1, iM_2 N_2 T_1; \omega).$$

By the definition of the sets Σ_M, for $[iM_2 N_2 T_1, (i+1)M_2 N_2 T_1) \notin \Sigma_{M_2}$, we have the following inequalities

$$0 \le \frac{1}{M_2 N_2 T_1} \sum_{s=0}^{M_2-1} \ln e_k(X_{i,s+1,s}) - \frac{1}{M_2 N_2 T_1} \ln e_k(X_i) \le \frac{\varepsilon}{8n^3},$$

$$0 \le \frac{1}{M_2 N_2 T_1} \sum_{s=0}^{M_2-1} \ln e_{k-1}(X_{i,s+1,s}) - \frac{1}{M_2 N_2 T_1} \ln e_{k-1}(X_i) \le \frac{\varepsilon}{8n^3}.$$

Denote by $M_{\sigma,i}$ the set of those $\omega \in \Omega$ for which the following inequality holds

$$\sum_{s=0}^{M_2-1} \|I - X_{i,s+1,s} Y_{\sigma,i,s+1,s}^{-1}(\omega)\| \le f_\sigma \varepsilon^{-1} M_2.$$

By the definition of f_σ, we have $\mathbb{P}(M_{\sigma,i}) > 1 - \varepsilon$. We fix an arbitrary $\omega \in M_{\sigma,i}$ and set $\mathscr{V}_s := \|I - X_{i,s+1,s} Y_{\sigma,i,s+1,s}^{-1}(\omega)\|$. Then by (16) and the choice of M_1, M_2, we have

$$\sum_{s=0}^{M_2-1} \mathscr{V}_s \le f_\sigma \varepsilon^{-1} M_2 < \eta \left\{ \exp\left(-\frac{\varepsilon}{2n^3} M_2 N_2 T_1 \right) \right\} M_2 < \eta \exp\left(-\frac{3\varepsilon}{8n^3} M_2 N_2 T_1 \right). \tag{18}$$

Fix a number i such that $[iM_2N_2T_1, (i+1)M_2N_2T_1) \notin \Sigma_{M_2}$. Denote by $\overline{\mathbf{R}}_i^k$ the linear subspace spanned by k first eigenvectors (in order of decreasing eigenvalues) of the matrix $X_i^* X_i$. For each fixed s we fix an orthonormal basis $x_{i,s,1}, \dots, x_{i,s,n}$ of $\mathbb{R}^n$ such that $x_{i,s,k} \in X_{i,s}\overline{\mathbf{R}}_i{}^k$, $k = 1, \dots, n$. Let $X_{i,s+1,s}^{(k)}$ denote the matrix representation with respect to the bases $x_{i,s,j}$ and $x_{i,s+1,j}$ ($j = 1, \dots, k$) of the restriction to $X_{i,s}\overline{\mathbf{R}}_i^k$ of the map given in the standard basis of $\mathbb{R}^n$ by the matrix $X_{i,s+s,s}$. Clearly, for $m \le k$,

$$e_m(X_{i,s+1,s}^{(k)}) \le e_m(X_{i,s+1,s}).$$

For $m \le k$, by the definitions of $\overline{\mathbf{R}}_i^k$ and $X_{i,s+1,s}^{(k)}$ and (5), we have

$$\prod_{s=0}^{M_2-1} e_m(X_{i,s+1,s}^{(k)}) \ge e_m\left(\prod_{s=0}^{M_2-1} X_{i+s+1,s}^{(k)} \right) = e_m(X_i). \tag{19}$$

From the definition of the set Σ_{M_2} and the choice of the number i, it follows that

$$e_k(X_i) \ge \left\{ \exp\left(-\frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) \right\} \prod_{s=0}^{M_2-1} e_k(X_{i,s+1,s}). \tag{20}$$

Let $\bar{x}$ be an arbitrary vector of $\overline{\mathbf{R}}_i^k$. We denote by γ_s the sinus of the angle between the vector $y_s = Y_{\sigma,i,s}(\omega)\bar{x}$ and the subspace $X_{i,s}\overline{\mathbf{R}}_i^k$, $\gamma_0 = 0$. Assume $\gamma_s \le \eta$ for $s < \bar{s}$. It is easily seen that

$$\gamma_{s+1} \le \varphi_{s+1} + \psi_{s+1},$$

where

$$\varphi_{s+1} = \sin \angle(Y_{\sigma,i,s+1,s}(\omega) y_s, X_{i,s+1,s} y_s) \le \kappa_s,$$

$$\psi_{s+1} = \sin \angle(X_{i,s+1,s} y_s, X_{i,s+1} \overline{\mathbf{R}}_i^k).$$

