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On the Existence of Solutions to Functional Differential Inclusions with Boundary Values*

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Dedicated to Professor Hoang Tuy on the occasion of his 70th birthday

Abstract. For a general class of functional differential inclusions with non-convex right-hand side, being the set of extreme points of a continuous closed convex set-valued map, the set of local solutions and that of global solutions are proved to be nonempty. Our proof is based essentially on the Baire category theorem.

1. Introduction

In this paper, we shall consider the functional differential inclusion of the form

$$\dot{x}(t) \in \partial G(t, x_t), \quad t \in [0, T], \tag{1}$$

$$x(\theta) = \varphi^0(\theta), \quad \theta \in [-h, 0], \tag{2}$$

where $\partial G(t, x_t)$ is the set of extreme points of the set $G(t, x_t), \varphi^0 \in C_E[-h, 0]$, E is a separable reflexive Banach space, and G is a given set-valued map from $[0, T] \times C_E[-h, 0]$ into E.

Under quite general assumptions on map G, we shall prove, by using the Baire category theorem, that the differential inclusion (1)–(2) admits local and global solutions. The idea of using the Baire category theorem has been proposed firstly by Cellina [5] for finite-dimensional differential inclusions. Subsequently, in [7, 8], a method based on the Baire category has been used in order to prove the existence of solutions to the Cauchy problem for non-convex, set-valued differential inclusions in Banach spaces. Further developments in this direction can be found in [3, 6].

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2. Preliminaries and Formulations of Main Results

Throughout this paper, the following notations will be used. Let E be a reflexive separable real Banach space and E^* its topological dual. For $T > 0, h > 0, r > 0$, we denote by $C_E[-h, T]$ and $C_E[-h, 0]$ the Banach spaces of continuous functions from $[-h, T]$ and $[-h, 0]$ to E, respectively, and $B(x, r)$ the ball in E of radius r centered at $x \in E$, $B = B(0, 1)$. For any $A \subset E$, $\bar{A}$ denotes the closure of $A, A^c = E \backslash A$ and ∂A stands for the set of all extreme points of A. The closed convex hull of A is denoted by $\overline{co}A$. By definition,

$$r_A = \sup\{\rho \geq 0 : \exists x \in A : B(x, \rho) \subset A\},$$
$$d(x, A) = \inf_{y \in A} \|x - y\|.$$

The Hausdorff distance between two subsets A, B in E is defined by

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\}.$$

For any $x(\cdot) \in C_E[-h, T]$ and any $t \in [0, T]$, we denote by x_t the element of the function space $C := C_E[-h, 0]$ defined by $x_t(\theta) = x(t + \theta), -h \leq \theta \leq 0$. Then the map $t \to x_t$ is continuous on $[0, T]$ and satisfies

$$\max_{t \in [0, T]} \|x_t(\cdot)\|_C = \max_{t \in [-h, T]} \|x(t)\|_E.$$

Assume $G : I \times C \to 2^E$ a set-valued map such that for each $t \in I$, $\varphi \in C$, $G(t, \varphi)$ is a closed convex set with nonempty interior in E and $\varphi^0 \in C$ a given initial function. Together with (1)–(2) we consider the following differential inclusion

$$\dot{x}(t) \in G(t, x_t), \quad t \in [0, T], \tag{3}$$
$$x(\theta) = \varphi^0(\theta), \quad \theta \in [-h, 0]. \tag{4}$$

We say that the function $x(\cdot) \in C_E[-h, T]$ with $x(\theta) = \varphi^0(\theta)$, $\theta \in [-h, 0]$ is a local solution of the Cauchy problem (1)–(2) (resp. (3)–(4)) if there exists $T_0 \in (0, T]$ such that $x(\cdot)$ is absolutely continuous on $[0, T_0]$ satisfying the differential inclusion (1) (resp. (3)) for a.e. $t \in [0, T_0]$. Moreover, if $T_0 = T$, then $x(\cdot)$ is said to be a global solution of the respective Cauchy problem.

The main results of this paper are the following two theorems.

