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Survey

Partial Differential Inequalities of Haar Type and Their Applications to the Uniqueness Problem*

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Dedicated to Prof. Yu. A. Dubinskii on the occasion of his 60th birthday

Abstract. Our recent results are surveyed on the so-called partial differential inequalities of Haar type and their applications to stability questions concerning global solutions of the Cauchy problem for nonlinear partial differential equations of the first order. Several more revisions have been made and some material are published for the first time in this paper.

1. Introduction

The purpose of this paper is to survey the most recent developments in first-order partial differential inequality theory and their applications to stability questions concerning global (semi)classical solutions of the nonfunctional Cauchy problem for evolution partial differential equations. These results have been published or are being published in [29–33]. Several of them have been revised and updated. Moreover, some material are presented here for the first time. For functional problems in hereditary setting, we refer the reader to [3–6] and the references therein.

Let us mention that the theory of ordinary differential inequalities was originated by Chaplygin [8] and Kamke [18], and then developed by Ważewski [36]. Its main applications to the Cauchy problem for (ordinary) differential equations concern questions such as: estimates of solutions and of their existence intervals, estimates of the difference between two solutions, criteria of uniqueness and of continuous dependence on initial data and right sides of equations for solutions, Chaplygin's method and

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approximation of solutions, etc. Results in this direction were also extended to (absolutely continuous) solutions of the Cauchy problem for countable systems of differential equations satisfying Carathéodory's conditions. We refer to Szarski [24] for a systematic study of such subjects.

As for the theory of partial differential inequalities, first achievements were obtained by Haar [16], Nagumo [22], and then by Ważewski [35]. Up to now the theory has attracted a great deal of attention. (The reader is referred to Deimling [13], Lakshmikantham and Leela [21], Szarski [24], Walter [34] for the complete bibliography.) We emphasize here that one of its applications to the Cauchy problem for first-order partial differential equations, viz. the Haar–Ważewski *uniqueness criterion* to be quoted in Theorem 2.1 next, is just for classical solutions and may only be used locally. (For more details, see the introductory comments in Sec. 2.) The present paper provides a new method, based on the theory of multifunctions and differential inclusions, to investigate the uniqueness problem. This method allows us to deal with *global solutions*, the condition on whose smoothness is relaxed significantly. As we shall show more concretely in Sec. 5, the equations to be considered satisfy certain conditions somewhat like Carathéodorys; and their *global semiclassical solutions* need only be absolutely continuous in the time variable. (For this, see also [27, 28].)

The structure of the paper is as follows. In Sec. 2 we introduce a so-called *differential inequality of Haar type* (see (2.6) later). An estimate via initial values for functions satisfying this differential inequality will be established (cf. (2.7)–(2.8)). As an application, Sec. 3 gives some uniqueness criteria for global classical solutions to the Cauchy problem for first-order nonlinear partial differential equations. In this way, moreover, the continuous dependence on the initial data of solutions can be examined. Section 4 concerns some generalizations to the case of *weakly-coupled systems*. Finally, in Section 5 we investigate the uniqueness problem for global semiclassical solutions.

From now on, n stands for a certain positive integer, $0 < T < +\infty$, and

$$\Omega_T \overset{\text{def}}{=} (0,T) \times \mathbb{R}^n = \{(t,x) \;:\; 0 < t < T,\; x \in \mathbb{R}^n\}.$$

The notation $\partial/\partial x$ will denote the gradient $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Let $|.|$ and $\langle .,. \rangle$ be the Euclidean norm and scalar product in $\mathbb{R}^n$, respectively.

Denote by $\mathrm{Lip}(\Omega_T)$ the set of all locally Lipschitz continuous functions $u = u(t,x)$ defined on Ω_T. Further, set $\mathrm{Lip}([0,T) \times \mathbb{R}^n) \overset{\text{def}}{=} \mathrm{Lip}(\Omega_T) \cap C([0,T) \times \mathbb{R}^n)$. For every function $u = u(t,x)$ defined on Ω_T, we put

$$\mathrm{Dif}(u) \overset{\text{def}}{=} \{(s,y) \in \Omega_T \;:\; u = u(t,x) \text{ is differentiable at } (s,y)\}.$$

We shall be concerned with the following class of Lipschitz continuous functions:

$$V(\Omega_T) \overset{\text{def}}{=} \{u(.,.) \in \mathrm{Lip}([0,T) \times \mathbb{R}^n) \;:\; \exists G \subset [0,T] \quad \mathrm{mes}(G) = 0, \quad \mathrm{Dif}(u) \supset \Omega_T \backslash (G \times \mathbb{R}^n)\}.$$

(Here, "mes" signifies the Lebesgue measure on $\mathbb{R}^1$.) In other words, a function $u = u(t,x)$ in $\mathrm{Lip}([0,T) \times \mathbb{R}^n)$ belongs to $V(\Omega_T)$ if and only if for almost all t, it is differentiable at any point (t,x).

2. Differential Inequality of Haar Type

First, several comments are called for in connection with the following classical Haar–Wazewski theorem.

Theorem 2.1. [16, 35] *Let $M \geq 0$, $a > 0$, $L_i \geq 0$, $c_i < d_i$ with $2L_i a \leq d_i - c_i$ $(i = 1, \ldots, n)$ and*

$$\Delta \overset{\text{def}}{=} \{(t, x) \;:\; 0 \leq t \leq a,\; c_i + L_i t \leq x_i \leq d_i - L_i t \quad (i = 1, \ldots, n)\}.$$

Let K be a compact set in $\{(u, p) \;:\; u \in \mathbb{R},\; p \in \mathbb{R}^n\}$ and $f = f(t, x, u, p)$ a function satisfying the condition

$$|f(t, x, u, p) - f(t, x, v, q)| \leq \sum_{i=1}^{n} L_i |p_i - q_i| + M|u - v| \tag{2.1}$$

for $(t, x, u, p), (t, x, v, q) \in \Delta \times K$. Finally, let $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ be functions in $C^1(\Delta)$ such that

$$\big(u_j(t, x), (\partial u_j/\partial x)(t, x)\big) \in K \quad for \quad (t, x) \in \Delta,\; j = 1, 2 \tag{2.2}$$

and that $u_1(0, x) = u_2(0, x)$ for $x \in [c_1, d_1] \times \cdots \times [c_n, d_n]$. If $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ are two solutions of the equation

$$\partial u/\partial t + f(t, x, u, \partial u/\partial x) = 0$$

in Δ, then $u_1(t, x) \equiv u_2(t, x)$.

Our comments will concern the Cauchy problem *in the large* for a general first-order partial differential equation as follows:

$$\partial u/\partial t + f(t, x, u, \partial u/\partial x) = 0 \quad \text{in} \quad \Omega_T, \tag{2.3}$$

$$u(0, x) = \phi(x) \quad \text{on} \quad \{t = 0,\; x \in \mathbb{R}^n\}, \tag{2.4}$$

where the Hamiltonian $f = f(t, x, u, p)$ is a function of $(t, x, u, p) \in \Omega_T \times \mathbb{R}^1 \times \mathbb{R}^n$ and $\phi = \phi(x)$ is a given function of $x \in \mathbb{R}^n$.

Assume the function $f = f(t, x, u, p)$ to be locally Lipschitz continuous with respect to (u, p) in the sense that, for any bounded set $K^* \subset \Omega_T \times \mathbb{R}^1 \times \mathbb{R}^n$, there exist nonnegative numbers $L_1, \ldots, L_n$ and M such that (2.1) holds for (t, x, u, p) and (t, x, v, q) in K^*. Let $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ be two C^1-solutions on the whole $\overline{\Omega}_T$ of (2.3)–(2.4). Then, Theorem 2.1 assures the equality $u_1(t, x) = u_2(t, x)$ in a neighborhood of $\{t = 0,\; x \in \mathbb{R}^n\}$ in $\overline{\Omega}_T$. The question arises as to whether this equality can be extended to the entire domain Ω_T. It seems to us that there is no standard procedure for joining a point $(t^0, x^0) \in \Omega_T$ to the hyperplane $\{t = 0,\; x \in \mathbb{R}^n\}$ by consecutively gluing pyramids of the form

$$\Delta \overset{\text{def}}{=} \{(t, x) \;:\; a_1 < t \leq a_2,\; c_i + L_i t \leq x_i \leq d_i - L_i t \quad (i = 1, \ldots, n)\} \tag{2.5}$$

where

$$0 \le a_1 < a_2 < T, \; c_i < d_i, \; 0 \le 2L_i a_2 \le d_i - c_i \quad (i = 1, \ldots, n)$$

so that Theorem 2.1 can simultaneously work therein. The reason is that the relations between $L_1, \ldots, L_n$ and Δ are bilateral and somehow awkward. In fact, $L_1, \ldots, L_n$ are to some extent *overdetermined* by Δ. Specifically, for each application of the theorem in such a procedure, Δ must be ready-given. Thus, by (2.5), a tuple $(L_1, \ldots, L_n)$ is predetermined. However, the further condition (2.1) is needed for $(t, x, u, p), (t, x, v, q) \in \Delta \times K$, with K being a compact set in $\mathbb{R}^1 \times \mathbb{R}^n$ such that (2.2) holds. Therefore, $L_1, \ldots, L_n$ are necessarily Lipschitz constants with respect to $p_1, \ldots, p_n$ of the function $f = f(t, x, u, p)$ restricted to $\Delta \times K$. But these constants might unfortunately become large, say, substantially greater than the above-predetermined values $L_1, \ldots, L_n$. So we would reduce Δ, and hence possibly fail to touch the hyperplane $\{t = 0, \; x \in \mathbb{R}^n\}$ in such a procedure emanating from a point $(t^0, x^0) \in \Omega_T$.

All the preceding remarks suggest that we should make an attempt to develop the theory of first-order partial differential equations. The aim of this section is to prove the following. (We put off discussing its applications to the uniqueness problem to Secs. 3 and 5.)

Theorem 2.2. *Let $u = u(t, x)$ be a function in $V(\Omega_T)$. If there exist a nonnegative function $\mu = \mu(x)$ locally bounded on $\mathbb{R}^n$ and a nonnegative function $\ell = \ell(t)$ in $L^1(0, T)$ such that*

$$|\partial u(t, x)/\partial t| \le \ell(t) \cdot \big[(1 + |x|)|\partial u(t, x)/\partial x| + \mu(x)|u(t, x)|\big] \tag{2.6}$$

for almost every $t \in (0, T)$ and for all $x \in \mathbb{R}^n$, then

$$|u(t, x)| \le \exp\left[C(x) \int_0^t \ell(\tau) d\tau\right] \cdot \sup_{|y| \le (1+|x|) \exp \int_0^t \ell(\tau)d\tau - 1} |u(0, y)|, \tag{2.7}$$

where

$$C(x) \stackrel{\text{def}}{=} \sup\big\{|\mu(y)| \; : \; |y| \le (1 + |x|) \exp \int_0^T \ell(\tau) d\tau - 1\big\}. \tag{2.8}$$

Remark 1. We would like to call (2.6) a "differential inequality of Haar type" because its form looks like that of Haar's differential inequality (see [24, Corollary 37.1]).

