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Compactification Methods in a Control Problem of Jump Processes under Partial Observations

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Abstract. In this paper, we are concerned with a control problem of the transition intensity of a Markovian jump process (X_t) with values in a finite set when the observable process is a diffusion. By enlarging the director space, we prove that there exists optimal controls. The separated control problem is also considered and it is shown that there exists an optimal Markovian filter.

1. Introduction

In this paper, we deal with the control problem of a jump Markovian process (X_t) with values in a finite set $I = \{1, 2, \dots, m\}$. We control the intensity of jump times and jump amplitudes of this process by minimizing an objective function of the form (1.1) when the observation (Y_t) satisfies the equation

$$dY_t = h(t, X_t) + dW_t\ ; \quad Y_0 = 0,\ 0 \le t \le T,$$

where W_t is a Brownian motion and h is a certain function.

More precisely, let A be a compact metric space and $\lambda(t, i, j, a)$ a non-negative function defined on $[0, T] \times I \times I \times A$. The phrase "control of jump times and jump amplitudes" means that if $N^i(t),\ i \in I$ is the number of entrances into the state i during the interval $[0, t]$, then the admissible control is a stochastic process, with values in A, $\mathcal{F}_t^Y$ – adapted, namely (u_t), such that

$$N^i(t) - \int_s^t \lambda(h, X_h, i, u_h)\, dh\ ,\ \ i = 1, 2, \dots, m$$

are martingales after s and the distribution of X_s is p, where $\mathcal{F}_t^Y = \sigma\,(Y_h,\ h \le t)$ and $p = (p_1, p_2, \dots, p_m)\ ;\ p_i \ge 0, \sum p_i = 1$. Our aim is to find an optimal control (u_t^*) which minimizes a cost function having the form

$$J(s, p, u) = E\Big[\int_0^T c(t, X_t, u_t)\, dt + g(X_T)\Big], \tag{1.1}$$

i.e.,

$$J(s, p, u^*) = J^*(s, p) = \inf\{J(s, p, u) : u \text{ is admissible control}\}.$$

This problem has been formulated by many authors (see [1, 6] for example). The controls so defined are called strictly admissible and it seems to be difficult to prove the existence of an optimal control among strictly admissible classes except for some special cases. In [1], Bismut has supposed that the dimension of the observation process (Y_t) is rather large to ensure that the filter equation is not degenerated. Without this hypothesis, the existence problem so far is still open because the coefficients of the filter equation do not satisfy convex conditions and we are unable to use the so-called Kushner technique with the Skorohod topology to show the existence of optimal controls.

Therefore, to overcome this difficulty, in this article we follow the idea of Ekeland in deterministic controls, so-called "compactification methods", which is applied in stochastic controls by many authors (see [9, 10, 12] for example) to study this problem. This means that we introduce a larger class of controls, namely, relaxed controls, to compactify the class of admissible controls by means of not increasing the value function. Based on the compacity of the set of relaxed controls it is easy to solve our problem.

In comparing with the results in [9, 10], by virtue of the finiteness of the state space of the signal process, the problem that we deal with here is not subject to the uniqueness of the filter equation, thus we can keep the third space as in Definition 2. This means that our relation is smaller than in [9, 10]. So far we are still unable to extend this result to the general case.

The article is organized as follows: in Sec. 1, we formulate the classical problem with the notion of admissible controls. Section 2 states the relaxed control problem in terms of martingale problems. Instead of considering the admissible class of controls we introduce the laws of control processes with values in a space of measures. The admissible controls can be regarded as a subset of relaxed controls. It is proved that this extension does not change the value function. We also establish the measurability and continuity of the solution of martingale problems in the initial conditions and controls. The advantage of this method is that under relaxed actions, the set of controls is convex compact. Therefore, by using the techniques of passing to limits, it is easy to show the existence of an optimal control. In Section 3, we are concerned with the separated problem. By virtue of the finiteness of the state space of the signal process, we do not deal with the uniqueness of solutions of filter equations. Thus, it is possible to consider this equation on the initial space formulated in Sec. 2. After that, by using the same method as for the control problem of degenerate diffusions to be studied in [10], we can show the existence of an optimal Markovian filter.

Although our problem is formulated for signal process with values in a finite set, it is easy to transfer it to the case of denumerable states.

2. Statement of the Problem

2.1. Let us formulate the problem on the canonical space. Suppose I is a finite set, say, $I = \{1, 2, \dots, m\}$. Let X be the set of all càdlàg functions from $[0, T]$ into I. We denote by (X_t) the coordinate process defined on X by $X_t(x) = x_t$ for any $x \in X$, $t \in [0, T]$.