Now we give an estimation for the number ψ_{s+1}. Denote by x_l the vector $X_{i,s+1,s} x_{i,s,l}$ ($l = 1, \dots, k$). From the definitions of $x_{i,s,l}$, $X_{i,s+1,s}^{(k)}$ and $e_k(X)$, it follows that

$$G_{x_1,\dots,x_k} = e_k(X_{i,s+1,s}^{(k)}).$$

Let u_s be the vector such that $u_s \perp X_{i,s}\overline{\mathbf{R}}_i^k$ and $y_s - u_s \in X_{i,s}\overline{\mathbf{R}}_i^k$, and v_s be the vector such that $v_s \perp X_{i,s+1}\overline{\mathbf{R}}_i^k$ and $X_{i,s+1,s} y_s - v_s \in X_{i,s+1}\overline{\mathbf{R}}_i^k$. Then $\gamma_s = \frac{\|u_s\|}{\|y_s\|}$ and $\varphi_{s+1} = \frac{\|v_s\|}{\|X_{i,s+1,s} y_s\|}$. By (4), we have

$$e_{k+1}(X_{i,s+1,s}) \ge \frac{G_{X_{i,s+1,s} y_s, x_1, \dots, x_k}}{G_{y_s, x_{i,s,1}, \dots, x_{i,s,k}}} = \frac{\|v_s\| G_{x_1,\dots,x_k}}{\|u_s\|}$$

$$= \frac{\varphi_{s+1} \|X_{i,s+1,s} y_s\| e_k(X_{i,s+1,s}^{(k)})}{\gamma_s \|y_s\|}.$$

Therefore,

$$\varphi_{s+1} \leq \gamma_s e_{k+1}(X_{i,s+1,s})\|y_s\| \cdot \|X_{i,s+1,s}y_s\|^{-1}\{e_k(X^{(k)}_{i,s+1,s})\}^{-1}.$$

From the way η was chosen, the assumption $\gamma_s \leq \eta$, and from (3), it follows that

$$\|X_{i,s+1,s}y_s\| \cdot \|y_s\|^{-1} \geq d_k(X^{(k)}_{i,s+1,s}) \exp\left(-\frac{\varepsilon N_2 T_1}{8n^3}\right).$$

Hence, by the inequality $e_{k+1}(X) \leq e_k(X) d_k(X)$, we have $\gamma_{s+1} \leq \kappa_s + g_s\gamma_s$, where

$$g_s = e_k(X_{i,s+1,s}) d_k(X_{i,s+1,s})[e_k(X^{(k)}_{i,s+1,s}) d_k(X^{(k)}_{i,s+1,s})]^{-1} \exp\left(\frac{\varepsilon N_1 T_1}{8n^3}\right) > 1$$
$$(s = 0, 1, \ldots, M_2 - 1).$$

Hence, for $s < \bar{s}$ by (18), (19) and (20), the choice of i and Σ_{M_2}, we have

$$\begin{aligned}
\gamma_{s+1} &\leq \left(\sum_{j=0}^{s} \kappa_j\right) \prod_{j=1}^{s} g_s \\
&\leq \eta \exp\left(-\frac{3\varepsilon}{8n^3} M_2 N_2 T_1\right) \prod_{j=0}^{M_2-1} e_k(X_{i,s+1,s}) d_k(X_{i,s+1,s}) \\
&\quad \times [e_k X^{(k)}_{i,s+1,s}) d_k(X^{(k)}_{i,s+1,s})]^{-1} \exp\left(\frac{\varepsilon N_2 T_1}{8n^3}\right) \\
&\leq \eta \exp\left(-\frac{\varepsilon}{4n^3} M_2 N_2 T_1\right) \prod_{j=0}^{M_2-1} \{e_k(X_{i,s+1,s})[e_k(X^{(k)}_{i,s+1,s})]^{-1}\}^2 \\
&\leq \eta \exp\left(-\frac{\varepsilon}{4n^3} M_2 N_2 T_1\right) \left(\prod_{j=0}^{M_2-1} e_k(X_{i,s+1,s})\right)^2 (e_k(Xi))^{-2} \\
&\leq \eta \exp\left(-\frac{\varepsilon}{4n^3} M_2 N_2 T_1\right) \exp\left(\frac{\varepsilon}{4n^3} M_2 N_2 T_1\right) = \eta.
\end{aligned}$$