Remark 2. We show by the following example that the Lipschitz continuity of $u = u(t, x)$ is essential in Theorem 2.2.

Let $J \subset [0, 1]$ be the *Cantor set*, i.e., the set of all numbers of the form

$$t \stackrel{\text{def}}{=} \sum_{k=1}^{+\infty} c_k / 3^k \tag{2.9}$$

where each c_k is either 0 or 2. The set J is complete, nowhere dense on $\mathbb{R}^1$, and is of Lebesgue measure 0.

We define a function $w = w(t)$, which is called the *Cantor ladder*, in the following way (see [14]). For $t \in J$ given by (2.9), we take

$$w(t) \stackrel{\text{def}}{=} \sum_{k=1}^{+\infty} \varepsilon_k/2^k \quad \text{where} \quad \varepsilon_k \stackrel{\text{def}}{=} c_k/2.$$

If (α, β) is an open maximal interval in $(0, 1)\backslash J$, then $\alpha, \beta \in J$ and $w(\alpha) = w(\beta)$. We set for $t \in (\alpha, \beta)$: $w(t) \stackrel{\text{def}}{=} w(\alpha) = w(\beta) = \text{const}$. It follows that $w = w(t)$ is continuous on $[0, 1]$ and that $dw/dt = 0$ almost everywhere in $(0, 1)$. In fact, $dw(t)/dt = 0$ for $t \in (0, 1)\backslash J$.

Setting $u(t, x) \stackrel{\text{def}}{=} w(t)$ for $(t, x) \in \Omega_1$, we easily see that $u = u(t, x)$ belongs to $C^1\Big(\big((0, 1)\backslash J\big) \times \mathbb{R}^n\Big) \cap C\big([0, 1] \times \mathbb{R}^n\big)$ with $u(0, x) \equiv 0$,

$$\partial u(t, x)/\partial t = 0 \quad \forall\, (t, x) \in \big((0, 1)\backslash J\big) \times \mathbb{R}^n.$$

The function $u = u(t, x)$ thereby satisfies all the conditions of Theorem 2.2 except for the Lipschitz continuity. This explains why (2.7) does not hold: $u(t, x) \not\equiv 0$.

Proof of Theorem 2.2. For an arbitrary point $(t^0, x^0) \in \Omega_T$, we must prove that

$$|u(t^0, x^0)| \le \exp\left[C(x^0) \int_0^{t^0} \ell(t)dt\right] \cdot \sup_{|y| \le (1+|x^0|) \exp \int_0^{t^0} \ell(t)dt - 1} |u(0, y)|. \tag{2.10}$$

Let $\overline{B}_r = \overline{B}_r^n \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : |x| \le r\}$, $r \ge 0$. Denote by $\Sigma_I(t^0, x^0)$ the set of all absolutely continuous functions $x = x(t)$ from $I \stackrel{\text{def}}{=} [0, t^0]$ into $\mathbb{R}^n$ which satisfy almost everywhere in I the differential inclusion $dx(t)/dt \in \overline{B}_{\ell(t)\cdot(1+|x(t)|)}$ subject to the constraint $x(t^0) = x^0$.

From [7, Theorem VI-13], it follows that $\Sigma_I(t^0, x^0)$ is a nonempty compact set in $C(I, \mathbb{R}^n)$. The sets $Z(t, t^0, x^0) \stackrel{\text{def}}{=} \big\{x(t) : x(.) \in \Sigma_I(t^0, x^0)\big\}$ and $\Gamma(t^0, x^0) \stackrel{\text{def}}{=} \big\{(\tau, x) : \tau \in I,\ x \in Z(\tau, t^0, x^0)\big\}$ are therefore compact sets in $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively, for all $t \in I$. Moreover, by the converse of Ascoli's theorem, the multifunction

$$Z(., t^0, x^0) : I \rightsquigarrow \mathbb{R}^n$$

is continuous. ■

We now define a real-valued function $g = g(t)$ on I by setting

$$g(t) \stackrel{\text{def}}{=} \max\big\{|u(t, x)| : x \in Z(t, t^0, x^0)\big\}.$$

Then according to the maximum theorem (see [2, Theorem 1.4.16]), the fact that $u = u(t, x)$ is continuous on $\Gamma(t^0, x^0)$ implies that $g = g(t)$ is continuous on I. In addition, we have:

Lemma 2.3. *For an arbitrary number $\theta \in (0, t^0)$, the function $g = g(t)$ is absolutely continuous on $[\theta, t^0]$.*

The following assertion will also be needed.

Lemma 2.4. *We have for every $t \in I$ the inclusion*

$$Z(t, t^0, x^0) \subset \overline{B}_{(1+|x^0|)\exp\int_t^{t^0} \ell(\tau)d\tau - 1}. \tag{2.11}$$

Proof of Lemma 2.4. For each $\eta > 0$, let

$$m_\eta(t) \stackrel{\text{def}}{=} (1 + |x^0| + \eta) \exp \int_t^{t^0} \ell(\tau) d\tau - 1.$$

The function $m_\eta = m_\eta(t)$ is absolutely continuous, positive on I with the derivative $dm_\eta(t)/dt = -\ell(t) \cdot (1 + m_\eta(t))$. To prove (2.11) we have only to show that

$$|x(t)| < m_\eta(t) \quad \forall t \in I \tag{2.12}$$

for all $x = x(t)$ in $\Sigma_I(t^0, x^0)$ and for all $\eta > 0$.

Since $m_\eta(t^0) > |x^0| = |x(t^0)|$, there exists a number $\delta \in (0, t^0)$ such that $m_\eta(t) > |x(t)|$ whenever $t \in (t^0 - \delta, t^0]$.

Assume that (2.12) was false, so that there exists $t' \in [0, t^0)$ such that $m_\eta(t') \le |x(t')|$. Setting $t^1 \stackrel{\text{def}}{=} \sup\{t \in [0, t^0) \; : \; m_\eta(t) \le |x(t)|\} < t^0$, we would have:

$$|x(t^1)| = m_\eta(t^1); \; |x(t)| < m_\eta(t) \quad \forall t \in (t^1, t^0],$$

and

$$\begin{aligned} dm_\eta(t)/dt &= -\ell(t) \cdot \big(1 + m_\eta(t)\big) \le -\ell(t) \cdot \big(1 + |x(t)|\big) \\ &\le -\big|dx(t)/dt\big| \le d|x(t)|/dt \end{aligned}$$

almost everywhere in (t^1, t^0). On the other hand,

$$\int_{t^1}^{t^0} \frac{dm_\eta(t)}{dt} dt > \int_{t^1}^{t^0} \frac{d|x(t)|}{dt} dt$$

if and only if

$$m_\eta(t^0) - m_\eta(t^1) = m_\eta(t^0) - |x(t^1)| > |x(t^0)| - |x(t^1)|.$$

Hence we obtain a contradiction. This proves Lemma 2.4. ∎

Proof of Lemma 2.3. Since $u = u(t, x)$ is locally Lipschitz continuous in Ω_T, there exists a number $L \ge 0$ such that

$$|u(t^1, x^1) - u(t^2, x^2)| \le L\big(|t^1 - t^2| + |x^1 - x^2|\big)$$
$$\forall \, (t^1, x^1), \; (t^2, x^2) \in \big([\theta, t^0] \times \mathbb{R}^n\big) \cap \Gamma(t^0, x^0).$$

By the absolute continuity of the Lebesgue integral, Lemma 2.3 will be proved if we can show that

$$|g(t^1) - g(t^2)| \le L\Big[|t^1 - t^2| + \big(1 + |x^0|\big) \cdot \int_{[t^1, t^2]} \ell(t)dt \cdot \exp \int_\theta^{t^0} \ell(t)dt\Big] \tag{2.13}$$
$$\forall \, t^1, t^2 \in [\theta, t^0].$$

Now let

$$g(t^1) \geq g(t^2) \quad \text{and} \quad g(t^1) = |u(t^1, x(t^1))|$$

for some $x = x(t)$ in $\Sigma_I(t^0, x^0)$. Since $x(t^2) \in Z(t^2, t^0, x^0)$, we have

$$\begin{aligned} 0 &\leq g(t^1) - g(t^2) = |u(t^1, x(t^1))| - g(t^2) \\ &\leq |u(t^1, x(t^1))| - |u(t^2, x(t^2))| \leq |u(t^1, x(t^1)) - u(t^2, x(t^2))| \\ &\leq L\big[|t^1 - t^2| + |x(t^1) - x(t^2)|\big] = L\Big[|t^1 - t^2| + \Big|\int_{[t^1,t^2]} \frac{dx}{dt}(t)dt\Big|\Big] \\ &\leq L\Big[|t^1 - t^2| + \int_{[t^1,t^2]} \ell(t) \cdot (1 + |x(t)|)dt\Big]. \end{aligned}$$

Therefore, (2.13) follows from Lemma 2.4. The proof is then complete. ■

Going back to the proof of Theorem 2.2, we now set

$$h(t) \stackrel{\text{def}}{=} \int_0^t \ell(\tau)d\tau \quad \text{for} \quad t \in [0, T].$$

By Lemma 2.4 and the definition of $g = g(t)$, the inequality (2.10) will be obtained if we show that

$$g(t) \leq g(0) \cdot \exp\big[C(x^0) \cdot h(t)\big] \quad \forall t \in [0, t^0]. \tag{2.14}$$

For every $\eta > 0$, let

$$g_\eta(t) \stackrel{\text{def}}{=} \big[g(0) + \eta\big] \cdot \exp\Big\{\big[C(x^0) + \eta\big] \cdot \big[h(t) + \eta t\big]\Big\}.$$

To get (2.14), it suffices to prove that

$$g(t) < g_\eta(t) \quad \forall t \in [0, t^0]. \tag{2.15}$$

Let $\omega(t) \stackrel{\text{def}}{=} g_\eta(t) - g(t)$, where η is temporarily fixed. Then (2.15) is equivalent to $\omega(t) > 0 \quad \forall t \in [0, t^0]$. Obviously, $\omega(0) = \eta > 0$. We shall show that $\omega(t) \geq \omega(0) \quad \forall t \in [0, t^0]$. Assume this is false, so there exists $t' \in (0, t^0]$ such that $\omega(t') < \omega(0)$.

It is well known that there exists a set $G_1 \subset (0, T)$ of Lebesgue measure 0 with the property that

$$dh(t)/dt = \ell(t) \quad \forall t \in (0, T) \backslash G_1.$$

By the hypothesis of Theorem 2.2, we find a set $G_2 \subset (0, T)$ also of Lebesgue measure 0 such that $\Omega_T \backslash (G_2 \times \mathbb{R}^n) \subset \text{Dif}(u)$ and that (2.6) holds for all $t \in (0, T)\backslash G_2, x \in \mathbb{R}^n$.

