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A Lower Bound for Index of Reducibility of Parameter Ideals in Local Rings*

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Dedicated to Professor Hoang Tuy on the occasion of his 70th birthday

Abstract. Let $r_A(M) = \sup\{\ell(0:m)_{M/q}, q$ is a parameter ideal of $M\}$ the type of the A-module M (see [5]). We give in this paper a lower bound for the type of a local ring in terms of local cohomology modules which is a generalization of a result of Goto-Suzuki [5] for generalized Cohen-Macaulay rings. As a consequence, we derive some criteria for the Cohen-Macaulayness of local rings by the types.

1. Introduction

Let (A, m) be a local ring with the maximal ideal m. For an m-primary ideal a, it is well known that the number of irreducible ideals which appear in an irredundant irreducible decomposition of a, denoted, by $N(a)$, is equal to $\ell((0:m)_{A/a})$, where $\ell(N)$ will be denoted for the length of the A-module N. The number $N(a)$ is called the index of reducibility of a and Northcott showed that if A is a Cohen-Macaulay ring, then the index of reducibility of parameter ideals is an invariant of A [8]. The first systematic treatment of the index of reducibility is due to Goto-Suzuki [5], and in this paper, an invariant called type of an A-module M and denoted by $r_A(M)$, is introduced as follows:

$$r_A(M) = \sup\{\ell((0:m)_{M/q}) \mid q \text{ is a parameter ideal of } M\}.$$

One of the main results of [5] is the bound for the type of a generalized Cohen-Macaulay (abbr. g.CM) module (i.e., a module M such that the local cohomology $H^i_m(M)$ is finitely generated for any $i \neq \dim M$) in terms of local cohomology. However, there are various examples of rings, which is not g.CM, but have finite type and even constant index of reducibility ([5], Examples (4.7)). Motivated by

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this fact, in this paper, we present a lower bound for the type of a general local ring in terms of local cohomology and derive, as a consequence, some Cohen-Macaulay criteria by the type of local rings.

Our main result is the following theorem which generalizes [5, Theorem 2.3].

Theorem 1.1. *Suppose (A, m) is a Noetherian local ring with the maximal ideal m and M is a finitely generated A-module of dimension d. We write $\hat{M}$ for the m-completion of M and set $p(M) = \dim(A/a_0(\hat{M}) \ldots a_{d-1}(\hat{M}))$, where $a_i(\hat{M})$ denotes the annihilator of the ith local cohomology module of M. Then we have*

$$r_A(M) \geq \sum_{i=p(M)}^{d} \binom{d - p(M)}{i - p(M)} \ell((0 : m)_{H^i_m(M)}).$$

Let us recall that $\ell((0 : m)_{H^i_m}(M)) < \infty$ since $H^i_m(M)$ is always an Artinian module. We should also mention that the invariant $p(M)$ above is just an invariant called *polynomial type* of M which was introduced in [2].

In Sec. 2, we shall prepare some results on p-standard system of parameters (abbr. p-standard s.o.p) and local cohomology which will be used to prove the main result. Theorem 1.1 will be proved in Sec. 3. In the latter half of this final section, we derive some corollaries of the main theorem and give examples to make sure of the best possibility of our result. The reader is referred to [7] for a general background and [11] for, definition and basic properties of the dualizing complexes.

2. Preliminaries

Throughout this paper, A denotes a Noetherian local ring with the maximal ideal m and M a finitely generated A-module of dimension d. We will write $a_i(M)$ for the annihilator of $H^i_m(M)$ and $a(M) = a_0(M) \cdots a_{d-1}(M)$.

Let $x = \{x_1, \ldots, x_d\}$ be a system of parameters (s.o.p for short) on M. We recall that p-standard s.o.p was first studied in [1]. The notion of p-standard s.o.p was given in [3] as follows.

Definition 2.1. *An s.o.p $x = \{x_1, \ldots, x_d\}$ of an A-module M is called p-standard s.o.p if*

(i) $x_d \in a(M)$;

(ii) $x_i \in a(M/(x_{i+1} + \cdots + x_d)M), i = 1, \ldots, d - 1$.