Theorem 2.1. *Let $U \subset C_E[-h, 0]$ be an open subset, $\varphi^0 \in U$ be given and $G : I \times U \to 2^E$ a set-valued map of closed convex values with nonempty interior in E. Moreover, assume the following hypotheses are satisfied:*

(i) *For each $\varphi \in U, G(\cdot, \varphi)$ is measurable on I.*

(ii) *For each $\varphi \in U$ and any $\varepsilon > 0$, there exists a neighborhood V_φ of φ such that, for a.e. $t \in I$,*

$$H(G(t, \varphi), G(t, \varphi')) < \varepsilon, \quad \forall \varphi' \in V_\varphi \cap U.$$

(iii) *There exists* $\delta \geq 0$ *such that, for a.e.* $t \in I$,

$$r_G(t, \varphi^0) > \delta \geq 0.$$

(iv) *There exists an integrable function* $\alpha(t) \geq 0$ *on* I *(or briefly,* $\alpha(\cdot) \in \mathscr{L}^1_{R_+}(I)$*) such that, for a.e.* $t \in I$ *and all* φ *in a bounded subset* $Q \subset C_E[0, T]$,

$$G(t, \varphi) \subset \alpha(t)B.$$

Then the Cauchy problem (1)–(2) *admits a local solution on* $[0, T]$.

Theorem 2.2. *Let* $G : I \times C \to 2^E$ *be a set-valued map of closed convex values with nonempty interior in* E *and* $\varphi^0 \in E$ *be given. Assume* G *satisfies the following hypotheses:*

(i) *For each* $\varphi \in C$, $G(\cdot, \varphi)$ *is measurable on* I.

(ii) *For each bounded set* $U \subset C_E[-h, 0]$ *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for all* $\varphi \in U$ *and all* $\varphi' \in B(\varphi, \delta) \cap U$,

$$H(G(t, \varphi), G(t, \varphi')) < \varepsilon.$$

(iii) *For each bounded set* $U \subset C_E[-h, 0]$, *there exists* $\rho_U > 0$ *such that, for a.e.* $t \in I$,

$$\inf_{\varphi \in U} r_{G(t,\varphi)} > \rho_U.$$

(iv) *There exists an integrable function* $\alpha(\cdot) \in \mathscr{L}^1_{R_+}(I)$ *such that, for all* $\varphi \in C_E[-h, 0]$ *and for a.e.* $t \in I$,

$$G(t, \varphi) \subset (1 + \|\varphi\|)\alpha(t)B.$$

Then the Cauchy problem (1)–(2) *admits a global solution on* $[0, T]$.

We note that the condition of Theorem 2.1(ii) is equivalent to the following

(ii) *For each compact set* $K \subset U$ *and any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for a.e.* $t \in I$,

$$H(G(t, \varphi), G(t, \varphi')) < \varepsilon, \quad \forall \varphi' \in B(\varphi, \delta) \cap U, \ \varphi \in K. \tag{5}$$

Indeed, Theorem 2.1(ii) implies that, for any $\varphi \in K$, there exist $\delta_\varphi > 0$ and a subset $I_\varphi \subset I$ of complete measure such that, for all $t \in I_\varphi$,

$$H(G(t, \varphi), G(t, \varphi')) < \frac{\varepsilon}{2}, \quad \forall \varphi' \in B(\varphi, \delta_\varphi).$$

Assume $\{B(\varphi_i, \delta_{\varphi_i})\}_{i=1}^n$ is a finite covering of K. Set

$$\delta = \min_{1 \leq i \leq n} \frac{\delta_i}{2}, \quad I_{K,\varepsilon} = \bigcap_{i=1}^n I_{\varphi_i}.$$

Then it is clear that $I_{K,\varepsilon}$ is of complete measure and (5) holds for all $t \in I_{K,\varepsilon}$.

3. Proof of Results

First, we recall two well-known facts (see [6]) which will be used in the proof of the main results.

Lemma 3.1. *Let $(\Omega, \mathscr{A})$ be a complete measurable space and Γ be a measurable multi-valued map from Ω into a complete metric space (E, d) and $\rho(\cdot)$ be a measurable function from Ω into R_+ such that, for each $\omega \in \Omega$,*

$$\Gamma(\omega) = \overline{\operatorname{int}\Gamma(\omega)}, \quad \text{and} \quad r_\Gamma(\omega) > \rho(\omega).$$

Then there exists a measurable function $S : \Omega \to E$ such that

$$d(S(\omega), \Gamma^c(\omega)) > \rho(\omega), \quad \forall \omega \in \Omega.$$

Lemma 3.2. *Let E be a Banach space and M, M_1 be closed convex sets with nonempty interior in E such that M^c, M_1^c are nonempty and $H(M, M_1) < +\infty$. Then*

(i) $H(M^c, M_1^c) \leq H(M, M_1) = H(\partial M, \partial M_1)$.