Since an absolutely continuous mapping preserves the measure of null sets, Lemma 2.3 implies

$$\text{mes}\Big(\omega\big(G \cap [\theta, t^0]\big)\Big) = 0 \quad \forall \theta \in (0, t^0),$$

where $G \stackrel{\text{def}}{=} G_1 \cup G_2$. So

$$\text{mes}\Big(\omega\big(G \cap [0, t^0]\big)\Big) = \lim_{\theta \searrow 0} \text{mes}\Big(\omega\big(G \cap [\theta, t^0]\big)\Big) = 0. \tag{2.16}$$

From (2.16) and the continuity of $\omega = \omega(t)$ on I, we conclude that there is a number λ with

$$\max\{0, \omega(t')\} < \lambda < \omega(0) \quad \text{and} \quad \lambda \in \omega[0, t']\backslash\omega\big(G \cap [0, t^0]\big).$$

Let

$$t_* \stackrel{\text{def}}{=} \inf\big\{t \in [0, t'] \;:\; \omega(t) = \lambda\big\}.$$

It is obvious that $\omega(t_*) = \lambda,\ t_* \in (0, t')\backslash G$, and that $\omega(t) > \lambda\ \ \forall t \in [0, t_*)$.

Suppose

$$g(t_*) = |u(t_*, x_*)| = \varepsilon \cdot u(t_*, x_*), \quad \varepsilon \stackrel{\text{def}}{=} \operatorname{sign} u(t_*, x_*)$$

for some $x_* \in Z(t_*, t^0, x^0)$. Then one may find a function ${}^*x = {}^*x(t)$ in $\Sigma_I(t^0, x^0)$ so that ${}^*x(t_*) = x_*$. Choose a unit vector $e \in \mathbb{R}^n$ with

$$\left\langle e, \varepsilon \cdot \frac{\partial u}{\partial x}(t_*, x_*)\right\rangle = -\left|\frac{\partial u}{\partial x}(t_*, x_*)\right|. \tag{2.17}$$

The system (of n ordinary differential equations)

$$\frac{dy}{ds}(s) = (1 + |y(s)|) \cdot e$$

has a C^1-solution $y = y(s)$ on $\mathbb{R}^1$ satisfying the condition $y\big(h(t_*)\big) = x_*$. Let $x(t) \stackrel{\text{def}}{=} y\big(h(t)\big)$ for $t \in [0, T]$. Of course, $x = x(t)$ is absolutely continuous on $[0, T]$, $x(t_*) = x_*$, and

$$\frac{dx}{dt}(t) = \frac{dh}{dt}(t) \cdot \frac{dy}{ds}(h(t)) = \ell(t) \cdot (1 + |x(t)|) \cdot e \quad \forall t \in (0, T)\backslash G_1.$$

The function ${}_*x = {}_*x(t)$ defined by

$${}_*x(t) \stackrel{\text{def}}{=} \begin{cases} x(t) & \text{if } 0 \le t \le t_*, \\ {}^*x(t) & \text{if } t_* \le t \le t^0 \end{cases}$$

belongs to $\Sigma_I(t^0, x^0)$. Hence,

$$x(t) \in Z(t, t^0, x^0) \quad \forall t \in [0, t_*].$$

This implies

$$\varepsilon \cdot u(t, x(t)) \le |u(t, x(t))| \le g(t) = g_\eta(t) - \omega(t) < g_\eta(t) - \lambda \tag{2.18}$$

for all $t \in [0, t_*)$. Besides that,

$$\varepsilon \cdot u(t_*, x(t_*)) = |u(t_*, x_*)| = g(t_*) = g_\eta(t_*) - \omega(t_*) = g_\eta(t_*) - \lambda. \tag{2.19}$$

Furthermore, since $t_* \in (0, T)\backslash G$, we see that:

(i) $u = u(t, x)$ is differentiable at (t_*, x_*),

(ii) $x = x(t)$ is differentiable at t_* with

$$\frac{dx}{dt}(t_*) = \ell(t_*) \cdot (1 + |x_*|) \cdot e,$$

(iii) $g_\eta = g_\eta(t)$ is differentiable at t_* with

$$\frac{dg_\eta}{dt}(t_*) = \left[C(x^0) + \eta\right] \cdot \left[\ell(t_*) + \eta\right] \cdot g_\eta(t_*).$$

So it follows from (2.18)–(2.19) that

$$\frac{d}{dt}\Big[\varepsilon \cdot u(t, x(t))\Big]\Big|_{t=t_*} \geq \frac{dg_\eta}{dt}(t_*).$$

Consequently,

$$\varepsilon \cdot \frac{\partial u}{\partial t}(t_*, x(t_*)) + \left\langle \frac{dx}{dt}(t_*), \varepsilon \cdot \frac{\partial u}{\partial x}(t_*, x(t_*)) \right\rangle \geq \left[C(x^0) + \eta\right] \cdot \left[\ell(t_*) + \eta\right] \cdot g_\eta(t_*).$$

Hence,

$$\varepsilon \cdot \frac{\partial u}{\partial t}(t_*, x_*) + \ell(t_*)(1 + |x_*|) \cdot \left\langle e, \varepsilon \cdot \frac{\partial u}{\partial x}(t_*, x_*) \right\rangle \geq \left[C(x^0) + \eta\right] \cdot \left[\ell(t_*) + \eta\right] \cdot \left[|u(t_*, x_*)| + \lambda\right].$$

Because $\eta > 0$ and $\lambda > 0$, the last inequality together with (2.17) imply

$$\left|\frac{\partial u}{\partial t}(t_*, x_*)\right| > \ell(t_*) \cdot \left[(1 + |x_*|) \cdot \left|\frac{\partial u}{\partial x}(t_*, x_*)\right| + C(x^0) \cdot |u(t_*, x_*)|\right]. \tag{2.20}$$

On the other hand, since $x_* \in Z(t_*, t^0, x^0)$, Lemma 2.4 yields

$$|x_*| \leq (1 + |x^0|) \exp \int_{t_*}^{t^0} \ell(\tau) d\tau - 1 \leq (1 + |x^0|) \exp \int_0^T \ell(\tau) d\tau - 1.$$

Therefore, the formula (2.8) gives $C(x^0) \geq |\mu(x_*)|$, which shows that (2.20) contradicts (2.6). It follows that there exists no $t' \in [0, t^0]$ with $\omega(t') < \omega(0)$. Thus, $\omega(t) \geq \omega(0) > 0$ for all $t \in [0, t^0]$; the inequality (2.15) is thereby proved. This completes the proof of Theorem 2.2. ∎

3. Uniqueness of Global Classical Solutions to the Cauchy Problem

The advantage of Theorem 2.2, as we have mentioned in the introduction, is that it allows us to discuss the so-called *global semiclassical solutions*, which are just absolutely continuous in time variable, for first-order nonlinear partial differential equations with time-measurable Hamiltonian. This will be taken up in Sec. 5, where an answer to a problem of Kruzhkov [20] is given. In the present section we restrict ourselves to the case of C^1-solutions, dealing with some applications of Theorem 2.2 to stability questions concerning the Cauchy problem *in the large* for partial differential equations of the first order. Even in this "classical case", using the *a priori* estimate (2.7)–(2.8) of Theorem 2.2, we find some new uniqueness criteria (posed on the Hamiltonian $f = f(t, x, u, p)$) for global C^1-solutions of (2.3)–(2.4). Criteria of continuous dependence on initial data for such solutions may also be obtained. Let us first repeat the definition of solutions to be considered.

Definition. *A function $u = u(t,x)$ in $C^1(\Omega_T) \cap C\big([0,T) \times \mathbb{R}^n\big)$ is called a global C^1-solution to the Cauchy problem (2.3)–(2.4) if it satisfies (2.3) everywhere in Ω_T and (2.4) for all $x \in \mathbb{R}^n$.*

As was shown in the introductory comments of Sec. 2, for the uniqueness of global C^1-solutions, the following result may be invoked instead of Theorem 2.1.

Theorem 3.1. *Suppose $f = f(t,x,u,p)$ satisfies the following condition: there exist nonnegative numbers L, M such that*

$$|f(t,x,u,p) - f(t,x,v,q)| \leq L(1+|x|)|p-q| + M|u-v| \tag{3.1}$$

for all $(t,x,u,p), (t,x,v,q) \in \Omega_T \times \mathbb{R}^1 \times \mathbb{R}^n$. If $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ are global C^1-solutions to the problem (2.3)–(2.4), then $u_1(t,x) \equiv u_2(t,x)$ in Ω_T.

Proof. Consider the function $u = u(t,x) \overset{\text{def}}{=} u_1(t,x) - u_2(t,x)$. Then $u(0,x) \equiv 0$. Furthermore, by (3.1) and the definition of global C^1-solutions, we have

$$\begin{aligned}
\left|\frac{\partial u}{\partial t}(t,x)\right| &= \left| f\Big(t,x,u_1(t,x),\frac{\partial u_1}{\partial x}(t,x)\Big) - f\Big(t,x,u_2(t,x),\frac{\partial u_2}{\partial x}(t,x)\Big)\right| \\
&\leq L(1+|x|)\left|\frac{\partial u_1}{\partial x}(t,x) - \frac{\partial u_2}{\partial x}(t,x)\right| + M|u_1(t,x)-u_2(t,x)| \\
&= L(1+|x|)\left|\frac{\partial u}{\partial x}(t,x)\right| + M|u(t,x)|
\end{aligned}$$

for all $(t,x) \in \Omega_T$. Now it follows from Theorem 2.2 that $u(t,x) \equiv 0$ in Ω_T. This proves the theorem. ■

The next sharpening (and its corollary) of Theorem 3.1 will give some useful uniqueness criteria for global C^1-solutions with bounded derivatives.

Theorem 3.2. *Suppose $f = f(t,x,u,p)$ satisfies the following condition: for any compact sets $K_1 \subset \mathbb{R}^1$, $K_2 \subset \mathbb{R}^n$, there exist a nonnegative number L_{K_2} and a nonnegative function $\mu_{K_1,K_2} = \mu_{K_1,K_2}(x)$ locally bounded on $\mathbb{R}^n$ such that (3.1) with L_{K_2} and $\mu_{K_1,K_2}(x)$ in place of L and M, respectively, holds for all $(t,x,u,p), (t,x,v,q) \in \Omega_T \times K_1 \times K_2$. If $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ are global C^1-solutions to the problem (2.3)–(2.4) with*

$$\sup_{(t,x)\in\Omega_T}\left|\frac{\partial u_j}{\partial x}(t,x)\right| < +\infty \quad (j=1,2),$$

then $u_1(t,x) \equiv u_2(t,x)$ in Ω_T.

