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Multipoint Boundary-Value Problems for Transferable Differential-Algebraic Equations I—Linear Case

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Abstract. In this paper, the solvability of linear multipoint BVPs for DAEs is studied. It will be shown that if multipoint boundary conditions are stated properly, then the bounded linear operator generated by a multipoint BVP is continuously invertible. Otherwise, it is a Noether operator of a negative index. A formula representing general solutions of linear multipoint BPVs for transferable DAEs is obtained.

1. Introduction

This paper is motivated by a series of works of März and her colleagues on two-point BVPs for DAEs (see [1, 2] for an exhaustive bibliography) and Sweet's results on multipoint BVPs for ODEs [3, 4]. It is also closely related to our papers on nonlinear BVPs at resonance [5–8]. It will be proved that März results remain true for regular multipoint BVPs. In irregular cases, our results are essentially new, even for two-point BVPs. On the other hand, when DAEs are regular implicit ODEs, we obtain again Sweet's results.

As is well known, DAEs arise in various applications, especially in describing dynamical processes with constraints.

In some cases, it may be of interest to consider more general boundary conditions than endpoint ones for DAEs.

We shall omit all discussions concerning motivation for the problem studied below. Further examples and comments can be found in the literature (see [1–4, 9]).

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We are interested in the following linear multipoint BVP for a transferable differential algebraic system:

$$A(t)x'(t) + B(t)x(t) = q(t), \quad T \in J := [t_0, T], \tag{1}$$

$$\Gamma x = \gamma, \tag{2}$$

where $A, B \in C(J, \mathbb{R}^{n\times n})$ are continuous matrix-valued functions, $q \in C := C(J, \mathbb{R}^n)$, $\gamma \in \mathbb{R}^n$ and $\Gamma : C \to \mathbb{R}^n$ is a bounded linear operator.

By the Riesz representation theorem, there exists a matrix-valued function of bounded variation $\eta \in BV(J, \mathbb{R}^{n\times n})$ such that $\Gamma x = \int_{t_0}^T d\eta(t)x(t)$.

In the remainder of this section, we state some known facts about transferable DAEs (see [1, 2]).

The DAE (1) is called transferable if:

(1) There exist continuously differentiable projector-functions $P, Q \in C^1(J, \mathbb{R}^{n\times n})$ so that $P(t) = I - Q(t)$, $Q^2(t) = Q(t)$, $\operatorname{Im} Q(t) = \operatorname{Ker} A(t)$ for all $t \in J$.

(2) The matrix $G := A + BQ$ is nonsingular for all $t \in J$. Denote $Q_s(t) := Q(t)G^{-1}(t)B(t)$; $P_s(t) = I - Q_s(t)$; $S(t) := \{\xi \in \mathbb{R}^n : B(t)\xi \in \operatorname{Im} A(t)\}$. The transferability implies the decomposition $\mathbb{R}^n = S(t) \oplus \operatorname{Ker} A(t)$. Moreover, Q_s is a projection onto *Ker* $A(t)$ along $S(t)$.

Since $Ax' = APx' = A(Px)' - AP'x$, we should ask for solutions of (1) belonging to the Banach space

$$\chi := \{x \in C : Px \in C^1(J, \mathbb{R}^n)\}$$

with the norm $\|x\| := \|x\|_\infty + \|(Px)'\|_\infty$.

It has been proved that χ is invariant with respect to the choice of the projector functions P, Q. Let Y be the fundamental solution matrix of the ordinary IVP:

$$Y' = (P'P_s - PG^{-1}B)Y, \quad Y(t_0) = I,$$

and X be the fundamental solution matrix, whose columns belong to χ, satisfying:

$$AX' + BX = 0, \quad P(t_0)(X(t_0) - I) = 0.$$

It holds that $X(t) = P_s(t)Y(t)P(t_0)$, and $\operatorname{Im} X(t) = S(t)$, $\operatorname{Ker} X(t) = \operatorname{Ker} A(t_0)$. Finally, a solution of the IVP:

$$Ax' + Bx = q, \quad P(t_0)(x(t_0) - x_0) = 0$$

can be represented by:

$$x(t) = X(t)x_0 + X(t)\int_{t_0}^{t} Y^{-1}(s)P(s)h(s)ds + Q(t)G^{-1}(t)q(t), \tag{3}$$

where $h(t) := P(t)(I + P'(t))G^{-1}(t)q(t)$.