We now gather the properties of p-standard s.o.p needed for proving our main results.

Proposition 2.2. [3] *The following statements hold to be true:*

(i) *If A has a dualizing complex, then there always exists a p-standard s.o.p.*

(ii) *If $x = \{x_1, \ldots, x_d\}$ is p-standard s.o.p, then x is a f-sequence of M, in the sense of* [4], *i.e., $x_i \notin p$, $\forall p \in Ass(M/(x_1, \ldots, x_{i-1}) M) \setminus \{m\}$, $\forall i = 1, \ldots, d$.*

(iii) *If $x = \{x_1, \ldots, x_d\}$ is a p-standard s.o.p, then there exists an integer N such that $M/(x_1^n, \ldots, x_{p(M)}^n)M$ is a g.CM module for all $n > N$.*

The following results of Schenzel [10] will be used to derive some corollaries of the main theorem.

Proposition 2.3. [10] *Suppose A has a dualizing complex and M is an equi-dimensional A-module, denote $K^i(M) = \mathrm{Hom}(H_m^i(M), E(k))$, where $E(k)$ is injective envelope of the residue field k of A. Then for an integer $r \geq 1$, the following conditions are equivalent:*

(i) *M satisfies Serre condition S_r;*

(ii) *$\dim K^i(M) \leq i - r$, for all i with $0 \leq i < \dim M$. Here we stipulate that $\dim M = -\infty$ for the zero module.*

3. Proof of the Main Theorem and its Corollaries

Proof of Theorem 1.1. It is known that $r_A(M) = r_A(\hat{M})$. On the other hand, $H_m(\hat{M}) = H_m^i(M) \otimes_A \hat{A}$, hence,

$$\mathrm{Hom}_A(A/m, H_m^i(M)) = \mathrm{Hom}_{\hat{A}}(\hat{A}/\hat{m}, H_m^i(\hat{M}))$$

since the one on the left is killed by m, and so is not affected by tensoring with $\hat{A}$. So all involved in the inequality of Theorem 1.1 do not change if we pass to the completion and we may assume A is complete. Since A has a dualizing complex, by Proposition 2.2(i), we may choose a p-standard s.o.p $x = \{x_1, \ldots, x_d\}$. As we know that $r(M) \geq r(M/(x_1^n, \ldots, x_{p(M)}^n)M)$ $\forall n > 0$ and by Proposition 2.2(iii), $M/(x_1^n, \ldots, x_{p(M)}^n)M$ is a g.CM module, we can apply the evaluation in [5, Theorem 2.3] for $M/(x_1^n, \ldots, x_{p(M)}^n)M$ to obtain the following inequality

$$r_A(M) \geq \sum_{i=0}^{d-p(M)} \binom{d-p(M)}{i} \ell((0:m)_{H_m^i(M/(x_1^n,\ldots,x_{p(M)}^n)M)}) \qquad (*)$$

for all n large enough ($n \gg 0$). On the other hand, by Proposition 2.2(ii), $x_1, \ldots, x_{p(M)}$ is also an f-sequence of M. Now using [9, Corollary 3.5] for $\{x_1, \ldots, x_{p(M)}\}$, we have

$$H_m^{i+p(M)}(M) = \varinjlim_n H_m^i(M/(x_1^n, \ldots, x_{p(M)}^n)M), \ \forall i \geq 0.$$

Since the direct limit commutes with the Ext-functors [11, 3.3.7], from the equality above, we get

$$\begin{aligned}\varinjlim_n (0:m)_{H_m^i(M/(x_1^n,\ldots,x_{p(M)}^n)M)} &= \varinjlim_n \mathrm{Hom}(A/m, H_m^i(M/(x_1^n, \ldots, x_{p(M)}^n)M)) \\ &= \mathrm{Hom}(A/m, H_m^{i+p(M)}(M)), \ \forall i \geq 0.\end{aligned}$$

Since $\ell(\mathrm{Hom}(A/m, H_m^{i+p(M)}(M))) < \infty$, the homomorphisms of the direct limit must become surjective for n large enough. This fact yields

$$\ell((0:m)_{H^i_m(M/(x^n_1,\dots,x^n_{p(M)})M)}) \geq \ell(\mathrm{Hom}(A/m, H^{i+p(M)}_m(M))), \quad \forall i \geq 0,\ n \gg 0.$$

Combining the inequality above and $(*)$, we obtain the required inequality.