Proof. Let $u = u(t,x)$ be as in the proof of Theorem 3.1 and let

$$r \overset{\text{def}}{=} \max_{j=1,2}\sup_{(t,x)\in\Omega_T}\left|\frac{\partial u_j}{\partial x}(t,x)\right| < +\infty,\ K_2 \overset{\text{def}}{=} \overline{B}_r^n \subset \mathbb{R}^n,\ L \overset{\text{def}}{=} L_{K_2}, \tag{3.2}$$

$$X^k \stackrel{\text{def}}{=} \underbrace{(-k,k)\times\cdots\times(-k,k)}_{n \text{ times}} \subset \mathbb{R}^n \quad (k=1,2,\ldots). \tag{3.3}$$

For an arbitrarily fixed $T' \in (0,T)$, we consider the sequence $\{P^k\}_{k=1}^{+\infty}$ of the following parallelepipeds:

$$P^k \stackrel{\text{def}}{=} (0,T')\times X^k = \{(t,x) \,:\, 0<t<T',\; x\in X^k\}.$$

Obviously, $P^1 \subset P^2 \subset \ldots \subset P^k \subset \ldots$ and $\bigcup_{k=1}^{+\infty} P^k = \Omega_{T'}$. Next, take

$$s^k \stackrel{\text{def}}{=} \max_{j=1,2} \max_{(t,x)\in\overline{P}^k} |u_j(t,x)|, \quad K_1^k \stackrel{\text{def}}{=} [-s^k, \quad s^k] \subset \mathbb{R}^1. \tag{3.4}$$

We now define a function $\mu = \mu(x)$ by setting

$$\mu(x) \stackrel{\text{def}}{=} \begin{cases} \mu_{K_1^1,K_2}(x) & \text{if } x\in X^1, \\ \mu_{K_1^{k+1},K_2}(x) & \text{if } x\in X^{k+1}\backslash X^k \text{ (for } k=1,2\ldots). \end{cases} \tag{3.5}$$

It follows that $\mu = \mu(x)$ is locally bounded on $\mathbb{R}^n$. Moreover, (3.2)–(3.5) together with the hypothesis of the theorem imply

$$\begin{aligned} \left|\frac{\partial u}{\partial t}(t,x)\right| &= \left| f\Big(t,x,u_1(t,x),\frac{\partial u_1}{\partial x}(t,x)\Big) - f\Big(t,x,u_2(t,x),\frac{\partial u_2}{\partial x}(t,x)\Big)\right| \\ &\le L(1+|x|)\left|\frac{\partial u_1}{\partial x}(t,x) - \frac{\partial u_2}{\partial x}(t,x)\right| + \mu(x)|u_1(t,x)-u_2(t,x)| \\ &= L(1+|x|)\left|\frac{\partial u}{\partial x}(t,x)\right| + \mu(x)|u(t,x)| \quad \text{in } \Omega_{T'}. \end{aligned}$$

(We may check this inequality first for (t,x) in P^1, and then for (t,x) in each $P^{k+1}\backslash P^k$.) Theorem 2.2 therefore shows that $u(t,x) \equiv 0$ in $\Omega_{T'}$. Since $T' \in (0,T)$ is arbitrarily chosen, the conclusion follows. ■

Corollary 3.3. *Let $f = f(t,x,u,p)$ belong to $C^1(\overline{\Omega}_T \times \mathbb{R}^1 \times \mathbb{R}^n)$ and be such that the function*

$$\nu = \nu(t,p) \stackrel{\text{def}}{=} \sup_{(x,u)\in\mathbb{R}^n\times\mathbb{R}^1} \left|\frac{\partial f}{\partial p}(t,x,u,p)/(1+|x|)\right|$$

is finite and continuous on $[0,T]\times\mathbb{R}^n$. If $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ are global C^1-solutions to the Cauchy problem (2.3)–(2.4) with

$$\sup_{(t,x)\in\Omega_T} \left|\frac{\partial u_j}{\partial x}(t,x)\right| < +\infty \quad (j=1,2),$$

then $u_1(t,x) \equiv u_2(t,x)$ in Ω_T.

Proof. For any convex compact sets $K_1 \subset \mathbb{R}^1$, $K_2 \subset \mathbb{R}^n$ we see, by assumption, that

$$L_{K_2} \stackrel{\text{def}}{=} \max_{(t,p)\in[0,T]\times K_2} \nu(t,p) < +\infty,$$

and that the function

$$\mu_{K_1,K_2} = \mu_{K_1,K_2}(x) \stackrel{\text{def}}{=} \max_{(t,u,p)\in[0,T]\times K_1\times K_2} \left|\frac{\partial f}{\partial u}(t,x,u,p)\right|$$

is continuous, and hence locally bounded on $\mathbb{R}^n$. It is easy to check that (3.1) with L_{K_2} and $\mu_{K_1,K_2}(x)$ in place of L and M, respectively, holds for any $(t,x,u,p), (t,x,v,q) \in \Omega_T \times K_1 \times K_2$. The corollary thereby follows from Theorem 3.2. ∎

We conclude this section with the following result of continuous dependence on initial data for global C^1-solutions. (Here the continuity is with respect to the topology of uniform convergence on compact sets.)

Theorem 3.4. *Suppose $f = f(t,x,u,p)$ satisfies the condition (3.1) in Theorem 3.1. Let $u_j = u_j(t,x)$ $(j = 1,2)$ be global C^1-solutions to the equation (2.3) with the initial conditions*

$$u_j(0,x) = \phi_j(x) \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^n\},$$

where $\phi_j = \phi_j(x)$ $(j = 1,2)$ are given functions of class C^0 on $\mathbb{R}^n$. Then

$$|u_1(t,x) - u_2(t,x)| \le \exp(Mt) \cdot \sup_{|y|\le(1+|x|)\exp(Lt)-1} |\phi_1(y) - \phi_2(y)|$$

for all $(t,x) \in \Omega_T$.

The proof of this theorem will be left to the reader.

4. Generalizations to the Case of Weakly Coupled Systems

We now examine how the case of systems of first-order partial differential inequalities or equations can be treated by the preceding method. Let m be a positive integer. Consider the class

$$V^m(\Omega_T) \stackrel{\text{def}}{=} \underbrace{V(\Omega_T) \times \cdots \times V(\Omega_T)}_{m \text{ times}}.$$

Each element of $V^m(\Omega_T)$ is therefore a vector function, namely,

$$u = u(t,x) = (u_1(t,x), \ldots, u_m(t,x))$$

from $\Omega_T \subset \mathbb{R}^{n+1}$ into $\mathbb{R}^m$ such that $u_j = u_j(t,x)$ belongs to $V(\Omega_T)$ for every $j \in \{1, \ldots, m\}$.

First, the following result may be proved in much the same way as Theorem 2.2.

Theorem 4.1. *Let $u = u(t,x)$ be a vector function in $V^m(\Omega_T)$. If there exist a nonnegative function $\mu = \mu(x)$ locally bounded on $\mathbb{R}^n$ and a nonnegative function $\ell = \ell(t)$ in $L^1(0,T)$ such that*

$$|\partial u_j(t,x)/\partial t| \le \ell(t) \cdot \big[(1+|x|)|\partial u_j(t,x)/\partial x| + \mu(x) \max_{k=1,\dots,m} |u_k(t,x)|\big] \quad (j=1,\dots,m) \tag{4.1}$$

for almost every $t \in (0,T)$ and for all $x \in \mathbb{R}^n$, then

$$\max_{j=1,\dots,m} |u_j(t,x)| \le \exp\left[C(x)\int_0^t \ell(\tau)d\tau\right] \cdot \sup_{|y|\le(1+|x|)\exp\int_0^t \ell(\tau)d\tau - 1} \max_{j=1,\dots,m} |u_j(0,y)|,$$

where $C(x)$ is given by the formula (2.8).

Proof. For an arbitrary point $(t^0, x^0) \in \Omega_T$, we must prove that

$$\max_{j=1,\dots,m} |u_j(t^0,x^0)| \le \exp\left[C(x^0)\int_0^{t^0} \ell(t)dt\right] \cdot \sup_{|y|\le(1+|x^0|)\exp\int_0^{t^0} \ell(t)dt - 1} \max_{j=1,\dots,m} |u_j(0,y)|. \tag{4.2}$$

Let us continue using the notations $I \stackrel{\text{def}}{=} [0,t^0]$, $\Sigma_I(t^0,x^0)$, $Z(.,t^0,x^0)$, $h(.)$, G_1 introduced in the proof of Theorem 2.2 and then define

$$g(t) \stackrel{\text{def}}{=} \max_{k=1,\dots,m} g^k(t) \tag{4.3}$$

for $t \in I$, where

$$g^k(t) \stackrel{\text{def}}{=} \max\left\{|u_k(t,x)| \ : \ x \in Z(t,t^0,x^0)\right\} \quad (k=1,\dots,m). \tag{4.4}$$

It follows from Lemma 2.3 that, for any number $\theta \in (0,t^0)$, each function $g^k = g^k(t)$ is absolutely continuous on $[\theta, t^0]$ and so is the function $g = g(t)$. Moreover, they are all continuous on the whole I. Still as in the proof of Theorem 2.2, we see that (4.2) will be obtained if we can show that (2.14) holds. To this end, setting $\omega(t) \stackrel{\text{def}}{=} g_\eta(t) - g(t)$, with $\eta > 0$ temporarily fixed and

$$g_\eta(t) \stackrel{\text{def}}{=} [g(0)+\eta] \cdot \exp\Big\{\big[C(x^0)+\eta\big] \cdot \big[h(t)+\eta t\big]\Big\},$$

we need only claim that $\omega(t) \ge \omega(0)\ (=\eta > 0)$ for $t \in I$. On the contrary, suppose there exists $t' \in (0,t^0]$ with $\omega(t') < \omega(0)$.