2. Regular Multipoint BVPs for DAEs

Denote the so-called shooting matrix $\int_{t_0}^T d\eta(t)X(t)$ by D and let

$$\mathscr{R}_0 := \operatorname{Im}\Gamma = \left\{\int_{t_0}^{T} d\eta(t)x(t) : x \in C\right\} \subset \mathbb{R}^n.$$

Theorem 2.1. *The BVP* (1), (2) *has a unique solution for any* $q \in C$ *and* $\gamma \in \mathscr{R}_0$ *if and only if the shooting matrix D satisfies conditions:*

$$\operatorname{Ker} D = \operatorname{Ker} A(t_0), \tag{4}$$

$$\operatorname{Im} D = \mathscr{R}_0. \tag{5}$$

Proof. Consider an operator $\mathscr{L} : \mathscr{X} \to \mathscr{Y} := C \times \mathscr{R}_0$, generated by multipoint BVP (1), (2) such that:

$$\mathscr{L}x := \begin{pmatrix} Lx \\ \Gamma x \end{pmatrix}, \text{ where } Lx := Ax' + Bx.$$

The norm of $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in C \times \mathscr{R}_0$ is defined as $\left\| \begin{pmatrix} q \\ \gamma \end{pmatrix} \right\| := \|q\|_\infty + |\gamma|$, and $|\cdot|$ denotes an arbitrary norm of $\mathbb{R}^n$. Clearly, $\mathscr{L}$ is a bounded linear operator and

$$\operatorname{Ker} \mathscr{L} = \{X(t)x_0 : x_0 \in \operatorname{Ker} D\}. \tag{6}$$

Now observe that the inclusions:

$$\operatorname{Ker} A(t_0) \subset \operatorname{Ker} D, \quad \operatorname{Im} D \subset \mathscr{R}_0 \tag{7}$$

hold trivially.

First suppose (4) and (5) are satisfied. Then from the fact that $\operatorname{Ker} X(t) = \operatorname{Ker} A(t_0)$ and relations (4), (5), (6), it follows that $\operatorname{Ker} \mathscr{L} = \{0\}$, i.e., $\mathscr{L}$ is injective. By virtue of (3), (5) and using the definition of $\mathscr{R}_0$, we have $\forall \begin{pmatrix} q \\ \gamma \end{pmatrix} \in \mathscr{Y}$, $\gamma - \int_{t_0}^T d\eta(t) X(t) \int_{t_0}^t Y^{-1}(s)h(s)ds - \int_{t_0}^T d\eta(t)Q(t)G^{-1}(t)q(t) \in \mathscr{R}_0 = \operatorname{Im} D$.

Consequently, we can find $x_0 \in \mathbb{R}^n$ such that

$$Dx_0 = \gamma - \int_{t_0}^T d\eta(t)X(t) \int_{t_0}^t Y^{-1}(s)P(s)h(s)ds - \int_{t_0}^T d\eta(t)Q(t)G^{-1}(t)q(t).$$

Putting

$$x(t) := X(t)x_0 + X(t) \int_{t_0}^t Y^{-1}(s)P(s)h(s)ds + Q(t)G^{-1}(t)q(t)$$

we get $Lx = q$ and $\Gamma x = \gamma$. From the last relations, it implies that

$$\mathscr{L}x = \begin{pmatrix} q \\ \gamma \end{pmatrix}. \tag{8}$$

Thus, $\mathscr{L}$ is a surjective operator. The well-known Banach theorem ensures that $\mathscr{L}$ possesses a bounded inverse and the unique solution of (8) satisfies estimate:

$$\|x\| \le K(\|q\|_\infty + |\gamma|). \tag{9}$$

Now let $\mathscr{L}$ be continuously invertible. For any $x_0 \in \operatorname{Ker} D$, we define $x(t) = X(t)x_0$ then $Lx = 0$ and $\Gamma x = Dx_0$ or $\mathscr{L}x = 0$. The unique solvability of (8) implies that $x(t) = X(t)x_0 = 0$ or $x_0 \in \operatorname{Ker} X(t) = \operatorname{Ker} A(t_0)$. Combining (7) with the last inclusion, we get $\operatorname{Ker} A(t_0) = \operatorname{Ker} D$. Further, for an arbitrary $\gamma \in \mathscr{R}_0$, there exists a unique solution x, such that $\mathscr{L}x = \begin{pmatrix} 0 \\ \gamma \end{pmatrix}$. This means that $x = X(t)x_0$, and $\gamma = \Gamma x = Dx_0$, hence, $\gamma \in \operatorname{Im} D$. From (7) it implies $\mathscr{R}_0 = \operatorname{Im} D$. Q.E.D.