Because I is a finite set, then the Skorohod topology can be introduced into X in the following way. The sequence $\{\alpha_n\}_{n\geq 1}$ in X converges to an element α in X if and only if there exists a sequence of time changes $\{\lambda_n\}$ where $\lambda_n : [0, T] \to [0, T]$ are continuous, strictly increasing, $\lambda_n(0) = 0$, $\lambda_n(T) = T$ such that
(a)

$$\lim_{n\to\infty} \sup_{t\neq s} \left\{ \log \frac{\lambda_n(t) - \lambda_n(s)}{t - s} \right\} = 0, \tag{2.1}$$

(b) There exists an $M > 0$ such that, for any $n > M$, we have $\alpha_n \leq \lambda_n \equiv \alpha$ (see [2, p. 171]). Moreover, let $\tau_k(x)$ be the kth-jump time of the element $x \in X$, i.e.,

$$\tau_0(x) = 0;\ \tau_{k+1}(x) = \inf\{t > \tau_k(x) : x_{t-} \neq x_t\},$$

where $x_{t-} = \lim_{h\to 0+} x_{t-h}$ and suppose $N(x)$ is the number of sums of x on $[0,\ T]$. Then it is easy to see that the sequence $\{\alpha_n\}_{n\geq 1}$ converges to α if and only if there exists an $M > 0$ such that, for any $n > M$, we have

$$N(\alpha_n) = N(\alpha);\ \lim_{n\to\infty} \tau_k(\alpha_n) = \tau_k(\alpha) \tag{2.2}$$

for any $k = 1,\ 2, \ldots,\ N(\alpha)$.

The space X endowed with this topology is separable and complete. Let $\mathcal{X}$ be the Borel-field on X and let $\mathcal{X}_t = \sigma(X_s : s \leq t)$ be the filtration generated by the process (X_t). It is well known that $\mathcal{X}_t$ is right-continuous.

Suppose A is a metric compact space, called the space of actions. As in the deterministic case, we give a convex formulation of the problem by introducing "randomized" controls with values in the space of generalized actions. We denote by V the set of measures on $[0,\ T] \times A$ having the form $dt.q(t, da)$ where dt is the Lebesgue measure on $[0,\ T]$ and $q(t, da)$ is a probability on A for any $t \in [0,\ T]$. Because A is compact then V is a weakly compact set. We endow V with a family of topologies of weak convergence (V_t) where V_t is checked on the functions which are continuous in $(t,\ a)$ with the support on $[0,\ t] \times A$. It is obvious that (V_t) is an increasing sequence. We denote by $\mathcal{V}_t$ the Borel field generated by the open sets of V_t.

Let $\lambda : [0,\ T] \times I \times I \times A \to R^1$ be a non-negative continuous function such that $\lambda(t, i, i, a) = 0$ for any $(t, i, a) \in [0,\ T] \times I \times A$. The function λ plays the role of the intensity of jump times of (X_t) which depends on a control action $a \in A$. To simplify notations, let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be two measurable spaces and $f : E \times F \to R$ a measurable function, then we set

$$\int f(x, y)\, Q(x, dy) \triangleq f(x, Q_x) \triangleq f(x, Q) \triangleq Q_x(f) \tag{2.3}$$

for any probability kernel $Q(x, dy)$ from E into F.

For any $q = dt\, q(t, da) \in V$ and for any $f : I \to R$, we set

$$(A_t^q f)_i = \sum_{j\in I} \int_A \lambda(t, i, j, a)\big(f(j) - f(i)\big)\, q(t, da) \tag{2.4}$$

which is called the generator of (X_t) associated with the action q.

Definition 1. *A probability measure Q on $(X, \mathcal{X})$ is said to be a solution of the martingale problem associated with (s, p, q) where $s < T$, $q \in V$ and p is a probability on I (in brief: M. P. associated with (s, p, q)), if*

$$C_t^f = f(X_t) - f(X_s) - \int_s^t (A_u^q f)_{X_u}\, du \tag{2.5}$$

is a $(\mathcal{X}_t, Q)$-martingale such that

$$Q(X_u = i : u \leq s) = p_i \quad \text{for any } i \in I, \tag{2.6}$$

where $p = (p_1,\ p_2, \ldots,\ p_m)$.

For any $i \in I$, let $N^i(t)$ be the number of exits from i of the process (X_t) on the interval $[0,\ t]$, i.e.,

$$N^i(t) = \sum_{s<t} 1_{\{X_{s-}=i\}} \cdot 1_{\{X_s \neq i\}}, \tag{2.7}$$

where 1_B denotes the indicator function of the set B. The following result is well-known and the proof can be found in [1] or [4].

Lemma 1. *If $S^q_{(s,p)}$ is the solution of M. P. associated with (s, p, q), then for any $i \in I$,*

$$M^i(t) = N^i(t) - N^i(s) - \int_A \int_s^t \lambda(u, X_u, i, a)\, q(u, da)\, du \tag{2.8}$$

is a martingale under $S^q_{(s,p)}$.

The following lemmas establish the continuity and measurability of $S^q_{(s,p)}$ in (s, p, q) so that it allows us to use the technique of passing to limits or of enlarging spaces.

Lemma 2.
(1) *For any (s, p, q), there is a unique solution of M. P. associated with (s, p, q).*
(2) *The map $(s, p, q) \mapsto S^q_{(s,p)}$ is continuous.*

Proof. The first part follows from [1, Theorem 1.1] and the second is deduced from [14, Theorem 3.7, p. 389]. ■

Lemma 3. *If $B \in \mathcal{X}_t$, then the map $q \mapsto S^q_{(s,p)}$ is $\mathcal{V}_t$ measurable.*

Proof. From Lemma 2, it follows that if we formulate the martingale problem on $[0, t]$, then the map $q \mapsto S^q_{(s,p)}$ is $\mathcal{V}_t$-continuous. Hence, by approximation methods and the construction of σ-field $\mathcal{X}_t$, the result follows. ■

We now proceed to describe observations.