By the hypothesis of the theorem, one finds a set $G_2 \subset (0,T)$ of Lebesgue measure 0 such that

$$\Omega_T \backslash (G_2 \times \mathbb{R}^n) \subset \cap_{k=1}^m \mathrm{Dif}(u_k) \tag{4.5}$$

and that (4.1) is satisfied for any $t \in (0, T)\backslash G_2$, $x \in \mathbb{R}^n$. From the above, it follows that (2.16) still holds where $G \stackrel{\text{def}}{=} G_1 \cup G_2$; hence, there is a number λ with

$$\max\{0, \omega(t')\} < \lambda < \omega(0) \quad \text{and} \quad \lambda \in \omega[0, t']\backslash\omega\big(G \cap [0, t^0]\big).$$

Now take

$$(0, T)\backslash G \ni t_* \stackrel{\text{def}}{=} \inf\big\{t \in [0, t'] \ : \ \omega(t) = \lambda\big\}$$

and $1 \leq j \leq m$ such that

$$g(t_*) = g^j(t_*) = |u_j(t_*, x_*)| = \varepsilon \cdot u_j(t_*, x_*), \quad \varepsilon \stackrel{\text{def}}{=} \operatorname{sign} u_j(t_*, x_*) \tag{4.6}$$

for some $x_* \in Z(t_*, t^0, x^0)$. Next, choose a unit vector $e \in \mathbb{R}^n$ with

$$\left\langle e, \varepsilon \cdot \frac{\partial u_j}{\partial x}(t_*, x_*)\right\rangle = -\left|\frac{\partial u_j}{\partial x}(t_*, x_*)\right|. \tag{4.7}$$

Finally, let $y = y(s)$ be an $\mathbb{R}^n$-valued function continuously differentiable on $\mathbb{R}^1$ such that $y\big(h(t_*)\big) = x_*$ and $dy/ds = (1+|y|) \cdot e$, and let $x(t) \stackrel{\text{def}}{=} y\big(h(t)\big)$ for $t \in [0, T]$. Analysis similar to that in the proof of Theorem 2.2 shows that

$$\varepsilon \cdot u_j(t, x(t)) \leq |u_j(t, x(t))| \leq g(t) = g_\eta(t) - \omega(t) < g_\eta(t) - \lambda$$

for all $t \in [0, t_*)$, and that

$$\varepsilon \cdot u_j(t_*, x(t_*)) = |u_j(t_*, x_*)| = g(t_*) = g_\eta(t_*) - \omega(t_*) = g_\eta(t_*) - \lambda.$$

Consequently,

$$\frac{d}{dt}\Big[\varepsilon \cdot u_j(t, x(t))\Big]\Big|_{t=t_*} \geq \frac{dg_\eta}{dt}(t_*).$$

This would give

$$\begin{aligned}\left|\frac{\partial u_j}{\partial t}(t_*, x_*)\right| &> \ell(t_*) \cdot \left[\big(1 + |x_*|\big) \cdot \left|\frac{\partial u_j}{\partial x}(t_*, x_*)\right| + C(x^0) \cdot \max_{k=1,\ldots,m} |u_k(t_*, x_*)|\right] \\ &\geq \ell(t_*) \cdot \left[\big(1 + |x_*|\big) \left|\frac{\partial u_j}{\partial x}(t_*, x_*)\right| + |\mu(x_*)| \cdot \max_{k=1,\ldots,m} |u_k(t_*, x_*)|\right],\end{aligned}$$

a contradiction with (4.1). The proof is therefore complete. ∎

Remark. Theorem 4.1 can be used to investigate the stability of global solutions to the Cauchy problem for *weakly-coupled systems*, i.e., systems of first-order partial differential equations of the form

$$\partial u_j/\partial t + f_j(t, x, u, \partial u_j/\partial x) = 0 \ \text{ in } \Omega_T \ (j = 1, \ldots, m), \tag{4.8}$$

$$u(0, x) = \big(\phi_1(x), \ldots, \phi_m(x)\big) \ \text{ on } \{t = 0, \ x \in \mathbb{R}^n\}. \tag{4.9}$$

The systems (4.8)–(4.9) are of special hyperbolic type because each equation contains first-order derivatives of only one unknown function. Since (classical) solutions of elliptic equations do not depend continuously (with respect to the topology of uniform convergence on compact sets) on initial data, theorems of the *non-stationary type* that we have studied in this paper cannot be expected to apply to partial differential equations or inequalities of elliptic type. (First results on second-order partial differential inequalities of parabolic and hyperbolic types were obtained by Nagumo and Simoda [23] and by Westphal [37].)

For our next discussion, we need to extend the notion of *comparison equation* given in Szarski [24] to the Carathéodory case. Consider an ordinary differential equation

$$w' = \rho(t, w), \tag{4.10}$$

where the function $\rho = \rho(t, w)$ is defined on $D_+ \overset{\text{def}}{=} (0, +\infty) \times [0, +\infty) = \{(t, w) : t > 0,\ w \geq 0\}$. The following Carathéodory conditions are always assumed.

(1) *For almost every* $t \in (0, +\infty)$, *the function* $[0, +\infty) \ni w \mapsto \rho(t, w)$ *is continuous.*

(2) *For each* $w \in [0, +\infty)$, *the function* $(0, +\infty) \ni t \mapsto \rho(t, w)$ *is measurable.*

(3) *For any* $r \in (0, +\infty)$, *there exists a function* $m_r = m_r(t)$ *in* $L^1_{loc}(0, +\infty)$ *with*

$$|\rho(t, w)| \leq m_r(t) \quad \forall\, w \in [0, r]$$

for almost every $t \in (0, +\infty)$.

In this situation we call (4.10) a *Carathéodory differential equation* on D_+. A *solution* of it *on an interval* $I \subset (0, +\infty)$, with $\operatorname{int} I \neq \emptyset$, means a function $w = w(t) \geq 0$ absolutely continuous on each compact interval $J \subset I$ (*absolutely continuous on* I for short) such that

$$w'(t) = \rho(t, w(t))$$

almost everywhere in I. We refer to Coddington and Levinson [9] for what concerns the local existence of a solution of (4.10) through any given point $(t^0, w^0) \in \operatorname{int} D_+$. Moreover, every such solution can be extended (as a solution) over a [left, right] maximal interval of existence.

Definition. *A Carathéodory differential equation (4.10), with* $\rho(t, w) \geq 0$ *on* D_+ *and* $\rho(t, 0) = 0$ *for almost all* $t > 0$, *will be called a comparison equation if* $w = w(t) \equiv 0$ *is in every interval* $(0, \gamma)$ *the only solution satisfying the condition* $\lim_{t \to 0} w(t) = 0$.

Remark. Let $\ell = \ell(t)$ be a nonnegative function Lebesgue integrable on each bounded interval $(0, \gamma) \subset \mathbb{R}$, and $\sigma = \sigma(w)$ a function of class $C[0, +\infty)$ such that $\sigma(0) = 0$, $\sigma(w) > 0$ as $w > 0$, and $\int_0^\delta (1/\sigma(w))dw = +\infty$ for every $\delta > 0$. Then (cf. [24, Example 14.2])

$$w' = \ell(t)\sigma(w) \tag{4.11}$$

is a comparison equation. In fact, assume the contrary that (4.11) admits a nonzero solution $w = w(t)$ on some interval $(0, \gamma)$ with $\lim_{t\to 0} w(t) = 0$. Letting $w(0) \overset{\text{def}}{=} 0$, from this we easily find a nonempty subinterval $(t^1, t^2]$ of $(0, \gamma)$ such that $w(t^1) = 0$ and $w(t) > 0$ for all $t \in (t^1, t^2]$. It follows that

$$\int_0^{w(t^2)} \frac{dv}{\sigma(v)} = \int_{t^1}^{t^2} \frac{w'(t)}{\sigma(w(t))} dt = \int_{t^1}^{t^2} \ell(t) dt < +\infty,$$

a contradiction. Therefore, (4.11) must be a comparison equation. Motivated by this fact, we propose the following:

Proposition 4.2. *Let $\sigma = \sigma(w)$ be of class $C[0, +\infty)$, and $\ell = \ell(t) \geq 0$ be Lebesgue integrable on each bounded interval $(0, \gamma) \subset \mathbb{R}$ with $\int_0^{+\infty} \ell(t)dt = +\infty$.*
(i) *If (4.11) is a comparison equation, then so is the equation*

$$w' = \sigma(w). \tag{4.12}$$

(ii) *Conversely, under the condition $\operatorname{ess\,inf}_{t\in(0,+\infty)} \ell(t) > 0$, if moreover (4.12) is a comparison equation, then so is (4.11).*

Proof.
(i) Let $w^1 = w^1(t)$ be a solution of (4.12) on some interval $(0, \gamma^1)$ with $\lim_{t\to 0} w^1(t) = 0$. Find a number $\gamma^2 > 0$ such that

$$\gamma^1 = \int_0^{\gamma^2} \ell(\tau)d\tau. \tag{4.13}$$

Setting $w^2(t) \overset{\text{def}}{=} w^1\big(\int_0^t \ell(\tau)d\tau\big)$, we see that $w^2 = w^2(t)$ is a solution of (4.11) on $(0, \gamma^2)$ with $\lim_{t\to 0} w^2(t) = 0$. By assumption, $w^2(t) \equiv 0$ on $(0, \gamma^2)$. Hence, $w^1(t) \equiv 0$ on $(0, \gamma^1)$. This shows that (4.12) is a comparison equation.
(ii) Let $(0, +\infty) \ni t \mapsto \hat{\ell}(t)$ be the inverse of $(0, +\infty) \ni t \mapsto \int_0^t \ell(\tau)d\tau$, and $w^2 = w^2(t)$ be a solution of (4.11) on some interval $(0, \gamma^2)$ with $\lim_{t\to 0} w^2(t) = 0$. First, define a number $\gamma^1 > 0$ by (4.13). Then setting $w^1(t) \overset{\text{def}}{=} w^2(\hat{\ell}(t))$, we also see that $w^1 = w^1(t)$ is a solution of (4.12) on $(0, \gamma^1)$ with $\lim_{t\to 0} w^1(t) = 0$ (cf. [13, Proposition 3.4(c)]). The rest of the proof runs as before. ■

In the sequel, for each function $g = g(t)$ defined and continuous in a certain interval $(0, t^0)$, let P_g denote the open set $\{t \in (0, t^0) : g(t) > 0\}$. Here is an elementary property of comparison equations:

Proposition 4.3. *Let (4.10) be a comparison equation and $g = g(t)$ a given function absolutely continuous on some interval $(0, t^0)$ such that $\lim_{t\to 0} g(t) \leq 0$ and that $g'(t) \leq \rho(t, g(t))$ almost everywhere in P_g. Then $g(t) \leq 0$ for all $t \in (0, t^0)$.*

Proof. On the contrary, suppose there exists $t^1 \in (0, t^0)$ with $w^1 \stackrel{\text{def}}{=} g(t^1) > 0$. Setting $g(0) \stackrel{\text{def}}{=} \lim_{t\to 0} g(t)$ and $t^2 \stackrel{\text{def}}{=} \sup\{t \in [0, t^1) \ : \ g(t) = 0\}$, we see that $0 \le t^2 < t^1$, $g(t^2) = 0$ and $(t^2, t^1) \subset P_g$. Hence, by assumption,

$$g'(t) \le \rho(t, g(t)) \quad \text{almost everywhere in } (t^2, t^1). \tag{4.14}$$

Now take

$$\hat{\rho}(t, w) \stackrel{\text{def}}{=} \begin{cases} \rho\big(t, \max\{0, g(t)\}\big) & \text{if } t^2 < t < t^0, \ w \ge \max\{0, g(t)\}, \\ \rho(t, w) & \text{if } t^2 < t < t^0, \ 0 \le w < \max\{0, g(t)\}. \end{cases} \tag{4.15}$$

The above-mentioned Carathéodory conditions (1)–(3) are clearly satisfied for $\hat{\rho} = \hat{\rho}(t, w)$ on $(t^2, t^0) \times [0, +\infty)$. Let $w = w(t)$ be a solution through (t^1, w^1) of (4.10) with $\hat{\rho}$ in place of ρ, and let $(t^3, t^1] \subset (t^2, t^1]$ be its left maximal interval of existence. We next claim that

$$(0 \le)\, w(t) \le g(t) \quad \forall\, t \in (t^3, t^1]. \tag{4.16}$$

Assume (4.16) is false. Then one would find a nonempty interval $(t^4, t^5) \subset (t^3, t^1)$ such that

$$w(t) > g(t) \quad \forall\, t \in (t^4, t^5), \tag{4.17}$$

with

$$w(t^5) = g(t^5). \tag{4.18}$$

It follows from (4.14)–(4.15) and (4.17) that $g'(t) \le \rho(t, g(t)) = \hat{\rho}(t, w(t)) = w'(t)$ almost everywhere in (t^4, t^5). Thus (4.18) implies that $g(t) \ge w(t)$ for all $t \in (t^4, t^5)$, which contradicts (4.17). So (4.16) must hold.