As a direct consequence of Theorem (2.1), we obtain the following result on the solvability of the DAE (1) with the multipoint BVP:

$$\Gamma x := \sum_{i=1}^{m} D_i x(t_i) = \gamma, \tag{10}$$

where $t_0 \leq t_1 < t_2 < \cdots < t_m \leq T$ and $D_i \in \mathbb{R}^{n \times n}$ $(i = 1, \ldots, m)$ are given constant matrices.

Corollary 2.1. *The BVP* (1), (10) *is uniquely solvable on* $\mathscr{X}$ *for any* $q \in C$ *and* $\gamma \in \operatorname{Im}(D_1, D_2, \ldots, D_m)$ *if and only if the shooting matrix* $D = \sum_{i=1}^{m} D_i X(t_i)$ *has the properties:*

$$\operatorname{Ker} D = \operatorname{Ker} A(t_0); \ \operatorname{Im} D = \operatorname{Im}(D_1, D_2, \ldots, D_m).$$

3. Irregular Multipoint BVPs for DAEs

In this section, the solvability of (1), (2) in irregular cases, where conditions (4), (5) are not fulfilled, is studied. Consider a bounded linear operator $\mathscr{L}$ acting from $\mathscr{X}$ into $\mathscr{Y} := C \times \mathbb{R}^n$.

Let $\dim \operatorname{Ker} A(t_0) = \nu$, $\dim \operatorname{Ker} D = p$ and $\{w_i^0\}_1^\nu$ be an orthonormal basis of $\operatorname{Ker} A(t_0) \subset \operatorname{Ker} D$. We extend $\{w_i^0\}_1^\nu$ to an orthonormal basis of $\operatorname{Ker} D$, i.e., $(w_i^0)^T w_j^0 = \delta_{ij}$, $(i, j = 1, \ldots, p)$, where in what follows the superscript T will denote transpose vectors or matrices. Then $\{w_i^0\}_{\nu+1}^p$ is a basis of $\operatorname{Ker} D \cup (\operatorname{Ker} A(t_0))^\perp$ and

$$\operatorname{Ker} \mathscr{L} = \{X(t)x_0 : x_0 \in \operatorname{Ker} D \cap (\operatorname{Ker} A(t_0))^\perp\}. \tag{11}$$

We define vector-valued functions $\varphi_i(t) = X(t)w_i^0$ $(i = \nu + 1, \ldots, p)$ and a column matrix $\Phi(t) = (\varphi_{\nu+1}(t), \ldots, \varphi_p(t) \in \mathbb{R}^{n \times (p-\nu)}$.

Lemma 3.1. *The vector-valued functions* $\varphi_i(t)$ *are linearly independent, hence, the Gram matrix* $M := \int_{t_0}^T \Phi^T(t)\Phi(t)dt$ *is nonsingular. Further,*

$$\operatorname{Ker} \mathscr{L} = \{\Phi(t)a : a \in \mathbb{R}^{p-\nu}\}.$$

Proof. Let $\sum_{i=\nu+1}^{p} \alpha_i \varphi_i = 0$ and put $x_0 = \sum_{i=\nu+1}^{p} \alpha_i w_i^0$, then we have $X(t)x_0 = 0$. Clearly, $x_0 \in \operatorname{Ker} A(t_0) \cap D \cap (\operatorname{Ker} A(t_0))^\perp$, therefore, $x_0 = 0$. The linear independence of $\{w_i^0\}_{\nu+1}^p$ implies that $\alpha_i = 0$ $(i = \nu + 1, \ldots, p)$. The remaining statements are obvious. Q.E.D.

Now we define a bounded linear projector $\mathscr{U} : \mathscr{X} \to \mathscr{X}$ by:

$$(\mathscr{U}x)(t) = \Phi(t)M^{-1}\int_{t_0}^{T}\Phi^T(s)x(s)ds.$$

Lemma 3.2. (1) $\mathscr{U}^2 = \mathscr{U}$, $\operatorname{Im} U = \operatorname{Ker}\mathscr{L}$.
(2) $\mathscr{X} = \operatorname{Ker}\mathscr{U} \oplus \operatorname{Ker}\mathscr{L}$.