(ii) *For each $\varepsilon > 0$ and for any $x \in C$, we have*

$$d(x, \partial(\overline{M + \varepsilon B}) \leq d(x, \partial M) + \varepsilon, \quad \forall x \in M.$$

Now, we proceed with establishing several technical results which will enable us to use the Baire category in proving Theorem 2.1.

Lemma 3.3. *Let G satisfy the hypotheses of Theorem 2.1. Then there exist a number $T_0 \in (0, T]$ and a function $x(\cdot) : [-h, T_0] \to E$, continuous on $[-h, T_0]$, absolutely continuous on $[0, T_0]$, such that $x(\theta) = \varphi^0(\theta)$, $-h \leq \theta \leq 0$, and*

$$\operatorname*{ess\ inf}_{t \in [0, T_0]} d(\dot{x}(t), G^c(t, x_t)) > 0.$$

Proof. By Theorem 2.1(iii) and Lemma 3.1, there exists a measurable function $u_0 : [0, T] \to E$, such that

$$d(u_0(t), G^c(t, \varphi^0)) > \delta, \quad \text{a.e. on } I.$$

Set

$$x^0(t) = \begin{cases} \varphi^0(0) + \int_0^t u_0(\tau) d\tau, & \text{for } t \in I \\ \varphi^0(t), & \text{for } t \in [-h, 0]. \end{cases}$$

By Theorem 2.1(ii) and Lemma 3.2.a, there exists a neighborhood $V_{\varphi^0} \subset U$ of φ^0, such that for a.e. $t \in I$ we have

$$H(G^c(t, \varphi), G^c(t, \varphi^0)) < \frac{\delta}{2}, \quad \forall \varphi \in V_{\varphi^0},$$

and hence,

$$d(u_0(t), G^c(t, \varphi^0)) \geq d(u_0(t), G^c(t, \varphi^0)) - H(G^c(t, \varphi), G^c(t, \varphi^0)) > \frac{\delta}{2}, \ \forall \varphi \in V_{\varphi^0}.$$

On the other hand, since the map $t \to x_t^0$ is continuous at $t = 0$ and $x_0^0 = \varphi^0$, there exists $T_0 > 0$ such that $x_t^0 \in V_{\varphi^0}$, for all $t \in (0, T_0]$. Therefore,

$$d(u_0(t), G^c(t, x_t^0)) > \frac{\delta}{2}, \ \forall t \in [0, T_0].$$

This completes the proof. ∎

For the sake of simplicity, we introduce the following notations: $I_1 = [0, T_1]$ is a fixed subinterval of I; S_1 (resp. S_2) is the set of all solutions to the Cauchy problems (1)–(2) (resp. (3)–(4)); S_0 is the set of continuous functions $x(\cdot) : [-h, T_1] \to E$ such that $x(\cdot)$ is absolutely continuous on $I_1, x_0 = \varphi^0$,

$$\operatorname*{ess\,inf}_{t \in I_1} d(\dot{x}(t), G^c(t, x_t)) > 0$$

and $x(\cdot)$ satisfies the inclusion $G(t, x_t) \subset \alpha_1(t)B$, for a.e. $t \in I_1$ and with some function $\alpha_1 \in \mathscr{L}^1_{R_+}(I_1)$.