We proceed to show that $t^3 = t^2$. In fact, if $(0 \le)\, t^2 < t^3$, then (4.15) together with Carathéodory's condition (3), where $r \stackrel{\text{def}}{=} \max\{g(t) \ : \ t \in [t^3, t^1]\}$, prove that the limit $\lim_{t\to t^3} w(t)$ exists and is finite. Hence, $w = w(t)$ could be extended (as a solution of (4.10) with $\hat{\rho}$ in place of ρ) over an interval $(t^6, t^1] \supset [t^3, t^1]$, which is impossible.

Finally, (4.15)–(4.16) shows that $w = w(t)$ is indeed a solution through (t^1, w^1) of (4.10) on $(t^2, t^1]$ with $\lim_{t\to t^2} w(t) = g(t^2) = 0$. Setting $w(t) \stackrel{\text{def}}{=} 0$ for $t \in [0, t^2]$, we obtain a nonzero solution of (4.10) on $(0, t^1)$ which tends to 0 as t goes to 0. Thus we arrive at a contradiction. This completes the proof. ■

We can now combine the method of Sec. 2 with the technique of Carathéodory comparison equations and prove the following.

Theorem 4.4. *Let $u = u(t, x)$ be a vector function in $V^m(\Omega_T)$ with $u_j(0, x) \equiv 0$ $(j = 1, \dots, m)$, and (4.10) a comparison equation. If there exists a nonnegative function $\ell = \ell(t)$ in $L^1(0, T)$ such that*

$$\begin{aligned} |\partial u_j(t, x)/\partial t| &\le \ell(t)(1 + |x|) \cdot |\partial u_j(t, x)/\partial x| \\ &\quad + \rho\big(t, \max_{k=1,\dots,m} |u_k(t, x)|\big) \quad (j = 1, \dots, m) \end{aligned} \tag{4.19}$$

for almost every $t \in (0, T)$ and for all $x \in \mathbb{R}^n$, then $u_j(t, x) \equiv 0$ in Ω_T $(j = 1, \dots, m)$.

Proof. For an arbitrary point $(t^0, x^0) \in \Omega_T$, it suffices to prove that

$$\max_{j=1,\dots,m} |u_j(t^0, x^0)| = 0. \tag{4.20}$$

We shall continue using the notations $I \stackrel{\text{def}}{=} [0, t^0]$, $\Sigma_I(t^0, x^0)$, $Z(., t^0, x^0)$, $h(.)$, G_1 introduced in the proof of Theorem 2.2 (and also, of Theorem 4.1) and letting $g = g(t)$, $g^k = g^k(t)$ be as in (4.3)–(4.4). Obviously, (4.20) will be obtained if one can verify that $g(t^0) = 0$. Since $g = g(t)$ is a nonnegative function absolutely continuous on $(0, t^0]$, with $\lim_{t\to 0} g(t) = g(0) = 0$ (by assumption), Proposition 4.3 shows that we need only claim that

$$g'(t) \le \rho(t, g(t)) \quad \text{almost everywhere in } (0, t^0). \tag{4.21}$$

By the hypothesis of the theorem, one finds a set $G_2 \subset (0, T)$ of Lebesgue measure 0 such that (4.5) and (4.19) are fulfilled for any $t \in (0, T)\backslash G_2$, $x \in \mathbb{R}^n$. Assume without loss of generality that $g = g(t)$ is differentiable at any point of $(0, t^0)\backslash G$, where $G \stackrel{\text{def}}{=} G_1 \cup G_2$. Now fix an arbitrary point $t_* \in (0, t^0)\backslash G$ and take $1 \le j \le m$ such that (4.6) holds for some $x_* \in Z(t_*, t^0, x^0)$. Next, choose a unit vector $e \in \mathbb{R}^n$ satisfying (4.7). Let $y = y(s)$ be an $\mathbb{R}^n$-valued function continuously differentiable on $\mathbb{R}^1$ such that $y\big(h(t_*)\big) = x_*$ and $dy/ds = (1+|y|)\cdot e$, and let $x(t) \stackrel{\text{def}}{=} y\big(h(t)\big)$ for $t \in [0, T]$. Of course (see the proof of Theorem 2.2), $x = x(t)$ is absolutely continuous on $[0, T]$, $x(t_*) = x_*$, and

$$\frac{dx}{dt}(t) = \ell(t)\cdot(1 + |x(t)|)\cdot e \quad \forall t \in (0, T)\backslash G. \tag{4.22}$$

Moreover,

$$x(t) \in Z(t, t^0, x^0) \quad \forall t \in [0, t_*].$$

This together with (4.3)–(4.4) implies

$$\varepsilon \cdot u_j(t, x(t)) \le |u_j(t, x(t))| \le g^j(t) \le g(t) \quad \text{for all } t \in [0, t_*). \tag{4.23}$$

Besides that, by (4.6),

$$\varepsilon \cdot u_j(t_*, x(t_*)) = |u_j(t_*, x_*)| = g^j(t_*) = g(t_*). \tag{4.24}$$

Therefore, since $t_* \in (0, t^0)\backslash G$, it may be deduced from (4.23)–(4.24) that

$$g'(t_*) \le \frac{d}{dt}\Big[\varepsilon \cdot u_j(t, x(t))\Big]\Big|_{t=t_*}.$$

Consequently, by (4.6)–(4.7), (4.19) and (4.22), we conclude that

$$\begin{aligned}
g'(t_*) &\le \varepsilon\cdot(\partial u_j(t_*, x_*)/\partial t) + \left\langle \frac{dx}{dt}(t_*), \varepsilon\cdot\frac{\partial u_j}{\partial x}(t_*, x(t_*))\right\rangle \\
&\le |\partial u_j(t_*, x_*)/\partial t| - \ell(t_*)(1+|x_*|)\cdot|\partial u_j(t_*, x_*)/\partial x| \\
&\le \rho\big(t_*, \max_{k=1,\dots,m} |u_k(t_*, x_*)|\big) = \rho\big(t_*, |u_j(t_*, x_*)|\big) = \rho\big(t_*, g(t_*)\big).
\end{aligned}$$

Finally, because G has measure 0 and $t_* \in (0, t^0)\backslash G$ is arbitrarily chosen, (4.21) must hold. This completes the proof. ■

Theorem 4.5. *Let $u = u(t,x)$ be a vector function in $V^m(\Omega_T)$ with $u_j(0,x) \equiv 0$ ($j = 1,\dots,m$), and (4.12) be a comparison equation. If there exist a nonnegative function $\mu = \mu(x)$ locally bounded on $\mathbb{R}^n$ and a nonnegative function $\ell = \ell(t)$ in $L^1(0,T)$ such that*

$$|\partial u_j(t,x)/\partial t| \le \ell(t)\Big[(1+|x|)\cdot|\partial u_j(t,x)/\partial x| + \mu(x)\sigma\big(\max_{k=1,\dots,m}|u_k(t,x)|\big)\Big] \quad (j=1,\dots,m) \tag{4.25}$$

for almost every $t \in (0,T)$ and for all $x \in \mathbb{R}^n$, then $u_j(t,x) \equiv 0$ in Ω_T ($j = 1,\dots,m$).

Proof. For an arbitrary point $(t^0, x^0) \in \Omega_T$, it suffices to prove (4.20). Let us continue using the method (and notations) introduced in the proof of Theorem 4.4. We may extend the function $\ell = \ell(t)$ over the whole $(0,+\infty)$ and assume $\operatorname{ess\,inf}_{t\in(0,+\infty)} \ell(t) > 0$. Then by (4.25) (instead of (4.19)) we get

$$g'(t) \le C\ell(t)\sigma(g(t)) \quad \text{almost everywhere in } (0,t^0)$$

(instead of (4.21)) for some positive constant C. By Proposition 4.2(ii), the Carathéodory differential equation

$$w' = C\ell(t)\sigma(w)$$

is also a comparison equation. Thus (4.20) is straightforward as before. ■

5. Uniqueness of Global Semiclassical Solutions to the Cauchy Problem

The present section is in principle a continuation of the previous three. However, it was actually originated in the following problem posed by Kruzhkov [20].

Let a C^1-function $\omega = \omega(t,x)$ satisfy in the strip $\Pi_T \stackrel{\text{def}}{=} [0,T] \times \mathbb{R}^1$ the inequality

$$|\partial\omega(t,x)/\partial t| \le N|\partial\omega(t,x)/\partial x|, \quad N = \text{const.} \ge 0, \tag{5.1}$$

and the initial condition

$$\omega(0,x) \equiv 0 \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^1\}. \tag{5.2}$$

Then it is easy to show (cf. Haar–Ważewski's Theorem 2.1) that $\omega(t,x) \equiv 0$ in Π_T. Therefore, the Cauchy problem for the first-order nonlinear equation

$$\partial u/\partial t + f(\partial u/\partial x) = 0,$$

where $f = f(p)$ is of class $C^1(\mathbb{R}^1)$, cannot have more than one solution in Π_T, say, in the class of C^1-functions with bounded derivatives. As Kruzhkov already remarked, the same conclusion may be drawn without appeal to the differentiability of $\omega = \omega(t, x)$ (respectively, of the solution) or the validity of (5.1) (respectively, of the equation) at the points in any given finite union of straight lines $\{t = \text{const.},\ x \in \mathbb{R}\} \subset \Pi_T$. The following question arises naturally: to what extent can the condition on the smoothness of $\omega = \omega(t, x)$ and on the validity of inequality (5.1) in the entire strip Π_T be weakened? For example, the Cauchy problem for the equation $\partial u/\partial t + (\partial u/\partial x)^2 = 0$ with the zero initial condition $u(0, x) \equiv 0$ has a continuum of piecewise smooth solutions in Π_T, such as $u_\alpha = u_\alpha(t, x) \stackrel{\text{def}}{=} \min\{0, \alpha|x| - \alpha^2 t\}$, $\alpha = \text{const.} \geq 0$. Note that each function $\omega = u_\alpha(t, x)$ satisfies the corresponding inequality $|\partial\omega/\partial t| \leq \alpha|\partial\omega/\partial x|$ almost everywhere in Π_T. Therefore, it is interesting to find intermediate classes (as wide as possible) between $C^1(\Pi_T)$ and $\text{Lip}(\Pi_T)$, in which only the zero function can simultaneously satisfy (5.1) and (5.2). These questions can be generalized to the multi-dimensional case.