Proof. For an arbitrary $x \in \mathscr{X}$, putting $a := M^{-1}\int_{t_0}^{T}\Phi^T(t)x(t)dt$, we have $\mathscr{U}x = \Phi(t)a$. Therefore, $\mathscr{U}^2x = \mathscr{U}(\Phi a) = \Phi(t)M^{-1}\left(\int_{t_0}^{T}\Phi^T(s)\Phi(s)ds\right)a = \Phi(t)\,M^{-1}Ma = \Phi(t)a = \mathscr{U}x.$

Further, for any $x \in \mathscr{X}$, $\mathscr{U}x = \Phi(t)a \in \operatorname{Ker}\mathscr{L}$, hence, $\operatorname{Im}\mathscr{U} \subset \operatorname{Ker}\mathscr{L}$. Conversely, if $x \in \operatorname{Ker}\mathscr{L}$, then $x(t) = \Phi(t)a$ and $\mathscr{U}x = \Phi(t)M^{-1}Ma = \Phi(t)a = x(t)$. Thus, $x \in \operatorname{Im}\mathscr{U}$ or $\operatorname{Im}\mathscr{U} = \operatorname{Ker}\mathscr{L}$.

Now suppose $x \in \operatorname{Ker}\mathscr{L} \cap \operatorname{Ker}\mathscr{U}$, then $0 = \mathscr{U}x = x$. Observe that, for every $x \in \mathscr{X}$, $\varphi := \mathscr{U}x \in \operatorname{Im}\mathscr{U} = \operatorname{Ker}\mathscr{L}$ and $\Psi := x - \varphi \in \operatorname{Ker}\mathscr{U}$, we come to the decomposition $\mathscr{X} = \operatorname{Ker}\mathscr{U} \oplus \operatorname{Ker}\mathscr{L}$. Q.E.D.

Lemma 3.3. *The inclusion* $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Im}\mathscr{L}$ *holds if and only if, for every* $w \in \operatorname{Ker} D^T$,

$$w^T\gamma - \int_{t_0}^{T} w^T d\eta f - \int_{t_0}^{T} w^T d\eta Q G^{-1} q = 0, \tag{12}$$

where $f(t) = X(t)\int_{t_0}^{T} Y^{-1}(s)P(s)h(s)ds$ *and* $h = P(I + P')G^{-1}q$.

Proof. Clearly that $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Im}\mathscr{L}$, if and only if there exists $x \in \mathscr{X}$, such that $\mathscr{L}x = \begin{pmatrix} q \\ \gamma \end{pmatrix}$, i.e.,

$$x(t) = X(t)x_0 + X(t)\int_{t_0}^{t} Y^{-1}(s)h(s)ds + Q(t)G^{-1}(t)q(t) \tag{13}$$

and

$$\gamma = Dx_0 + \int_{t_0}^{T} d\eta(t)f(t) + \int_{t_0}^{T} d\eta(t)Q(t)G^{-1}(t)q(t). \tag{14}$$

Since $Dx_0 \in \operatorname{Im} D = (\operatorname{Ker} D^T)^{\perp}$, for every $w \in \operatorname{Ker} D^T$, (14) implies that

$$w^T\gamma = w^T\int_{t_0}^{T} d\eta f + w^T\int_{t_0}^{T} d\eta Q G^{-1} q. \qquad \text{Q.E.D.}$$

Denote by $\{w_i\}_1^p$ an orthonormal basis of $\operatorname{Ker} D^T$, i.e., $w_i^T w_j = \delta_{ij}$ $(i, j = 1, \ldots, p)$. For a given $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \mathscr{Y} := C \times \mathbb{R}^n$, we define a bounded linear projector $\mathscr{V}$: $\mathscr{Y} \to \mathscr{Y}$ given by the formula:

$$\mathscr{V}\begin{pmatrix} q \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \sum_{i=1}^{p} c_i w_i \end{pmatrix},$$

where

$$c_i = w_i^T \gamma - \int_{t_0}^{T} w_i^T d\eta f - \int_{t_0}^{T} w_i^T d\eta Q G^{-1} q \quad (i = 1, \dots, p). \tag{15}$$

Lemma 3.4. (1) $\mathscr{V}^2 = \mathscr{V}$, $\operatorname{Ker} \mathscr{V} = \operatorname{Im} \mathscr{L}$.
(2) $\mathscr{Y} = \operatorname{Im} \mathscr{L} \oplus \operatorname{Im} \mathscr{V}$.