Lemma 3.4. *Let G satisfy the hypotheses of Theorem 2.1. Then S_2 is closed in $C_E[-h, T_1]$ with respect to the topology of uniform convergence.*

Proof. Let $\{x^n(\cdot)\}$ be a sequence converging to the function $x(\cdot) \in C_E[-h, T_1]$. Since $x^n(\cdot) = \varphi^0(\cdot), \forall n$, on $[-h, 0]$, it follows that $x(\cdot)$ satisfies the initial condition (2). By the definition of S_2 and Theorem 2.1(iv), for a.e. $t \in I_1 = [0, T_1]$, we have $\dot{x}^n(t) \in G(t, x_t^n) \subset \alpha_1(t)B$ with $\alpha_1 \in \mathscr{L}^1_{R_+}(I_1)$. It follows that $\{\dot{x}^n(\cdot)\}$ is contained in a metrizable weakly compact subset of $\mathscr{L}^1_{R_+}(I_1)$. Therefore, we can assume, with no loss of generality, that $\{\dot{x}^n(\cdot)\}$ converges weakly to a function $u(\cdot) \in \mathscr{L}^1_{R_+}(I_1)$. By Mazur's Theorem, there exists a sequence $\{u^n(\cdot)\}$ defined by

$$u^n(t) = \sum_{i=1}^{l_n} \lambda_i^n \dot{x}^{n+i}(t), \quad \text{with} \quad \lambda_i^n \geq 0, \quad \sum_{i=1}^{l_n} \lambda_i^n = 1,$$

which converges to $u(\cdot)$ in the normed topology of $\mathscr{L}^1_{R_+}(I_1)$. It follows that, for a.e. $t \in I_1, \|u^n(t) - u(t)\|_E \to 0$ as $n \to \infty$. Therefore, we have

$$x(t) = \varphi^0(0) + \lim_{n \to \infty} \int_0^t u^n(s)\, ds,$$

which implies that $x(\cdot)$ is absolutely continuous on I_1 and $\dot{x}(t) = u(t)$ a.e. on I_1. According to $(\text{ii})_1$, for any $\varepsilon > 0$ and n large enough, we have

$$G(t, x_t^n) \subset \frac{\varepsilon}{2}B + G(t, x_t).$$

Hence, $u^n(t) \in \frac{\varepsilon}{2}B + G(t, x_t)$ a.e. on I_1. This implies

$$\dot{x}(t) = \lim_{n \to \infty} u^n(t) \in \varepsilon B + G(t, x_t), \quad \text{a.e. on} \quad I_1.$$

The proof is complete. ∎

Now, for $\sigma > 0$ we define the following subset in $C_E[-h, T_1]$

$$S^\sigma = \left\{ x(\cdot) \in \overline{S}_0 : \int_0^{T_1} d(\dot{x}(t), G^c(t, x_t))\, dt < \sigma \right\}.$$

We shall consider $\overline{S_0}$ as a metric space with the metric induced from $C_E[-h, T_1]$.

Lemma 3.5. *Let G satisfy all hypotheses of Theorem 2.1. Then for every $\sigma > 0$, S^σ is open in $\overline{S_0}$.*

Proof. Let $\{x^n(\cdot)\} \subset \overline{S}_0 \backslash S^\sigma$ and $x^n(\cdot)$ converges to $x(\cdot)$ in the metric space $\overline{S}_0$. Clearly, $x_0 = \varphi^0$ on $[-h, 0]$. Thus, it suffices to consider the situation on I_1. By the same reasonings as in the proof of Lemma 3.4, there exists a sequence $\{u^n(\cdot)\}$ of the form

$$u^n(t) = \sum_{i=1}^{l_n} \lambda_i^n \dot{x}^{n+i}(t), \quad \text{with} \quad \lambda_i^n \geq 0, \quad \sum_{i=1}^{l_n} \lambda_i^n = 1$$

converging a.e. on I_1 to $\dot{x}(t)$. According to $(\mathrm{ii})_1$, for n sufficiently large, $G(t, x_t^n) \subset \frac{\varepsilon}{T_1} B + G(t, x_t)$ a.e. on I_1. Therefore, $\dot{x}^n(t) = u^n(t) \in \frac{\varepsilon}{T_1} B + G(t, x_t)$ a.e. on I_1. By Lemma 3.2(ii) and the fact that the function $u \mapsto d(u, (G(t, x_t) + \frac{\varepsilon}{T_1} B)^c)$ is concave on $G(t, x_t) + \frac{\varepsilon}{T_1} B$, we can deduce