Let $(\Omega, \mathcal{Y}_t, P)$ be a probability space with filtration called the director space satisfying the usual conditions, and let (Y_t) be Brownian motion defined on $(\Omega, \mathcal{Y}_t, P)$ such that $\mathcal{Y}_t = \sigma(Y_s, s \leq t)$ for any $t \in T$.

We denote by $P_s(\cdot)$ the probability on $(\Omega, \mathcal{Y}_T)$ such that $Y_t - Y_s$ is a Brownian after s (i.e., $P_s(Y_t - Y_s = 0, t \leq s) = 1$; see [16]).

Let

$$\mathcal{F}_t = \bigcap_{u>t} \mathcal{Y}_u \otimes \mathcal{V}_u.$$

Definition 2. *A partially observable control rule (in brief, a control rule) with initial condition (s, p) is a term $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{R})$ where*

(1)

$$\overline{\Omega} = \Omega \times V \times X, \quad \overline{\mathcal{F}}_t = \bigcap_{u>t} \mathcal{Y}_u \otimes \mathcal{V}_u \otimes \mathcal{X}_u.$$

(2) $\overline{R} = P_s(d\omega).Q(\omega, dq).S^q_{(s,p)}(dx)$ such that $(\Omega \times V, \mathcal{F}_t, P_s.Q(\omega, dq))$ is a natural extension of $(\Omega, \mathcal{F}_t, P_s)$.

Remark.

(1) By Lemma 3, it is easy to see that $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{R})$ is a natural extension of $(\Omega, \mathcal{Y}_t, P_s)$. Therefore, (Y_t) is a Brownian motion on $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{R})$.

(2) For any $f : I \to R$, the process (C^f_t) defined by (2.5) is still a discontinuous martingale on $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{R})$ satisfying the relation

$$\langle C^f_t, Y_t \rangle \equiv 0 \quad \text{for any } t \in [0, T], \tag{2.10}$$

where $\langle ., . \rangle$ denotes the Meyer's process of two martingales.

In this definition, we do not mention processes because we consider the signal process (X_t) and the Brownian motion (Y_t) to be fixed. They are defined on $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{R})$ by the natural prolongation. In [9], a similar definition is called the "white control rule". To obtain observations, we use the so-called Girsanov transformation which leads us to weak solutions of the equation

$$dY_t = h(t, X_t)\,dt + d\,(\text{Brownian}).$$

2.2. Girsanov Transformation

Set

$$Z_t = \exp\left\{ \int_s^t h(u, X_u)\,dY_u - \frac{1}{2}\int_s^t h^2(u, X_u)\,du \right\}. \tag{2.11}$$

It is known that Z_t is a positive martingale satisfying the equation

$$dZ_t = Z_t.h(t, X_t)\,dY_t, \quad Z_t = 1 \text{ for } t \leq s. \tag{2.12}$$

Hence, we can define a new probability $\overline{R}^h$ on $(\overline{\Omega}, \overline{\mathcal{F}}_T)$ by

$$\overline{R}^h = Z_T.\overline{R} \tag{2.13}$$

which allows us to obtain the following classical result.

Proposition 1. *Under the probability* $\overline{R}^h$

(a) $W_t = Y_t - Y_s - \int_s^t h(u, X_u)\,du$ *is a Brownian motion.*

(b) (X_t) *has random intensity* $\lambda(t, i, j, q)$. *This means that*

$$N^i(t) - N^i(s) - \int_s^t \lambda(u, X_u, i, q)\,du$$

is a $\overline{R}^h$ *martingale.*

Proof. (a) is obvious. To obtain (b), we note that $\langle C_t^f, Y_t \rangle = 0$. Therefore, (C_t^f) is a $\overline{R}^h$-martingale and the result follows from Lemma 1. ■

2.3. The Cost of Control Rules

We denote by $\mathcal{R}(s, p)$ the set of control rules with initial condition (s, p). For each $\overline{R} \in \mathcal{R}(s, p)$, we consider the cost

$$J(s, p, \overline{R}) = \overline{R}^h \Big[\int_s^T c(t, X_t, q_t)\,dt + g(X_T) \Big], \tag{2.14}$$

where c and g are two real continuous functions and $Q[.]$ denotes the expectation under the probability Q.

Our aim is to minimize the cost function $J(s, p, .)$ over $\mathcal{R}(s, p)$, that is, to find an optimal control rule $R^* \in \mathcal{R}(s, p)$ such that

$$J(s, p, R^*) = J(s, p) = \inf\{J(s, p, R) : R \in \mathcal{R}(s, p)\}. \tag{2.15}$$

2.4. Existence of Optimal Control Rules

The compactification formulation mentioned above allows us to easily obtain optimal control rules. Let us recall that (see [15]) the sequence of control rules $\{\overline{R}_n\} \subset \mathcal{R}(s, p)$ converges in stable topology to $\overline{R}$ if and only if, for any $f \in M_{mc}^b(\Omega; V \times X)$ (i.e., $f(\omega, q, x)$ is measurable in ω, continuous in (q, x) and bounded), the sequence $\overline{R}_n(f) = \int_{\Omega \times V \times X} f(\omega, q, x)\,d\overline{R}_n$ converges to $\overline{R}(f) = \int f\,d\overline{R}$ as $n \to \infty$. By virtue of Lemma 2, $S_{(s,p)}^q$ is continuous in q. Hence, if $\overline{R}_n = R_n.S_{(s,p)}^q$, then the convergence of the sequence $\{\overline{R}_n\}$ to $\overline{R}$ is equivalent to the convergence of the sequence $\{R_n\}$ to R in stable topology on $\Omega \times V$. Moreover,

Proposition 2. *If* $\{\overline{R}_n\}$ *is a sequence of control rules convergent to* $\overline{R}$ *in stable topology, then the relative sequence* $\{\overline{R}_n^h\}$ *converges to* $\overline{R}^h$ *as* $n \to \infty$.