This implies $\gamma_{\bar{s}} \leq \eta$. Consequently, $\gamma_s \leq \eta$ for any $s \in \{0, 1, \ldots, M_2 - 1\}$. Hence, using the property of η, we get

$$\begin{aligned}
\|Y_{\sigma,i}(\omega)\bar{x}\| \cdot \|\bar{x}\|^{-1} &= \prod_{s=0}^{M_2-1} \frac{\|X_{i,s+1,s}y_s\|}{\|y_s\|} \cdot \frac{\|Y_{\sigma,i,s+1,s}(\omega)y_s\|}{\|X_{i,s+1,s}y_s\|} \\
&\geq \prod_{s=0}^{M_2-1} d_k(X^{(k)}_{i,s+1,s}) \exp\left(-\frac{\varepsilon}{8n^3} N_2 T_1\right) \frac{1}{\|X_{i,s+1,s} Y^{-1}_{\sigma,i,s+1,s}(\omega)\|} \\
&\geq \prod_{s=0}^{M_2-1} d_k(X^{(k)}_{i,s+1,s}) \exp\left(-\frac{\varepsilon}{8n^3} N_2 T_1\right)(1 + \kappa_s)^{-1} \\
&\geq \exp\left(-\sum_{s=0}^{M_2-1} \kappa_s - \frac{\varepsilon}{8n^3} M_2 N_2 T_1\right) \prod_{s=0}^{M_2-1} e_k(X^{(k)}_{i,s+1,s})[e_{k-1}(X^{(k)}_{i,s+1,s})]^{-1}
\end{aligned}$$

$$\geq \left\{ \exp\left(-\eta \exp\left(-\frac{3\varepsilon}{8n^3} M_2 N_2 T_1 \right) - \frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) \right\} e_k(X_i)$$

$$\times \prod_{s=0}^{M_2-1} [e_{k-1}(X_{i,s+1,s}^{(k)})]^{-1}$$

$$\geq \exp\left(-1 - \frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) e_k(X_i) \exp\left(-\frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) (e_{k-1}(X_i))^{-1}$$

$$\geq \exp\left(-\frac{\varepsilon}{8n^3} M_2 N_2 T_1 - \frac{\varepsilon}{8n^3} M_2 N_2 T_1 - \frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) d_k(X_i)$$

$$\geq \exp\left(-\frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) d_k(X_i).$$

Therefore, for $\bar{x} \in \bar{\mathbb{R}}_i^k$, $[iM_2N_2T_1, (i+1)M_2N_2T_1) \notin \Sigma_{M_2}$, $\omega \in M_{\sigma,i}$, the following inequality holds

$$\| Y_{\sigma,i}(\omega)\bar{x} \| \,.\, \|\bar{x}\|^{-1} \geq \exp\left(-\frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) d_k(X_i).$$

Hence, by (3), for $[iM_2N_2T_1, (i+1)M_2N_2T_1) \notin \Sigma_{M_2}$, $\omega \in M_{\sigma,i}$, we have

$$d_k(Y_{\sigma,i}(\omega)) \geq \exp\left(-\frac{\varepsilon}{8n^3} M_2 N_2 T_1 \right) d_k(X_i). \tag{21}$$

Denote by $\chi(\omega)$ the characteristic function of the set $M_{\sigma,i}$ in the space $(\Omega, \mathbb{P})$. From the definition of $M_{\sigma,i}$, it follows that $0 \leq \mathbf{E}\chi(\omega) \leq \varepsilon$. By virtue of (21) for $[iM_2N_2T_1, (i+1)M_2N_2T_1) \notin \Sigma_{M_2}$,

$$\frac{1}{M_2N_2T_1} \ln d_k(Y_{\sigma,i}(\omega)) \geq \left(\frac{1}{M_2N_2T_1} \ln d_k(X_i) - \frac{\varepsilon}{2n^3} \right) (1 - \chi(\omega))$$
$$+ \chi(\omega) \frac{\ln d_k(Y_{\sigma,i}(\omega))}{M_2N_2T_1}. \tag{22}$$