Proof. Since

$$\mathscr{V}^2 \begin{pmatrix} q \\ \gamma \end{pmatrix} = \mathscr{V} \begin{pmatrix} q \\ \sum_{i=1}^{p} c_i w_i \end{pmatrix} = \begin{pmatrix} 0 \\ \sum_{i=1}^{p} \tilde{c}_i w_i \end{pmatrix},$$

where

$$\tilde{c}_i = w_i^T \sum_{j=1}^{p} c_j w_j = c_i \quad (i = 1, \dots, p)$$

it follows that $\mathscr{V}^2 = \mathscr{V}$.

If $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Ker} \mathscr{V}$, then $\sum_{i=1}^{p} c_i w_i = 0$, hence, $c_i = 0$ $(i = 1, \dots, p)$.

For an arbitrary $w \in \operatorname{Ker} D^T$, we have $w = \sum_{i=1}^{p} \alpha_i w_i$, therefore,

$$w^T \gamma - \int_{t_0}^{T} w^T d\eta f - \int_{t_0}^{T} w^T d\eta Q G^{-1} q = \sum_{i=1}^{p} \alpha_i c_i = 0.$$

It follows from Lemma 3.3 that $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Im} \mathscr{L}$, or $\operatorname{Ker} \mathscr{V} \subset \operatorname{Im} \mathscr{L}$. Conversely, if $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Im} \mathscr{L}$, then by Lemma 3.3, $c_i = 0$ $(i = 1, \dots, q)$, hence,

$$\mathscr{V} \begin{pmatrix} q \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ \sum c_i w_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ i.e., } \begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Ker} \mathscr{V}.$$

Thus, $\operatorname{Im} \mathscr{L} \subset \operatorname{Ker} \mathscr{V}$, therefore, $\operatorname{Ker} \mathscr{V} = \operatorname{Im} \mathscr{L}$.

Now suppose $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Im} \mathscr{L} \cap \operatorname{Im} \mathscr{V}$, then there exists $\begin{pmatrix} q_1 \\ \gamma_1 \end{pmatrix} \in \mathscr{Y}$ such that $\begin{pmatrix} q \\ \gamma \end{pmatrix} = \mathscr{V} \begin{pmatrix} q_1 \\ \gamma_1 \end{pmatrix}$. Clearly,

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathscr{V} \begin{pmatrix} q \\ \gamma \end{pmatrix} = \mathscr{V}^2 \begin{pmatrix} q_1 \\ \gamma_1 \end{pmatrix} = \mathscr{V} \begin{pmatrix} q_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} q \\ \gamma \end{pmatrix}.$$

Taking any $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \mathscr{Y}$ and defining c_i $(i = 1, \dots, p)$ by (15), we get

$$\begin{pmatrix} 0 \\ \sum c_i w_i \end{pmatrix} = \mathscr{V} \begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Im} \mathscr{V}.$$

Further,

$$\mathscr{V}\begin{pmatrix} q \\ \gamma - \sum c_i w_i \end{pmatrix} = \begin{pmatrix} 0 \\ \sum \bar{c}_i w_i \end{pmatrix},$$

where

$$\bar{c}_i = w_i^T(\gamma - \sum c_j w_j) - \int_{t_0}^T w_i^T d\eta f - \int_{t_0}^T w_i^T d\eta Q G^{-1} q = c_i - c_i = 0$$

$(i = 1, \ldots, p)$.

It means $\begin{pmatrix} q \\ \gamma - \sum c_i w_i \end{pmatrix} \in \operatorname{Ker} \mathscr{V} = \operatorname{Im} \mathscr{L}$. Thus, the decomposition $\mathscr{Y} = \operatorname{Im} \mathscr{L} \oplus \operatorname{Im} \mathscr{V}$ is proved. Q.E.D.

Now we are able to state the main result of this paper.

Theorem 3.1. (1) *$\mathscr{L} : \mathscr{X} \to \mathscr{Y}$ is a bounded linear Noether operator and*

$$\operatorname{Ind} \mathscr{L} = \dim \operatorname{Ker} \mathscr{L} - \operatorname{codim} \operatorname{Im} \mathscr{L} = -\dim \operatorname{Ker} A(t_0).$$

(2) *Multipoint BVP* (1), (2) *with given data $q \in C$, $\gamma \in \mathbb{R}^n$ is solvable if and only if:*

$$\mathscr{W}\left(\gamma - \int_{t_0}^T d\eta f - \int_{t_0}^T d\eta Q G^{-1} q\right) = 0, \tag{16}$$

where $\mathscr{W} = \begin{pmatrix} w_1^T \\ \vdots \\ w_p^T \end{pmatrix}$, *$f(t) = X(t) \int_{t_0}^t Y^{-1}(s) P(s) h(s) ds$ and $h = P(I + P')G^{-1} q$.*