Now, using the theorem above, we can deduce some Cohen-Macaulay criteria as follows.

Corollary 3.1. *Suppose A has a dualizing complex, M is an equi-dimensional A-module and $r_A(M) + p(M) \leq d$. Then M is a CM module if one of the following conditions holds:*
(i) *M satisfies (S_1);*
(ii) *Depth $M > p(M)$.*

Proof. As A has a dualizing complex, the main theorem and [2, Lemma 2.6] show that $p(M)$ is just the polynomial type of M, which does not change under the completion. So we get $p(M) = \max\{\dim H^i_m(M),\ i = 0,\dots,d-1\}$. Now, applying Theorem 1.1, we get from the hypothesis that

$$\sum_{i=p(M)}^{d} \binom{d-p(M)}{i-p(M)} \ell((0:m)_{H^i_m(M)}) \leq d - p(M).$$

So $H^i_m(M) = 0$, $\forall i = p(M)+1,\dots,d-1$. If M satisfies (ii), then the assertion is obvious. If (i) holds then by Proposition 2.3, we get

$$\dim H^i_m(M) \leq i-1, \quad \forall i = 0,\dots,p(M).$$

Thus, M must be CM, otherwise, since $p(M) \geq 0$, it would lead to a contradiction that $p(M) \leq p(M) - 1$.

Corollary 3.2. *Suppose A has a dualizing complex, M is an equi-dimensional A-module and $r_A(M) \leq \binom{d-p(M)}{2}$. Then M is a CM module if M satisfies (S_2) and K_M satisfies (S_3), where K_M denotes the canonical module of M.*

Proof. By a similar argument as above, we deduce that

$$\sum_{i=p(M)}^{d} \binom{d-p(M)}{i-p(M)} \ell((0:m)_{H^i_m(M)}) \leq \binom{d-p(M)}{2}.$$

From this we conclude that $H^i_m(M) = 0$, $\forall i = p(M)+2,\dots,d-2$. Since K_M satisfies (S_3) and M satisfies (S_2), from [10, 3.2.3] we obtain $H^{d-1}_m(M) = 0$. Now by the same argument as in the proof of Corollary 3.1, we easily get the conclusion.

Remark. By [2, Theorem 4.1] we can see that if A is a homomorphic image of a CM ring, then our hypothesis $r_A(M) + p(M) \leq d$ is equivalent to the following condition:

For any $p \in \operatorname{Supp} M$ *with* $\dim Mp < r_A(M)$, M_p *is a CM module.*

So from Corollary 3.1(i), we obtain again [6, Theorem 3.1].

The following example, inspired by [5, Example 4.7], shows that there exist local rings A for which $p(A)$ takes any value between 1 and $\dim A$ and the equality in Theorem 1.1 holds.

Example. Let $R = k[[X_1, \dots, X_{d+2}]]$, $d \geq 2$, be the ring of formal power series in $X_1, \dots, X_{d+2}$ over a field k and

$$A = R/((X_1^a) + X_2^b(X_1, \dots, X_k))R, \quad a, b \geq 2,\ 3 \leq k \leq d+1.$$

There exists an exact sequence

$$0 \to ((X_1^a) + (X_2^b))R/((X_1^a) + X_2^b(X_1, \dots, X_k))R \to A \to R/(X_1^a, X_2^b) \to 0,$$

while we have

$$((X_1^a) + (X_2^b))R/((X_1^a) + X_2^b(X_1, \dots, X_k))R \cong R/(X_1, \dots, X_k).$$

Therefore, we can easily check that $H_m^i(A) = 0$ for all $i \neq d+2-k, d$ and $p(A) = d+2-k$. Now, by the same method as in [5, Example 4.7], we can deduce that $\ell((0 : m)_{A/q})) = 2$ for any parameter ideals q of A, so $r(A) = 2$ and it follows that the equality in Theorem 1.1 holds.

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