Proof. It suffices to show that $Z_T(\omega, x)$ is continuous in x for almost sure ω. Suppose that $\tau_k(x)$ and $N(x)$ are defined as in Subsec. 2.1. for all $x \in X$. It is obvious that $x \mapsto \int_s^T h(u, X_u(x))\, du$ is continuous in x. On the other hand, we have

$$\int_s^T h(u, X_u)\, dy_u = \sum_{i \in I} \int_s^T h(u, i) X^i(u)\, dY_u$$
$$= \sum_{i \in I} \sum_{k=1}^{N(x)} X^i \left(\frac{\tau_k(x) + \tau_{k-1}(x)}{2} \right) \int_{\tau_{k-1}(x)}^{\tau_k(x)} h(u, i)\, dY_u.$$

From Subsec. 2.1., it follows that the maps $x \mapsto X^i \left(\frac{\tau_k(x)+\tau_{k-1}(x)}{2} \right)$, $x \mapsto N(x)$, $x \mapsto \tau_k(x)$ are continuous. Hence

$$x \mapsto \int_s^T h(u, X_u)\, dY_u$$

is continuous. This implies that $Z_T(\omega, x)$ is continuous in x. ∎

Corollary 1. *There exists an optimal control rule which minimizes the cost function* $J(s, p, .)$.

Proof. Let $\{\overline{R}_n\}$ be a sequence of control rules such that $\lim_{n\to\infty} J(s, p, \overline{R}_n) = J(s, p)$. Suppose $\overline{R}_n = R_n.S^q_{(s,p)}$ for each n. Since V is compact, it follows that $\{R_n\}$ is relatively compact. Hence there is a subsequence $\{R_{n_k}\}$ convergent to an R^*. It is clear that $\overline{R}^* = R^*.S^q_{(s,p)}$ is a control rule. Therefore, the subsequence $\{\overline{R}_{n_k}\}$ converges to $\overline{R}^*$ in stable topology. By Proposition 2, it follows that the sequence $\{\overline{R}^h_{n_k}\}$ converges to $\overline{R}^{*h}$ as $n_k \to \infty$. Since $\int_0^T c(t, X_t, q)\, dt + g(X_T)$ is continuous in (q, x), we conclude that $\lim J(s, p, \overline{R}_{n_k}) = J(s, p, \overline{R}^*) = J(s, p)$. This means that $\overline{R}^*$ is an optimal control rule. The proof is complete. ∎

2.5. Comparison Between the Initial Problem and the Relaxed Control Problem

We have easily proved the existence of an optimal control for the relaxed problem. We now come back to the initial problem which is posed in Sec. 1 in order to compare the value functions of initial and relaxed problems.

If (u_t) is a strictly admissible control, then the map

$$\omega \mapsto (\omega, \delta_{u_t} dt)$$

induces a probability measure R on the space $\Omega \times V$ whose projection on Ω is P. It is easy to see that in this case

$$\overline{R} = R(d\omega, dq) \cdot S^q_{(s,p)}(dx)$$

is a partially observable control rule. This implies that

$$J(s,p) \leq J^*(s,p)$$

for any (s, p). The inverse relation is proved by a similar argument in [10, 13] with the aid of the so-called "chattering lemma" which says that every control rule can be approximated by a sequence of laws of admissible controls.

Thus to solve the initial problem, we have to relax the class of admissible controls. But this extension is the smallest in the sense that the initial value function and the relaxed one are equal.

3. Equation of the Dynamic Programming Principle

We follow the method dealt with in [11] to obtain the dynamic programming principle by using the stability of control rules by conditioning and concatenation.

Let $\overline{R} \in \mathcal{R}(s, p)$ be a control rule and τ a $\mathcal{F}_t$-stopping time bounded by T. We denote by $\mathcal{R}(\tau, \overline{R})$ the subset of $\mathcal{R}(s, p)$ whose restrictions on $\mathcal{F}_\tau$ coincide with the restriction of $\overline{R}$ on $\mathcal{F}_\tau$.

Stability by bifurcation:

Let $B \in \mathcal{F}_\tau$, $\overline{R}_2 \in \mathcal{R}(\tau, \overline{R}_1)$. We define a control rule $\overline{R}_3$ by

$$\overline{R}_3(C) = \overline{R}_2(B \cap C) + \overline{R}_1(B^c \cap C) \tag{3.1}$$

for any $C \in \mathcal{F}_T$, where B^c denotes the complement of the set B.