(3) *A general solution of* (1), (2) *can be represented as:*

$$x(t) = X(t)(\bar{x}_0 + \mathscr{W}_0 \alpha) + f(t) + Q(t)G^{-1}(t)q(t) + \Phi(t)a, \tag{17}$$

where $\mathscr{W}_0 = (w_{\nu+1}^0, \ldots, w_p^0)$ and

$$\bar{x}_0 = \hat{D}^{-1}\left(\gamma - \int_{t_0}^T d\eta f - \int_{t_0}^T d\eta Q G^{-1} q\right). \tag{18}$$

Further, $\hat{D}$ is a restriction of D onto $\operatorname{Im} D^T$,

$$\alpha = -M^{-1} \int_{t_0}^T \Phi^T(t)[X(t)\bar{x}_0 + f(t) + Q(t)G^{-1}(t)q(t)dt$$

and a is an arbitrary vector of $\mathbb{R}^{p-\nu}$.

Proof. By Lemmas 3.1 and 3.4, $\operatorname{Ker} \mathscr{L}$ is a $(p - \nu)$-dimensional subspace, $\operatorname{Im} \mathscr{L}$ is closed and $\operatorname{codim} \operatorname{Im} \mathscr{L} = \dim \operatorname{Im} \mathscr{V} = \dim \operatorname{Ker} D^T = \dim \operatorname{Ker} D = p$. Thus, $\mathscr{L}$ is a Noether operator with $\operatorname{Ind} \mathscr{L} = \dim \operatorname{Ker} \mathscr{L} - \operatorname{codim} \operatorname{Im} \mathscr{L} = (p - \nu) - p = -\nu = -\dim \operatorname{Ker} A(t_0)$. Further, relation (16) follows directly from (12) and Lemma 3.3.

Now let $\hat{\mathscr{L}}$ be a restriction of $\mathscr{L}$ onto $\operatorname{Ker} \mathscr{U}$. Since $\hat{\mathscr{L}}$ is a one-to-one and onto mapping from $\operatorname{Ker} \mathscr{U}$ to $\operatorname{Im} \mathscr{L}$, the Banach theorem ensures that $\hat{\mathscr{L}}$ possesses a bounded inverse. For any $\begin{pmatrix} q \\ \gamma \end{pmatrix} \in \operatorname{Im} \mathscr{L}$, there exists $x \in \mathscr{X}$ such that $\mathscr{L}x = \begin{pmatrix} q \\ \gamma \end{pmatrix}$. Decomposing x into a sum of $\bar{x} \in \operatorname{Ker} \mathscr{U}$ and $\tilde{x} \in \operatorname{Ker} \mathscr{L}$, we have $\tilde{x} = \Phi(t)a$ where a is an arbitrary vector of $\mathbb{R}^{p-\nu}$ and $\mathscr{L}x = \hat{\mathscr{L}}\bar{x}$.

Further, form (13), it follows that $\bar{x} = X(t)x_0 + z(t)$, where

$$z(t) := X(t)\int_{t_0}^{t} Y^{-1}P(s)h(s)ds + Q(t)G^{-1}(t)q(t).$$

By virtue of (14)

$$Dx_0 = \gamma - \int_{t_0}^{T} d\eta(t)z(t) \in \operatorname{Im} D.$$

Since $x_0 = \bar{x}_0 + \tilde{x}_0$, where $\bar{x}_0 \in \operatorname{Im} D^T$, $\tilde{x}_0 \in \operatorname{Ker} D$, it implies that $\hat{D}\bar{x}_0 = \gamma - \int_{t_0}^{T} d\eta(t)z(t)$. It leads to (18).

Observing that $\bar{x} \in \operatorname{Ker} \mathscr{U}$, we get $0 = \mathscr{U}(X(t)x_0 + z(t)) = \mathscr{U}(X(t)\tilde{x}_0 + X(t)\bar{x}_0 + z(t)) = 0$ or $\mathscr{U}X(t)\tilde{x}_0 = -\mathscr{U}(X(t)\bar{x}_0 + z(t))$.