Since $A(\cdot)$, is bounded, there exists a number $b_2 \in \mathbb{R}^+$ such that for $k \in \{1, \ldots, n\}$

$$\left\{ \mathbf{E}\left(\frac{\ln d_k(Y_{\sigma,i}(\omega))}{M_2N_2T_1} \right)^2 \right\}^{1/2} \leq b_2,$$

$$\left| \frac{1}{M_2N_2T_1} \ln d_k(X_i) \right| \leq b_2,$$

and the positive constant b_2 does not depend on M_2, N_2, i, T_1 and $\sigma \in (0,1)$. Consequently, the inequality (22) implies

$$\frac{1}{M_2N_2T_1} \mathbf{E} d_k(Y_{\sigma,i}(\omega)) \geq \left(\frac{1}{M_2N_2T_1} \ln d_k(X_i) - \frac{\varepsilon}{2n^3} \right) (1 - \varepsilon)$$
$$\geq \frac{1}{M_2N_2T_1} \ln d_k(X_i) - \varepsilon b_2 - \frac{\varepsilon}{2n^3}(1 - \varepsilon) - b_2\sqrt{\varepsilon}.$$

Hence, for $[iM_2N_2T_1,(i+1)M_2N_2T_1)\notin\Sigma_{M_2}$, we have

$$\mathbf{E}\frac{\ln d_k(Y_{\sigma,i}(\omega))}{M_2N_2T_1}\geq\frac{1}{M_2N_2T_1}\ln d_k(X_i)-b_3\sqrt{\varepsilon},\tag{23}$$

where the positive constant b_3 does not depend on M_2, N_2, T_1, i, k and $\sigma\in(0,1)$.

By the definition of M_2, $\mathscr{A}$, $\mathscr{A}_1$ and $\mathscr{A}_2$, the inequality (23) implies

$$\begin{aligned}
&\nu_k(\sigma,\ M_2N_2T_1)\\
&=\limsup_{\substack{m\to+\infty\\ m\in\mathbf{N}}}\frac{1}{mM_2N_2T_1}\sum_{i=0}^{m-1}\mathbf{E}\ln d_k(Y_\sigma((i+1)M_2N_2T_1,iM_2N_2T_1;\omega))\\
&\geq\limsup_{\substack{m\to+\infty,m\in\mathbf{N}\\ d(mM_2N_2,\mathscr{A}_2)\leq M_2N_2}}\frac{1}{mM_2N_2T_1}\sum_{i=0}^{m-1}\mathbf{E}\ln d_k(Y_{\sigma,i}(\omega))\\
&\geq\liminf_{\substack{m\to+\infty,m\in\mathbf{N}\\ d(mM_2N_2,\mathscr{A}_2)\leq M_2N_2}}\frac{1}{mM_2N_2T_1}\sum_{i=0}^{m-1}\mathbf{E}\ln d_k(Y_{\sigma,i}(\omega))\\
&\geq\liminf_{\substack{m\to+\infty,m\in\mathbf{N}\\ d(mM_2N_2,\mathscr{A}_2)\leq M_2N_2}}\frac{1}{mM_2N_2T_1}\sum_{i=0}^{m-1}(\ln d_k(X_i)-M_2N_2T_1b_3\sqrt{\varepsilon})-2b_2\varepsilon\\
&\geq\liminf_{\substack{m\to+\infty,m\in\mathbf{N}\\ d(mM_2N_2,\mathscr{A}_2)\leq M_2N_2}}\frac{1}{mM_2N_2T_1}\sum_{i=0}^{m-1}\ln d_k(X_i)-b_3\sqrt{\varepsilon}-2b_2\varepsilon.
\end{aligned}$$

Hence, by (7),

$$\nu_k(\sigma,M_2N_2T_1)\geq\Theta_k-2\varepsilon-b_3\sqrt{\varepsilon}-2b_2\varepsilon.$$

By Lemmas 2.2 and 2.3 and (17), this implies that for any $\sigma\in(0,1)$, we have

$$\begin{aligned}
\nu_k(\sigma)&\geq\nu_k(\sigma,M_2N_2T_1)-c\sqrt{\varepsilon}-\frac{1}{M_2N_2T_1}\ln\frac{\delta\sigma^{n^3}}{2}\\
&\geq\Theta_k-2\varepsilon-b_3\sqrt{\varepsilon}-2b_2\varepsilon-c\sqrt{\varepsilon}-\varepsilon\\
&\geq\Theta_k-b_4\sqrt{\varepsilon},
\end{aligned}$$

where the constant b_4 does not depend on σ, ε and k. Because ε is arbitrary, the theorem is proved. ■

Remark. By virtue of Lemma 2.3, the central exponent Θ_k of the initial system (1) gives a lower estimation also for central exponents $\Omega_k(\sigma)$ and $\Theta_k(\sigma)$ of the perturbed system (2).

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