Proposition 3. *$\overline{R}_3(.)$ defined by (3.1) is a control rule.*

Proof. It is obvious that $\overline{R}_3(d\omega, dq, dx) = P_s(d\omega).Q^3(\omega, dq).S^q_{(s,p)}(dx)$, where $Q^3(\omega, dq)$ is a probability kernel from Ω into V. Thus it remains to show that $(\Omega \times V, \mathcal{F}_t, P_s.Q^3)$ is a natural extension of $(\Omega, \mathcal{Y}_t, P_s)$. This means that we have to prove that, for any $\Gamma \in \mathcal{V}_t$, the map $\omega \mapsto Q^3(\omega, \Gamma)$ is $\mathcal{Y}_t$-measurable (see [3]).

Suppose $R_k = P_s(d\omega).Q^k(\omega, dq).S^q_{(s,p)}(dx)$ for $k = 1, 2$. First, we show that

$$Q^1_\omega(1_D) = Q^2_\omega(1_D) \text{ for any } D \in \mathcal{F}_\tau,\ P_s - a.s. \tag{3.2}$$

(see notations (2.3)). We set

$$\varphi_t(\omega) = \int_V 1_{D\cap\{\tau \leq t\}}\left[Q^1(\omega, dq) - Q^2(\omega, dq)\right].$$

Then, since $\overline{R}_1(.)$ and $\overline{R}_2(.)$ are two natural extensions of $(\Omega, \mathcal{Y}_t, P_s)$, it is easy to see that (φ_t) is a martingale. Moreover, since the filtration (Y_t) is continuous, then it follows that the martingale (φ_t) has the continuous trajectory. On the other hand, (φ_t), of course, is a process with bounded variation. Then it follows that $\varphi_t \equiv 0$. This means that (3.2) holds.

By this result, we can prove that the map $\omega \mapsto Q^3(\omega, \Gamma)$ is $\mathcal{Y}_t$-measurable. It is obvious that

$$Q^3(\omega, \Gamma) = Q^2(\omega, \Gamma) + \int_V 1_{B\cap\Gamma}[Q^2(\omega, dq) - Q^1(\omega, dq)].$$

Putting $D = B \cap \Gamma$, we obtain

$$\int 1_D[Q^1(\omega, dq) - Q^2(\omega, dq)] = \int 1_{D\cap\{\tau\le t\}}[Q^2(\omega, dq) - Q^1(\omega, dq)] + \int 1_{D\cap\{\tau>t\}}[Q^2(\omega, dq) - Q^1(\omega, dq)].$$

Since $D \cap \{\tau > t\} \in \mathcal{F}_\tau$, then the second term of the right-hand side is zero. Moreover, $B \cap \Gamma \cap \{\tau \le t\} \in \mathcal{F}_t$, therefore, the map $\omega \mapsto \int 1_{D\cap\{\tau\le t\}}[Q^2(\omega, dq) - Q^1(\omega, dq)]$ is $\mathcal{Y}_t$-measurable. Therefore, the $\mathcal{Y}_t$-measurability of $\omega \mapsto Q^3(\omega, \Gamma)$ follows. Proposition 3 is proved.

Definition 3. *Let τ be a $\mathcal{F}_t$-stopping time bounded by T and $\overline{R} \in \mathcal{R}(s, p)$ a control rule.*

(a) *We call*

$$\Gamma(\tau, \overline{R}) = \overline{R}^h[\Phi_s^T \mid \mathcal{F}_\tau]$$

the conditional cost of the control rule $\overline{R}$ given by $\mathcal{F}_\tau$.

(b) *The term*

$$J(\tau, P, \overline{R}) = \inf\{\Gamma(\tau, R) : R \in \overline{R}(\tau, P)\}$$

is called the optimal conditional cost of $\overline{R}$.

(c) *A control rule $\overline{R}^*$ is said to be $(\tau, \overline{R})$-conditionally optimal if $\overline{R}^* \in \mathcal{R}(\tau, \overline{R})$ and*

$$G(\tau, \overline{R}^*) = J(\tau, \overline{R}). \tag{3.3}$$

Proposition 4. (Principle of optimality) *If $\tau \le \rho \le T$ are two stopping times and $\overline{R} \in \mathcal{R}(s, p)$, then we have*

$$\overline{R}^h[J(\rho, \overline{R}) / \mathcal{F}_\tau] \ge J(\tau, \overline{R}). \tag{3.4}$$

The equality takes place in (3.4) for any $\tau \le \rho \le T$ if and only if $\overline{R}$ is an optimal control rule.

Proof. The proof of this proposition can be found in [11] by using Proposition 3. ∎

4. Separated Problem

We now proceed to study the separated problem, that is, to consider the controlled filter equation of the signal process (X_t) with respect to the observation (Y_t). The techniques that we are going to use here are similar to those in [9] to obtain an optimal Markovian filter. However, we can consider the filter equation on the space $(\overline{\Omega}, \overline{\mathcal{F}}_t, \overline{R})$ stated in the above-mentioned problem without transferring to the canonical space. It is possible to do that because the signal process in this case takes only a finite number of values and this process is completely described by the set of indicator functions: $\{X^i(t) = 1_{\{X_t=i\}},\ i \in I\}$. Thanks to this property, we do not need to deal with the unique solution problem of the filter equation.