The study of this problem suggests that we should single out the widest class between the class of continuously differentiable functions and the class of Lipschitz continuous functions in which the Cauchy problem for a first-order nonlinear partial differential equation has a unique global solution.

Our discussions in this section make an appeal to Theorems 2.2, 4.1, 4.4, 4.5. The condition on the validity of inequality (2.6) is clearly much weaker than that of (5.1) in the entire domains of the corresponding functions under consideration. Moreover, it should be noted (see Sec. 1) that

$$C^1(\Omega_T) \cap C([0, T) \times \mathbb{R}^n) \subset V(\Omega_T) \subset \text{Lip}([0, T) \times \mathbb{R}^n).$$

The smoothness requirement on functions in $V(\Omega_T)$ is really weak enough: roughly speaking, these functions need only be absolutely continuous in time variable. By the previous sections, the class $V(\Omega_T)$ would be nominated as best candidate for our discussion concerning the above questions for the Cauchy problem (2.3)–(2.4). We therefore arrive at the following definition of generalized solutions:

Definition. *A function $u = u(t, x)$ in $V(\Omega_T)$ is called a global semiclassical solution to (2.3)–(2.4) if it satisfies (2.3) for all $x \in \mathbb{R}^n$ and almost all $t \in (0, T)$ and if $u(0, x) = \phi(x)$ for all $x \in \mathbb{R}^n$.*

Here, the initial data $\phi = \phi(x)$ is a given continuous function on $\mathbb{R}^n$. The Hamiltonian $f = f(t, x, u, p)$ is always assumed to be measurable in $t \in (0, T)$ and continuous in $(x, u, p) \in \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$. In this section, we investigate the uniqueness of the above global semiclassical solution. Further, an answer to Kruzhkov's problem will be given.

The easy proof of the following uniqueness criterion will be left to the reader.

Theorem 5.1. *Suppose $f = f(t, x, u, p)$ satisfies the following condition: there exist a nonnegative function $\mu = \mu(x)$ locally bounded on $\mathbb{R}^n$ and a nonnegative function $\ell = \ell(t)$ in $L^1(0, T)$ such that*

$$|f(t, x, u, p) - f(t, x, v, q)| \leq \ell(t) \cdot [(1 + |x|)|p - q| + \mu(x)|u - v|] \qquad (5.3)$$

for almost every $t \in (0, T)$ and for all $(x, u, p), (x, v, q) \in \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$. If $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ are global semiclassical solutions to the Cauchy problem (2.3)–(2.4), then $u_1(t, x) \equiv u_2(t, x)$ in Ω_T.

Remark. Condition (5.3) is satisfied if and only if for some positive function $\ell = \ell(t)$ in $L^1(0, T)$, the function

$$\Omega_T \times \mathbb{R}^1 \times \mathbb{R}^n \ni (t, x, u, p) \mapsto f(t, x, u, p)/[\ell(t)(1 + |x|)]$$

is Lipschitz continuous with respect to p uniformly in $(t, x, u) \in \Omega_T \times \mathbb{R}^1$, and is Lipschitz continuous with respect to u uniformly in $(t, x, p) \in (0, T) \times X \times \mathbb{R}^n$ for every compact set $X \subset \mathbb{R}^n$ (i.e., uniformly globally in (t, p) and locally in x).

A useful uniqueness criterion for global semiclassical solutions with essentially bounded derivatives is given by the next sharpening.

Theorem 5.2. *Suppose $f = f(t, x, u, p)$ satisfies the following condition: for any compact sets $K_1 \subset \mathbb{R}^1$, $K_2 \subset \mathbb{R}^n$, there exist a nonnegative function $\ell_{K_2} = \ell_{K_2}(t)$ in $L^1(0, T)$ and a nonnegative function $\mu_{K_1,K_2} = \mu_{K_1,K_2}(x)$ locally bounded on $\mathbb{R}^n$ such that (5.3) with ℓ_{K_2} and μ_{K_1,K_2} in place of ℓ and μ, respectively, holds for almost every $t \in (0, T)$ and for all $(x, u, p), (x, v, q) \in \mathbb{R}^n \times K_1 \times K_2$. If $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ are global semiclassical solutions to the problem (2.3)–(2.4) with*

$$\operatorname*{ess\,sup}_{(t,x)\in\Omega_T} \left|\frac{\partial u_j}{\partial x}(t, x)\right| < +\infty \quad (j = 1, 2),$$

then $u_1(t, x) \equiv u_2(t, x)$ in Ω_T.

Remark. If $f = f(t, p)$ depends only on t, p and is of class C^1 on $[0, T] \times \mathbb{R}^n$, then the condition of Theorem 5.2 is satisfied. In this case Theorem 5.2 solves the problem of Kruzhkov (see Corollary 5.4 later).

To prove Theorem 5.2, we need the following:

Lemma 5.3. *Let $\psi = \psi(x)$ be a locally Lipschitz continuous function of x on $\mathbb{R}^n$. If it is differentiable in the whole $\mathbb{R}^n$, then*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|\frac{\partial \psi}{\partial x_i}(x)\right| = \sup_{x\in\mathbb{R}^n} \left|\frac{\partial \psi}{\partial x_i}(x)\right| \quad (i = 1, \dots, n).$$

Proof. Fix any $i \in \{1, \dots, n\}$. It suffices to treat the case when

$$s_i \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \left|\frac{\partial \psi}{\partial x_i}(x)\right| < +\infty.$$

Let us write $x = (x', x_i)$ instead of $x = (x_1, \dots, x_n)$, where

$$x' \stackrel{\text{def}}{=} (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Then for almost all (with respect to the $(n-1)$-dimensional Lebesgue measure) $x' \in \mathbb{R}^{n-1}$, we have

$$\left\| \frac{\partial \psi}{\partial x_i}(x', .) \right\|_{L^\infty(\mathbb{R}^1)} \leq s_i .$$

Since the function $\psi(x', .)$ is absolutely continuous on each bounded segment, it follows that

$$|\psi(x', x_i^1) - \psi(x', x_i^2)| = \left| \int_{x_i^1}^{x_i^2} \frac{\partial \psi}{\partial x_i}(x', x_i) dx_i \right| \leq s_i |x_i^1 - x_i^2| \tag{5.4}$$

for almost all (with respect to the $(n-1)$-dimensional Lebesgue measure) $x' \in \mathbb{R}^{n-1}$ and for all $x_i^1, x_i^2 \in \mathbb{R}^1$. From the continuity of $\psi = \psi(x)$ and from (5.4), we conclude that

$$|\psi(x', x_i^1) - \psi(x', x_i^2)| \leq s_i |x_i^1 - x_i^2|$$

for all $(x', x_i^1), (x', x_i^2) \in \mathbb{R}^n$. Therefore,

$$\left| \frac{\partial \psi}{\partial x_i}(x) \right| \leq s_i$$

for all $x \in \mathbb{R}^n$. This proves the lemma. ■

Proof of Theorem 5.2. According to the definition of $V(\Omega_T)$, Lemma 5.3 shows that

$$\sup_{x \in \mathbb{R}^n} \left| \frac{\partial u_j}{\partial x_i}(t, x) \right| = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \left| \frac{\partial u_j}{\partial x_i}(t, x) \right| \quad (j = 1, 2; \ i = 1, \ldots, n)$$

for almost all $t \in (0, T)$. Taking the essential supremum over $t \in (0, T)$, we find that

$$\operatorname*{ess\,sup}_{t \in (0,T)} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial u_j}{\partial x_i}(t, x) \right| = \operatorname*{ess\,sup}_{(t,x) \in \Omega_T} \left| \frac{\partial u_j}{\partial x_i}(t, x) \right|.$$

Consequently, by assumption,

$$r \stackrel{\text{def}}{=} \max_{j=1,2} \operatorname*{ess\,sup}_{t \in (0,T)} \sup_{x \in \mathbb{R}^n} \left| \frac{\partial u_j}{\partial x}(t, x) \right| < +\infty. \tag{5.5}$$

Let X^k be as in (3.3); $K_2 \stackrel{\text{def}}{=} \overline{B}_r \subset \mathbb{R}^n$; $\ell(.) \stackrel{\text{def}}{=} \ell_{K_2}(.)$. For an arbitrarily fixed $T' \in (0, T)$, we consider the sequence $\{P^k\}_{k=1}^{+\infty}$ of the following parallelepipeds:

$$P^k \stackrel{\text{def}}{=} (0, T') \times X^k = \left\{ (t, x) \ : \ 0 < t < T', \ x \in X^k \right\}.$$

Continue using (3.4). Then the function $\mu = \mu(x)$ given by (3.5) is locally bounded on $\mathbb{R}^n$.

We now consider the function $u = u(t,x) \stackrel{\text{def}}{=} u_1(t,x) - u_2(t,x)$. Of course, $u(0,x) \equiv 0$. Moreover, in view of (5.5), the hypothesis of the theorem implies

$$\begin{aligned}\left|\frac{\partial u}{\partial t}(t,x)\right| &= \left|f(t,x,u_1(t,x),\frac{\partial u_1}{\partial x}(t,x)) - f(t,x,u_2(t,x),\frac{\partial u_2}{\partial x}(t,x))\right|\\ &\le \ell(t)\cdot\left[(1+|x|)\left|\frac{\partial u_1}{\partial x}(t,x) - \frac{\partial u_2}{\partial x}(t,x)\right| + \mu(x)|u_1(t,x)-u_2(t,x)|\right]\\ &= \ell(t)\cdot\left[(1+|x|)\left|\frac{\partial u}{\partial x}(t,x)\right| + \mu(x)|u(t,x)|\right]\end{aligned}$$

for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,T')$. Therefore, Theorem 2.2 shows that $u(t,x) \equiv 0$ in $\Omega_{T'}$. Since $T' \in (0,T)$ is arbitrarily chosen, the proof is complete. ■

Corollary 5.4. *Let $f = f(t,x,u,p)$ be measurable in $t \in (0,T)$, continuous in $x \in \mathbb{R}^n$, and differentiable in $(u,p) \in \mathbb{R}^1 \times \mathbb{R}^n$ such that, for any compact set $K \subset \mathbb{R}^n$, the function*

$$\ell_K = \ell_K(t) \stackrel{\text{def}}{=} 1 + \sup_{(x,u,p)\in\mathbb{R}^n\times\mathbb{R}^1\times K} \left|\frac{\partial f}{\partial p}(t,x,u,p)/(1+|x|)\right|$$

is Lebesgue integrable on $(0,T)$, and the function

$$\nu_K = \nu_K(x,u) \stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{t\in(0,T)} \sup_{p\in K} \left|\frac{\partial f}{\partial u}(t,x,u,p)/\ell_K(t)\right|$$

is locally bounded on $\mathbb{R}^n \times \mathbb{R}^1$. If $u_1 = u_1(t,x)$ and $u_2 = u_2(t,x)$ are global semiclassical solutions to the Cauchy problem (2.3)–(2.4) with

$$\operatorname*{ess\,sup}_{(t,x)\in\Omega_T} \left|\frac{\partial u_j}{\partial x}(t,x)\right| < +\infty \quad (j=1,2),$$

then $u_1(t,x) \equiv u_2(t,x)$ in Ω_T.