As $\tilde{x}_0 \in \operatorname{Ker} D$, $\tilde{x}_0 = \sum_{i=1}^{p} \alpha_i w_i^0$, it follows that

$$X(t)\tilde{x}_0 = X(t)\left(\sum_{i=\nu+1}^{p} \alpha_i w_i^0\right) = \sum_{i=\nu+1}^{p} \alpha_i \varphi_i = \Phi(t)\alpha,$$

therefore,

$$\mathscr{U}X(t)\tilde{x}_0 = \mathscr{U}\Phi(t)\alpha = \Phi(t)\alpha = -\mathscr{U}(X(t)\bar{x}_0 + z(t)) =$$
$$= -\Phi(t)M^{-1}\int_{t_0}^{T} \Phi^T(s)(X(s)\bar{x}_0 + z(s))ds.$$

The last relations imply that

$$\alpha = -M^{-1}\int_{t_0}^{T} \Phi^T(s)(X(s)\bar{x}_0 + z(s))ds.$$

If there are two vectors $\alpha, \bar{\alpha} \in \mathbb{R}^{p-\nu}$ such that $\Phi(t)\alpha = \Phi(t)\bar{\alpha}$ for all $t \in J$, then $\Phi^T(t)\Phi(t)\alpha = \Phi^T(t)\Phi(t)\bar{\alpha}$, and hence, $M\alpha = M\bar{\alpha}$, therefore, $\alpha = \bar{\alpha}$. Thus, $x_0 = \bar{x}_0 + \sum_{i=\nu+1}^{p} \alpha_i w_i^0 + \hat{x}_0$, with an arbitrary $\hat{x}_0 \in \operatorname{Ker} A(t_0)$, and

$$\bar{x}(t) = X(t)x_0 + z(t) = X(t)\left\{\bar{x}_0 + \sum_{i=\nu+1}^{p} \alpha_i w_i^0\right\} + z(t).$$

We arrive at (17). Q.E.D.

When $A(t)$ is nonsingular, i.e., $\nu = 0$, $\mathscr{L}$ is a Fredholm operator (of index 0) and we obtain Sweet's results [3, 4].

4. Examples

Consider equation (1.1) with given data

$$A = \begin{pmatrix} 1 & -t & t^2 \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -(1+t) & t^2+2t \\ 0 & -1 & t-1 \\ 0 & 0 & 1 \end{pmatrix}, \tag{19}$$

$J := [0, 1]$ and $q \in C(J, \mathbb{R}^3)$.

The boundary conditions are given by

$$\int_0^1 x_1(s)ds = \gamma_1, \quad \int_0^1 x_2(s)ds = \gamma_2. \tag{20}$$

By definitions, we can find:

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & 0 & 0 \end{pmatrix}, \quad G := A + BQ = \begin{pmatrix} 1 & -t & t^2+t \\ 0 & 1 & -t-1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$G^{-1} = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & t+1 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_s := QG^{-1}B = Q, \quad P_s = P,$$

$$P' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad P'P_s = 0, \quad PG^{-1}B = \begin{pmatrix} 1 & -1 & 2t^2+t \\ 0 & 1 & t \\ 0 & 0 & 0 \end{pmatrix}.$$

The fundamental solution matrix Y satisfies the following equation:

$$Y' = \begin{pmatrix} -1 & 1 & -2t^2-t \\ 0 & -1 & -t \\ 0 & 0 & 0 \end{pmatrix} Y, \quad Y(0) = I,$$

therefore,

$$Y(t) = \begin{pmatrix} e^{-t} & te^{-t} & 2t-1-2t^2+(1-t)e^{-t} \\ 0 & e^{-t} & 1-t-e^{-t} \\ 0 & 0 & 1 \end{pmatrix},$$

hence,

$$X(t) = P_s(t)Y(t)P(0) = \begin{pmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Boundary conditions (20) can be written as

$$\Gamma x = \int_0^1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} dt = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ 0 \end{pmatrix}.$$

The shooting matrix in this case is of the form

$$D = \int_0^1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e^{-t} & te^{-t} & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 0 \end{pmatrix} dt = \begin{pmatrix} 1-e^{-1} & 1-2e^{-1} & 0 \\ 0 & 1-e^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Obviously,

$$\operatorname{Ker} D = \operatorname{Ker} A(0) = \operatorname{Span}\{(0,0,1)^T\}.$$
$$\operatorname{Im} D = \mathscr{R}_0 = \operatorname{Span}\{(1,0,0)^T, (0,1,0)^T\}.$$

According to Theorem 2.1, BVP (1), (2) with data (19), (20) is uniquely solvable for any $q \in C(J, \mathbb{R}^3)$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.