$$\begin{aligned}
\int_{I_1} d(\dot{x}(t), G^c(t, x_t))\, dt &\geq \int_{I_1} d\left(\dot{x}(t), \left(G(t, x_t) + \frac{\varepsilon}{T_1} B \right)^c \right) dt - \varepsilon \\
&\geq \lim_{n \to \infty} \int_{I_1} d\left(u^n(t), \left(G(t, x_t) + \frac{\varepsilon}{T_1} B \right)^c \right) dt - \varepsilon \\
&\geq \lim_{n \to \infty} \sum_{i=1}^{l_n} \lambda_i^n \int_{I_1} d\left(\dot{x}^{n+i}(t), \left(G(t, x_t) + \frac{\varepsilon}{T_1} B \right)^c \right) dt - \varepsilon \\
&\geq \lim_{n \to \infty} \sum_{i=1}^{l_n} \lambda_i^n \int_{I_1} d(\dot{x}^{n+i}(t), G^c(t, x_t^{n+i}))\, dt - \varepsilon \\
&\geq \sum_{i=1}^{l_n} \lambda_i^n \delta - \varepsilon = \delta - \varepsilon.
\end{aligned}$$

Since ε can be arbitrarily small, this implies that

$$\int_{I_1} d(\dot{x}(t), G^c(t, x_t))\, dt > \delta.$$

Thus, $x(\cdot) \in \overline{S_0} \backslash S^\sigma$, completing the proof. ■

Lemma 3.6. *For any $\sigma > 0$, the set S^σ is dense in $\overline{S_0}$.*

Proof. We shall prove that S^σ is dense in S_0. For arbitrary $x(\cdot) \in S_0$ and $\varepsilon > 0$, we set

$$r = ess \inf_{t \in I_1} d(\dot{x}(t), G^c(t, x_t)).$$

By the definition of the set $S_0, r > 0$. Define $\delta = \min\left\{\frac{r}{2}, \frac{2\sigma}{3T_1}\right\}$ and

$$G_1(t) = \{y \in E : \quad d(y, G^c(t, x_t)) \geq \delta\}.$$

By virtue of the hypotheses of Theorem 2.1, the multi-valued map $t \mapsto G(t, x_t)$ is measurable and hence, its graph is $\mathscr{L} \otimes \mathscr{B}(E)$-measurable itself (i.e., its graph belongs to the smallest σ-field containing all sets of the form $M \times A$ with $M \in \mathscr{L}$ and $A \in \mathscr{B}(E)$) [3]. It follows that the graph of the multi-valued map $t \mapsto G^c(t, x_t)$ also belongs to $\mathscr{L} \otimes \mathscr{B}(E)$, which in turn yields the measurability of this map itself [3]. Consequently, the map $t \mapsto d(y, G^c(t, x_t))$ is measurable for every $y \in E$. This implies that Graph $G_1 \in \mathscr{L} \otimes \mathscr{B}(E)$ and therefore, G_1 is measurable [3]. Moreover, the map G_1 takes closed convex values, with nonempty interiors, and for any $t \in I_1$ and $y \in \partial G_1(t)$, we have

$$d(y, G^c(t, x_t)) = \delta. \tag{6}$$

Now, choose $\rho > 0$ such that

$$H(G^c(t, \varphi), G^c(t, \varphi')) < \delta/2 \tag{7}$$

for all $\varphi \in \{x_t : t \in I_1\}, \varphi' \in B(\varphi, \rho) \cap U$ and all $t \in I_1$. Note that such a ρ exists, according to the hypothesis (ii)$_1$ and Lemma 3.2(i). Clearly, $\dot{x}(t) \in G_1(t)$ a.e. on I_1 and

$$\max_{t \in I_1} \left\| \int_0^t [\dot{x}(s) - u(s)] ds \right\| \leq \min\{\rho, \varepsilon\}. \tag{8}$$

Define

$$z(t) = \begin{cases} \varphi^0(0) + \int_0^t x(s)\, ds & \text{if } t \in [0, T_1] \\ \varphi^0(t) & \text{if } t \in [-h, 0]. \end{cases}$$

From (7) and (8), it follows that $z_t \in B(x_t, \rho) \cap U$ and hence,

$$H(G^c(t, x_t), G^c(t, z_t)) < \delta/2.$$

In view of (6), we can write

$$\begin{aligned} \frac{3}{2}\delta &\geq d(\dot{z}(t), G^c(t, x_t)) + H(G^c(t, x_t) G^c(t, z_t)) \\ &\geq d(\dot{z}(t), G^c(t, z_t)) \\ &\geq d(\dot{z}(t), G^c(t, x_t)) - H(G^c(t, x_t), G^c(t, z_t)) > \delta/2 \end{aligned}$$

a.e. on I_1.