4.1. Filtration Equation

Let (Z_t) be defined by (2.11) which satisfies the equation

$$\begin{cases} dZ_t = h(t, X_t).Z_t.dY_t, \\ Z_t \quad = 1 \ \text{ for any } t \le s. \end{cases} \tag{4.1}$$

It is known that there exists a universal version of (Z_t) such that (4.1) is fulfilled $\overline{R}-a.s.$ for any $\overline{R} \in \mathcal{R}(s, p)$ (this means Doleans–Dade's solution). Based on this solution, we can establish the filter equation for $\overline{R}^h[X^i(t)/\mathcal{F}_t]$.

Let $R \in \mathcal{R}(s, p)$ be fixed. By Definition 1, the process

$$C_t^f = f(X_t) - f(X_s) - \int_s^t (A_u^q f)_{X_u}\, du \tag{4.2}$$

is an $\overline{R}$-martingale which satisfies the relation

$$\langle C_t^f, Y_t \rangle = 0. \tag{4.3}$$

Hence, by using Ito's formula, it yields

$$Z_t f(X_t) = Z_s f(X_s) + \int_s^t Z_u (A_u^q f)_{X_u}\, du + \int_s^t Z_u f(X_u) h(u, X_u)\, du + \int_s^t Z_u\, dC_u^f. \tag{4.4}$$

Since (C_t^f) is a martingale with respect to $S_{(s,p)}^q$, we have

$$\int_X \Big[\int_s^t Z_u(\omega, x)\, dC_u^f \Big] S_{(s,p)}^q(dx) = 0. \tag{4.5}$$

On the other hand, using Lemma 3 (in Sec. 2), it follows that the map

$$(\omega, q) \mapsto \int_X Z_u\, f(X_u)\, h(u, X_u)\, S_{(s,p)}^q\, (dx)$$

is $\mathcal{F}_u$-measurable. Therefore,

$$\int_X \Big[\int_s^t Z_u\, f(X_u)\, h(u, X_u)\, dY_u \Big] S^q_{(s,p)}(dx)$$

$$= \int_s^t dY_u \Big[\int_X Z_u\, f(X_u)\, h(u, X_u)\, S^q_{(s,p)}(dx) \Big]. \tag{4.6}$$

Taking the integral of measure $S^q_{(s,p)}$ of both sides of (4.1) and noting (4.5) and (4.6), we get

$$\begin{aligned} \int_X Z_t\, f(X_t)\, S^q_{(s,p)}(dx) &= \int_X f(X_s)\, Z_s\, S^q_{(s,p)}(dx) \\ &+ \int_s^t dY_u \int_X Z_u\, f(X_u)\, h(u, X_u)\, S^q_{(s,p)} + \int_s^t du \int_X Z_u\, (A^q_u f)\, S^q_{(s,p)}(dx). \end{aligned} \tag{4.7}$$

Setting $f = \delta_i$, $i \in I$ and letting

$$\Pi^i(t, \omega, q) = \int_X X^i(t)\, Z_t(\omega, x)\, S^q_{(s,p)}(dx), \tag{4.8}$$

we obtain the equation

$$\begin{aligned} d\Pi^i(t) &= \Big[\sum_j \lambda(t, j, i, q)\, \Pi^j(t) - \alpha^i(t, q)\, \Pi^i(t) \Big] dt + h(t, i)\, \Pi^i(t)\, dY_t, \\ \Pi^i(t) &= p_i \ \text{ for } t \le s,\ i \in I, \end{aligned} \tag{4.9}$$

where $p = (p_1, p_2, \ldots, p_m)$.

Remark. The filter equation (4.9) has been obtained by many authors in different ways.

4.2. Cost of Controls in the Separated Problem

Let $\overline{R} = R.S^q_{(s,p)} \in \mathcal{R}(s, p)$, then it is easy to check that

$$\begin{aligned} J(s, p, \overline{R}) &= \overline{R}^h \Big[\int_s^T c(t, X_t, q_t)\, dt + g(X_T) \Big] \\ &= \overline{R} \Big[\int_s^T \sum_i c(t, i, q)\, \Pi^i\, dt + \sum_i g(i)\, \Pi^i(T) \Big] \\ &= R \Big[\int_s^T \sum_i c(t, i, q)\, \Pi^i\, dt + \sum_i g(i)\, \Pi^i(T) \Big] \\ &\triangleq \ell(s, p, R). \end{aligned}$$

Denote $\Pi(t) = (\Pi^1, \Pi^2, \ldots, \Pi^m)$, $C(t,a) = (c(t,1,a), \ldots, c(t,m,a))$ and $g = (g(1), g(2), \ldots, g(m))$. We are then able to formulate the separated problem as follows:

Let $\Omega = C([0, T], R)$; $\{Y_t - Y_s\}$ be a Brownian motion after s is defined on $(\Omega, \mathcal{Y}_t, P_s)$ such that $\mathcal{Y}_t = \sigma(Y_u : u \leq t)$.