Proof. Let us introduce the notation

$$\mu_{K_1,K_2}(x) \stackrel{\text{def}}{=} \sup_{u\in K_1} \nu_{K_2}(x,u)$$

for any convex compact sets $K_1 \subset \mathbb{R}^1$, $K_2 \subset \mathbb{R}^n$. Then it is easy to check that (5.3) with ℓ_{K_2} and μ_{K_1,K_2} in place of ℓ and μ, respectively, holds for almost every $t \in (0,T)$ and for all (x,u,p), $(x,v,q) \in \mathbb{R}^n \times K_1 \times K_2$. The corollary thereby follows from Theorem 5.2. ■

We leave it to the reader to prove the following criterion of continuous dependence on initial data for global semiclassical solutions.

Theorem 5.5. *Suppose $f = f(t, x, u, p)$ satisfies (5.3). Let $u_j = u_j(t, x)$ $(j = 1, 2)$ be global semiclassical solutions to (2.3) with*

$$u_j(0, x) = \phi_j(x) \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^n\},$$

where $\phi_j = \phi_j(x)$ $(j = 1, 2)$ are given functions continuous on $\mathbb{R}^n$. Then

$$|u_1(t, x) - u_2(t, x)| \le \exp\Big[C(x)\int_0^t \ell(\tau)d\tau\Big] \cdot \sup_{|y| \le (1+|x|)\exp\int_0^t \ell(\tau)d\tau - 1} |\phi_1(y) - \phi_2(y)|,$$

$C(x)$ being defined in (2.8).

Remark 1. The example in Remark 2 following Theorem 2.2 shows that the Lipschitz continuity of functions in the class $V(\Omega_T)$ also plays an essential role in the definition of global semiclassical solutions. The zero solution aside, this example gave no other global semiclassical solution to the Cauchy problem

$$\partial u/\partial t = 0 \quad \text{in} \quad \Omega_1,$$

$$u(0, x) = 0 \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^n\}.$$

Remark 2. Consider the Cauchy problem

$$\partial u/\partial t + (\partial u/\partial x)^2 = 0 \quad \text{in} \quad \{0 < t < T,\ x \in \mathbb{R}^1\}, \tag{5.6}$$

$$u(0, x) = 0 \quad \text{on} \quad \{t = 0,\ x \in \mathbb{R}^1\}. \tag{5.7}$$

By definition, if $u = u(t, x)$ is a global semiclassical solution to the problem, then for almost every $t \in (0, T)$, the function $u(t, .)$ is differentiable on $\mathbb{R}^1$. Obviously, (5.6)–(5.7) has a continuum of global solutions in the class of Lipschitz continuous functions, such as (see Fig. 1)

$$u_{\lambda,\varepsilon} = u_{\lambda,\varepsilon}(t, x) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } 0 \le t < T - \varepsilon, \\ \min\{0, \lambda|x| - \lambda^2(t - T + \varepsilon)\} & \text{if } T - \varepsilon \le t < T, \end{cases}$$

where $\lambda \ge 0$, $0 < \varepsilon < T$. For $\lambda > 0$, the differentiability of the function $u_{\lambda,\varepsilon}(t, .)$ fails somewhere (at $x = \pm\lambda(t - T + \varepsilon)$ and at $x = 0$) if and only if t belongs to the interval $(T - \varepsilon, T)$, whose Lebesgue measure is precisely ε (positive but as small as we please). Thus, the zero function $u_{0,\varepsilon} = u_{0,\varepsilon}(t, x)$ (i.e., $\lambda = 0$) is the unique global semiclassical solution to (5.6)–(5.7) in the class of functions with essentially bounded derivatives (cf. the remark following Theorem 5.2).

The results in this section can be generalized to the case of weakly coupled systems (4.8)–(4.9) by the use of Theorems 4.1, 4.4, 4.5. Here, the initial data $\phi = \phi(x) = (\phi_1(x), \ldots, \phi_m(x))$ is a given vector function continuous on $\mathbb{R}^n$. Each Hamiltonian $f_j = f_j(t, x, u, p^j)$ is always assumed to be measurable in $t \in (0, T)$ and continuous in $(x, u, p^j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. First, we give the definition [25,33] of global semiclassical solutions for the problem.

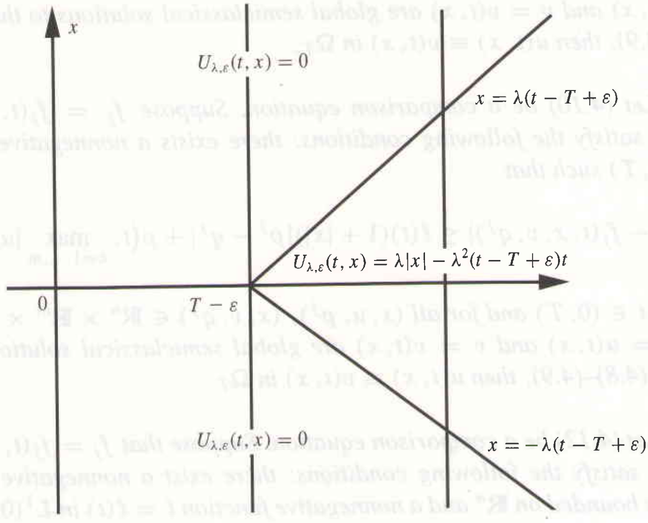

Fig. 1.

Definition. *A vector function $u = u(t,x)$ in $V^m(\Omega_T)$ is called a global semiclassical solution of (4.8)–(4.9) if it satisfies (4.8) for all $x \in \mathbb{R}^n$ and almost all $t \in (0,T)$ and if $u(0,x) = \phi(x)$ for all $x \in \mathbb{R}^n$.*

We can now formulate some stability results for global semiclassical solutions of the problem (4.8)–(4.9) and leave the proofs to the reader.

Theorem 5.6. *Suppose $f_j = f_j(t,x,u,p^j)$ $(j = 1,\ldots,m)$ satisfy the conditions as follows: there exist a nonnegative function $\mu = \mu(x)$ locally bounded on $\mathbb{R}^n$ and a nonnegative function $\ell = \ell(t)$ in $L^1(0,T)$ such that*

$$|f_j(t,x,u,p^j) - f_j(t,x,v,q^j)| \le \ell(t)\cdot[(1+|x|)|p^j - q^j| + \mu(x)\max_{k=1,\ldots,m}|u_k - v_k|] \tag{5.8}$$

for almost every $t \in (0,T)$ and for all $(x,u,p^j),\ (x,v,q^j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ $(j = 1,\ldots,m)$. Let $u = u(t,x)$, $\hat{u} = \hat{u}(t,x)$ be global semiclassical solutions to (4.8) with the following corresponding initial conditions:

$$u(0,x) = \phi(x),\ \hat{u}(0,x) = \hat{\phi}(x) \quad on \quad \{t = 0,\ x \in \mathbb{R}^n\},$$

where $\phi = \phi(x)$, $\hat{\phi} = \hat{\phi}(x)$ are given vector functions continuous on $\mathbb{R}^n$. Then

$$\max_{j=1,\ldots,m}|u_j(t,x) - \hat{u}_j(t,x)| \le$$

$$\exp\left[C(x)\int_0^t \ell(\tau)d\tau\right] \sup_{|y|\le(1+|x|)\exp\int_0^t \ell(\tau)d\tau - 1} \max_{j=1,\ldots,m}|\phi_j(y) - \hat{\phi}_j(y)|,$$

$C(x)$ being defined in (2.8).

Corollary 5.7. *Suppose $f_j = f_j(t, x, u, p^j)$ $(j = 1, \ldots, m)$ satisfy the conditions (5.8). If $u = u(t, x)$ and $v = v(t, x)$ are global semiclassical solutions to the Cauchy problem (4.8)–(4.9), then $u(t, x) \equiv v(t, x)$ in Ω_T.*

Theorem 5.8. *Let (4.10) be a comparison equation. Suppose $f_j = f_j(t, x, u, p^j)$ $(j = 1, \ldots, m)$ satisfy the following conditions: there exists a nonnegative function $\ell = \ell(t)$ in $L^1(0, T)$ such that*

$$|f_j(t, x, u, p^j) - f_j(t, x, v, q^j)| \le \ell(t)(1 + |x|)|p^j - q^j| + \rho\big(t, \max_{k=1,\ldots,m} |u_k - v_k|\big)$$

for almost every $t \in (0, T)$ and for all $(x, u, p^j), (x, v, q^j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ $(j = 1, \ldots, m)$. If $u = u(t, x)$ and $v = v(t, x)$ are global semiclassical solutions to the Cauchy problem (4.8)–(4.9), then $u(t, x) \equiv v(t, x)$ in Ω_T.

Theorem 5.9. *Let (4.12) be a comparison equation. Suppose that $f_j = f_j(t, x, u, p^j)$ $(j = 1, \ldots, m)$ satisfy the following conditions: there exist a nonnegative function $\mu = \mu(x)$ locally bounded on $\mathbb{R}^n$ and a nonnegative function $\ell = \ell(t)$ in $L^1(0, T)$ such that*

$$|f_j(t, x, u, p^j) - f_j(t, x, v, q^j)| \le \ell(t)\Big[(1 + |x|) \cdot |p^j - q^j| + \mu(x)\sigma\big(\max_{k=1,\ldots,m} |u_k - v_k|\big)\Big]$$

for almost every $t \in (0, T)$ and for all $(x, u, p^j), (x, v, q^j) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$ $(j = 1, \ldots, m)$. If $u = u(t, x)$ and $v = v(t, x)$ are global semiclassical solutions to the Cauchy problem (4.8)–(4.9), then $u(t, x) \equiv v(t, x)$ in Ω_T.

6. Concluding Remarks

The global existence and uniqueness of generalized solutions for convex Hamilton–Jacobi equations were well studied by several methods: variational method [10], method of envelopes [1], vanishing viscosity method [15, 19, ...], etc. The global theory for non-convex Hamilton-Jacobi equations has recently been considered by Crandall, Evans, Lions, and Ishii [11, 12, 17, ...], etc. They have introduced the notion "viscosity solutions" to define generalized solutions and characterized their properties. By these contributions, the global existence and uniqueness of generalized solutions have been established almost completely. However, it should be noted that viscosity solutions of partial differential equations are, as regular as possible, in general just continuous. They may therefore contain singularities. So what kinds of phenomena would appear when we extend the classical (local) solutions? In such a procedure, we must go back (for this, see [26]) to the Haar lemma. Of course, furthermore, the *a priori* estimates from the lemma (or something like it) are of much interest from various points of view.

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