Consequently,

$$\operatorname{ess\,inf}_{t \in I_1} d(\dot{z}(t), G^c(t, z_t)) > 0$$

and

$$\int_{I_1} d(\dot{z}(t), G^c(t, z_t))\, dt < \frac{3}{2}\delta T_1 < \sigma.$$

Thus, $z(\cdot) \in S^\sigma$. Moreover, from (8), it follows that $\|x(t) - z(t)\| < \varepsilon$, $\forall t \in I_1$, and $x_0 = z_0 = \varphi^0$. This completes the proof. ■

Proof of Theorem 2.1. We have to prove that $S_1 \neq \emptyset$. By Lemma 3.3, there exists $T_0 \in (0, T]$ such that $S_0 \neq \emptyset$, where S_0 is defined as above with $T_1 = T_0$. Therefore, by Lemma 3.4, $\overline{S_0}$ can be considered as a complete metric space (w.r.t. the topology of uniform convergence). From Lemmas 3.5 and 3.6, it follows that, for every $\sigma > 0$, $(S^\sigma)^c$ is a set of the first category in $\overline{S_0}$. Therefore, according to the Baire category theorem, we have

$$\bigcap_{p=1}^{\infty} S^{\frac{1}{p}} \neq \emptyset.$$

On the other hand, it is obvious that

$$\bigcap_{p=1}^{\infty} S^{\frac{1}{p}} \subset S_1.$$

Thus, S_1 is nonempty, as was to be shown. ■

To prove Theorem 2.2, we need the following

Lemma 3.7. *Let $x(\cdot)$ be a solution of the Cauchy problem* (3)–(4) *on the interval $[-h, T]$, then*

$$\|x(t)\| \leq (\|\varphi^0\| + 1)\exp\left(\int_0^T \alpha(s)\,ds\right) - 1,$$

for every $t \in [-h, T]$.

The proof of the above lemma can be found in [12].

Lemma 3.8. *Let G be a multi-valued map satisfying the hypotheses of Theorem 2.2. Then there exists a continuous function $x : [-h, T] \mapsto E$ such that $x_0 = \varphi^0$ on $[-h, 0]$, $x(\cdot)$ is absolutely continuous on $[0, T]$ and satisfies:*

$$\operatorname*{ess\,inf}_{t\in I} d(\dot{x}(t), G^c(t, x_t)) > 0.$$

Proof. Denote

$$R = (\|\varphi^0\| + 1)\exp\left(\int_0^T \alpha(s)\,ds\right) - 1$$

and $\rho = \rho_{B(0,R)}$. By Lemma 3.2(i) and the hypothesis of Theorem 2.2(ii), there exists δ satisfying

$$0 < \delta < (1 + R)\int_0^T \alpha(s)\,ds$$

such that, for any $\varphi \in B(0, R)$ and for a.e. $t \in I$, one has $H(G^c(t, \varphi), G^c(t, \varphi')) <$

$\rho/2, \forall \varphi' \in B(\varphi, \delta)$. On the other hand, since $\alpha(\cdot) \in \mathscr{L}^1_{R_+}(I)$, it follows that for any $\delta > 0$, one can choose an integer $m' \in N$ such that

$$\int_J \alpha(s)\, ds < \delta/(1+R)$$

for any interval $J \subset I$ with $\mu(J) < T/m'$.

By Lemma 3.1 and the condition Theorem 2.2(iii), there exists a measurable function $u_0 : [0, \ell'] \to E, \ell' = \frac{T}{m'}$ such that

$$d(u_0(t), G^c(t, \varphi^0)) > \rho \quad \text{a.e. on } [0, \ell'].$$

Now, we define

$$x(t) = \begin{cases} \varphi^0(0) + \int_0^t u_0(s)\, ds & \text{for } t \in [0, \ell'] \\ \varphi^0(t) & \text{for } t \in [-h, 0]. \end{cases}$$

Then, since the map $t \mapsto x_t$ from $[0, \ell']$ to $C_E[-h, 0]$ is continuous on $[0, \ell']$, for any $\delta > 0$, there exists $m'' \in N$ such that, for every $t \in [0, \ell']$ with $|t - 0| < T/m''$, we have $\|x_t - x_0\| = \|x_t - \varphi^0\| < \delta$.