Definition 4. *The separated problem is a term to describe*

(a) *A family of probability measures $\mathcal{R}(s)$ on $(\Omega \times V, \mathcal{F}_t)$ such that for any $R \in \mathcal{R}(s)$, $(\Omega \times V, \mathcal{F}_t, R)$ is a natural extension of $(\omega, \mathcal{Y}_t, P_s)$. Each element of $\mathcal{R}(s)$ is called a rule.*

(b) *For every $R \in \mathcal{R}(s)$, we consider the (unique) solution $\Pi(t, w, q, s, p)$ of (4.9) starting from p at $t = s$. Let us associate the rule R and the solution $\Pi(t)$ with a cost*

$$\ell(s, p, R) = R\left[\int_s^T \langle c(t, q)\,,\, \Pi(t)\rangle\, dt + \langle g\,,\, \Pi(T)\rangle\right] \tag{4.11}$$

where $\langle ., . \rangle$ denotes the scalar product on R^m.

Our aim is to minimize the cost function (4.11) over the set of rules $\mathcal{R}(s)$. The following lemma plays an important role in investigating the problem.

Lemma 4. *There exists a universal version of solution of (4.9) such that for any fixed (t, ω), the map $(q, p) \mapsto \Pi(t, \omega, q, p)$ is continuous.*

Proof. Suppose $\Pi(t, \omega, q, p)$ is defined as in (4.8) where Z_t is a universal version of the solution of (2.12). By Proposition 2 in Sec. 2, $Z_t(w, x)$ is continuous in x. Furthermore, if

$$D = \bigcup_{i \in I} \{x \in X : X^i(t) \text{ is discontinuous at } x\}$$

$$= \{x \in X : X_t \neq X_{t-}\} = \{x \in X : \sum_i dN^i(t) \neq 0\},$$

then it is known that $S^q_{(s,p)}(D) = 0$ for any (s, p, q). This implies that $Z_t(\omega, x)\, X^i(t)$ is continuous in x except for a $S^q_{(s,p)}$-negligible set. Hence, it follows that $\int_X Z_t X^i(t) S^q_{(s,p)}(dx)$ is continuous in (p, q). ∎

Since V is a compact set, it is clear that the set $\mathcal{R}(s)$ is convex and compact. In [9], the authors have proved that the map $s \mapsto \mathcal{R}(s)$ is continuous. Hence this leads to Theorem 1.

Theorem 1.

(1) *There exists an optimal control rule for the separated problem.*

(2) *Let*

$$\mathcal{R}^*(s, p) = \{R \in \mathcal{R}(s) : \ell(s, p, R) = \ell(s, p)\},$$

where $\ell(s, p) = \inf\{\ell(s, p, R) : R \in \mathcal{R}(s)\}$. Then $\mathcal{R}^*(s, p)$ is convex, compact and the map $(s, p) \mapsto R^*(s, p)$ is upper-semi-continuous (in brief: U.S.C).

Proof. Since the set $\mathcal{R}(s)$ is convex and compact, and by Lemma 4, $q \mapsto \int_0^T \langle C(t,q)\,,\ \Pi(t)\rangle\, dt$ $+ \langle g\,,\ \Pi(T)\rangle$ is continuous, it follows that $\ell(s,p,R)$ is continuous in R in stable topology. Therefore, $\ell(s,p,.)$ reaches the infimum on $\mathcal{R}(s)$. This means that the set of optimal rules is not empty. We yield (1).

We note that ℓ is continuous in (s,p,R). On the other hand, the map $s \mapsto \mathcal{R}(s)$ is continuous in s. So, in order to obtain (2), we follow [9, Theorem 5.7] to show that $R^*(s,p)$ is U.S.C. The proof is complete. ■

We now turn to the existence of an optimal Markovian rule. We are using the same techniques as in [9, 10]. The only small difference is that, here, we work on the space $\Omega \times V$, not in the canonical space of filtration. From the uniqueness of solution (4.9) we have

$$\Pi(t,s,p) = \Pi(t,u,\Pi(u,s,p)) \quad \text{for any } s \le u \le t.$$

The set of rules $\mathcal{R}(s)$ is stable by a conditional operator. This means that for any stopping time $\tau \le T$ and for any $R \in \mathcal{R}(s)$, let $R_{\overline{\omega}}(.)$ be a regular conditional distribution of R given $\mathcal{F}_\tau$. Then there exists an R-negligible set N such that if $\omega \notin N$ we have $R_{\overline{\omega}}(.) \in \mathcal{R}(\tau(\overline{\omega}))$. On the other hand, $\mathcal{R}(s)$ is also stable by concatenation: suppose that $\overline{\omega} \mapsto R_{\overline{\omega}}(.)$ is a measurable map from $\Omega \times V$ such that $R_{\overline{\omega}}(.) \in \mathcal{R}(\tau(\overline{\omega}))$ for any $\overline{\omega}$. Then we have $R \otimes R_\omega(.) \in \mathcal{R}(s)$ for any $R \in \mathcal{R}(s)$.

By virtue of these properties, we can establish the dynamic programming principle in a similar way as developed in [9, 10].

Theorem 2. *Let $\tau \le T$ be a stopping time. Then*

$$\ell(s,p) = \inf\left\{R\left[\int_s^\tau \langle c(t,q)\,,\ \Pi(t)\rangle\, dt + \ell(\tau,\Pi(\tau))\right] : R \in \mathcal{R}(s)\right\}.$$

Proof. See [9, 10, Sec. 4, part II]. ■

Corollary 2. *The set $\mathcal{R}^*(s,p)$ of optimal rules is stable by concatenation and conditioning.*

4.3. Existence of an Optimal Markovian Filter

The equation of the dynamic programming principle allows us to select an optimal Markovian filter by following Krylov's ideas.