Now suppose we are given another multipoint boundary condition:

$$\int_0^1 x_i(s)ds = \gamma_1 \quad (i = 1, 2, 3), \quad \text{i.e.,} \ d\eta = Idt. \tag{21}$$

In this case, the shooting matrix D remains the same as before, but $\operatorname{Im} D \simeq \mathbb{R}^2$, $\mathscr{R}_0 = \mathbb{R}^3$, hence, Theorem 2.1 cannot be implemented. Note that

$$Y^{-1}(t) = \begin{pmatrix} e^t & -te^t & (t^2-t+1)e^t - 1 \\ 0 & e^t & 1-(1-t)e^t \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$D^T = \begin{pmatrix} 1-e^{-1} & 0 & 0 \\ 1-2e^{-1} & 1-e^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

we have $\operatorname{Ker} D^T = \operatorname{Span}\{(0,0,1)^T\}$.

Let $w^T = (0,0,1)$. It is easy to show that $\int_0^1 w^T d\eta f = 0$, where

$$f = X(t)\int_0^t Y^{-1}(s)P(s)h(s)ds \quad \text{and} \quad w^T \int_0^1 d\eta Q G^{-1} q = \int_0^1 q_3(t)dt.$$

Thus, by Theorem 3.1, the necessary and sufficient condition for the solvability of BVP (1), (2) with data (19), (21) and $q \in C(J, \mathbb{R}^3)$, $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T \in \mathbb{R}^3$ is $\int_0^1 q_3(t)dt = \gamma_3$.

Finally, let us consider Eq. (1) with given data (19) and a three-point boundary condition:

$$D_1 x(0) + D_2 x\left(\frac{1}{2}\right) + D_3 x(1) = \gamma, \tag{22}$$

where

$$D_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{\frac{1}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D_3 = \begin{pmatrix} e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The shooting matrix is of the form

$$D = D_1 X(0) + D_2 X\left(\frac{1}{2}\right) + D_3 X(1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case all conditions of Theorem 2.1 are not fulfilled. Indeed,

$$\operatorname{Ker} D = \operatorname{Span}\{(1,0,0)^T, (0,0,1)^T\} \neq \operatorname{Ker} A(0) = \operatorname{Span}\{(0,0,1)^T\},$$
$$\operatorname{Im} D = \operatorname{Span}\{(1,0,0)^T\} \neq \mathscr{R}_0 = \mathbb{R}^3.$$

According to Theorem 3.1, the three-point BVP with data (19), (22) is solvable for $q \in C(J, \mathbb{R}^3)$ and $\gamma \in \mathbb{R}^3$ if, and only if, for any $w \in \operatorname{Ker} D^T = \operatorname{Span}\{(0,1,0)^T, (0,0,1)^T\}$ we have

$$w^T \gamma = w^T \left(D_1 g(0) + D_2 g\left(\frac{1}{2}\right) + D_3 g(1)\right), \tag{23}$$

where

$$g(t) = X(t) \int_0^t Y^{-1}(s) P(s)(I + P'(s)) G^{-1}(s) q(s) ds + Q(t) G^{-1}(t) q(t).$$

Taking $w = (0,1,0)^T$ and $w = (0,0,1)^T$, respectively and using (23), we get

$$\gamma_2 = e^{\frac{1}{2}} g_2\left(\frac{1}{2}\right) - g_2(0), \quad \gamma_3 = g_3\left(\frac{1}{2}\right) + g_3(1).$$

A simple computation shows that $g_3(t) = q_3(t)$ and $g_2(t) = t q_3(t) + e^{-t} \int_0^t e^s q_2(s) ds$. Thus, we come to the following necessary and sufficient conditions for the solvability of the above-mentioned three-point BVP;

$$\gamma_2 = \frac{1}{2} e^{\frac{1}{2}} q_3\left(\frac{1}{2}\right) + \int_0^{\frac{1}{2}} e^s q_2(s)\, ds, \quad \gamma_3 = q_3\left(\frac{1}{2}\right) + q_3(1).$$

4. Concluding Remarks

(1) The result of this paper can be applied to the investigation of nonlinear multi-point BVPs for transferable DAEs: $f(x', x, t) = 0$, $\int_0^T d\eta\, g(x(t), t) = 0$.
(2) Most of the results described here can be extended to higher index DAEs.

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