Set $m = \max\{m', m''\}, \ell = T/m$ and divide the interval $[0, T]$ into m equal parts by the points $0, \ell, 2\ell, \ldots, i\ell, \ldots, T;\ i = 1, 2, \ldots, m$. We shall interpolate the function $x(\cdot)$ on the whole interval $[-h, T]$ by induction. First, since $\|\varphi^0\| \le R$ and

$$u_0(s) \in G(s, \varphi^0) \subset \alpha(s)[1 + \|\varphi^0\|]B,$$

a.e. on $[0, \ell]$, we have

$$\|x(t) - \varphi^0(0)\| \le \int_0^\ell \alpha(s)[1+R]\, ds < \delta.$$

On the other hand, for each $t \in [0, \ell], \|x_t - x_0\| < \delta$ and hence, $x_t \in B(\varphi^0, \delta)$. It follows that

$$\begin{aligned} d(\dot{x}(t), G^c(t, x_t)) &= d(u_0(t), G^c(t, x_t)) \\ &\ge d(u_0(s), G^c(t, \varphi^0)) - H(G^c(t, \varphi^0), G^c(t, x_t)) > \frac{\rho}{2} \end{aligned}$$

a.e. on $[0, \ell]$.

Assuming the function $x(\cdot)$ has already been defined on $[0, i\ell]$ with $i < m$ and satisfying all required properties, we interpolate it on $[i\ell, (i+1)\ell]$ as follows. Let $u_i : [i\ell, (i+1)\ell] \to E$ be a measurable function such that

$$d(u_i(t), G^c(t, x_{i\ell})) > \rho$$

a.e. on $[i\ell, (i+1)\ell]$ (such a function u_i exists in view of the hypothesis Theorem 2.2(iii) and Lemma 3.1). On $[i\ell, (i+1)\ell]$, we define $x(t) = x(i\ell) + \int_{i\ell}^t u_i(s)\, ds$. Since $u_i(s) \in G(s, x_{i\ell})$ a.e. on $[i\ell, (i+1)\ell]$ and $\|x_{i\ell}\| < R$ (by Lemma 3.7), we obtain, for any $t \in [i\ell, (i+1)\ell]$,

$$\|x(t) - x(i\ell)\| \le \int_{i\ell}^{(i+1)\ell} \|u_i(s)\| ds \le \int_{i\ell}^{(i+1)\ell} \alpha(s)(1+R)\, ds < \delta.$$

Moreover, since $|t - i\ell| < \ell$ for all $t \in [i\ell, (i+1)\ell]$, we have $\|x_t - x_{i\ell}\| < \delta$. Consequently, $x_t \in B(x_{i\ell}, \delta)$ and

$$\begin{aligned} d(\dot{x}(t), G^c(t, x_t)) &= d(u_i(t), G^c(t, x_t)) \\ &\geq d(u_i(t), G^c(t, x_t)) - H(G^c(t, x_{i\ell}), G^c(t, x_t)) > \rho/2 \end{aligned}$$

a.e. on $[i\ell, (i+1)\ell]$.

Thus, by induction, the function $x(\cdot)$ with the required properties can be defined on the whole interval $[-h, T]$. This completes the proof. ∎

Proof of Theorem 2.2. It is clear that the multi-valued map G satisfies all the assumptions of Theorem 2.1 on $I \times B(0, R)$, with the number R defined as in Lemma 3.8. Again, as in the proof of Theorem 2.1, by using Lemmas 3.4, 3.5, 3.6, with $T_1 = T$ and applying the Baire category theorem to the set S_0 (which is nonempty, by Lemma 3.8), we deduce that $\bigcap_{p=1}^{\infty} S^{\frac{1}{p}} \neq \emptyset$. Thus, $S_1 \neq \emptyset$, with $T_1 = T$. This completes the proof. ∎

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