Theorem 3.

(1) *There exists a family of optimal rules $Q^*(s,p)$ defined on $(\Omega \times V,\ \mathcal{F}_T)$ whose restrictions on $\mathcal{F}_T$-generated by process $\{\Pi(t),\ t \in T\}$, consists of only one element, namely, $Q^*_{s,p}$ such that $(\Omega \times V,\ \Pi(t),\ Q^*_{s,p})$ is a strongly Markovian process.*

(2) *There exists a* $q^* : [0, T] \times R^m \to V$ *such that the Markovian process* $(\Omega \times V, \Pi(t), Q^*_{s,p})$ *has the generator*

$$L^{q^*} f(t, p) = \frac{\partial f}{\partial t} + \sum_{i \in I} \int_A \overline{\lambda}^i(t, p, a)\, q^*(t, p, da) \frac{\partial f}{\partial p_i} + \frac{1}{2} \sum_i h^2(t, i)\, p_i^2 \frac{\partial^2 f}{\partial {p_i}^2},$$

where

$$\overline{\lambda}^i(t, p, a) = \sum_{j \in I} \lambda(t, j, i, a)\, p_j - \alpha^i(t, a)\, p_i.$$

Proof. The proof is adapted to [14, p. 293] and is the same as in [10].

Let $\{t_k\}$ be a dense subset of $[0, T]$ and $\{\Phi_n\}$ a dense subset of $C_0[0, T] \times R^m$. Let $\{(t_m, \Phi_m)\}$ be an enumeration of $\{(t_k, \Phi_n) : n \geq 1,\ k \geq 1\}$. We define by induction the following control problems

$$\mathcal{R}_0^*(s, p) = \{R \in \mathcal{R}(s) : R \text{ optimal for the initial problem}\},$$
$$\mathcal{R}_{n+1}^*(s, p) = \{R \in \mathcal{R}_n^*(s, p) : R[\Phi_n(t_n, \Pi(t_n))] = U_n(s, p)\},$$
$$U_n(s, p) = \inf\left\{R[\Phi_n(t_n, \Pi(t_n))] : R \in \mathcal{R}_n^*(s, p)\right\}.$$

For each n, the set $\mathcal{R}_n^*(s, p)$ is non-empty, compact, convex and stable by conditioning and concatenation and the map $(s, p) \mapsto \mathcal{R}_n^*(s, p)$ is U.S.C. Therefore, their intersection is a compact non-empty set denoted by $Q^*(s, p)$, that is,

$$Q^*(s, p) = \bigcap_{n=1}^{\infty} \mathcal{R}_n^*(s, p).$$

It is easy to see that the restriction of $Q^*(s, p)$ on $\mathcal{F}_T^\Pi = \sigma(\Pi(t) : t \leq T)$ consists of only one element, namely, $Q^*_{s,p}$. The stability by conditioning implies the strongly Markovian property of the process $\Pi(t)$ and we get the first part.

In order to obtain the second one, we remark that under the probability $Q^*_{s,p}$, the process $\Pi(t)$ is a semi-martingale and a strong Markov process.

Therefore, according to Motoo's theorem (see [5]), $\Pi(t)$ admits a decomposition into a process with finite variation in the form $\lambda^*(t, \Pi(t))dt$ and a martingale additive functional whose increasing process is associated with the matrix $a^*(t, \Pi(t))dt$, i.e., the Markov process $(\Pi(t))$ has the generator

$$L^* f(t, p) = \frac{\partial f(s, p)}{\partial t} + \langle \lambda^*, \nabla_p f(s, p) \rangle + \nabla_p' a^*(t, p) \nabla_p f(s, p),$$

where ∇_p denotes the gradient operator in p.

This decomposition does not depend on the initial law. The uniqueness of the decomposition implies that

$$Q^*_{s,p}.dt \text{ a.s. } \lambda^*(t, \Pi(t)) = \overline{\lambda}(t, \Pi(t), q_t);$$
$$a^*(t, p) = \mathrm{diag}(h(t, 1)p_1, ..., h(t, m)p_m),$$

where $\overline{\lambda}^i(t,p,a) = \sum_{j\in I}\lambda(t,j,i,a)\,p_j - \alpha^i(t,a)\,p_i$ for $i = 1,2,...,m$ and $\overline{\lambda}(t,p,a) = (\overline{\lambda}^1(t,p,a),...,\overline{\lambda}^m(t,p,a))$.

Thus, if we denote by $v_{s,p}(\cdot)$ the potential kernel defined by

$$v_{s,p}(f) = Q^*_{s,p}\int_s^T \exp^{-(t-s)} f(t,\Pi(t))\,dt,$$

then $v_{s,p}$ a.s. we have $\lambda^*(t,p)$ belonging to $\overline{\mathrm{co}\overline{\lambda}(t,p,A)}$, the convex closed hull of $\overline{\lambda}(t,p,A)$. Let us denote $B = \{(t,p)$ such that $\lambda^*(t,p) \notin \overline{\mathrm{co}\overline{\lambda}(t,p,A)}\}$. B is a Borel subset negligible with respect to $v_{s,p}$ for each (s,p). The graph $B^c \cap \{(t,p,q); \overline{\lambda}(t,p,q) = \lambda^*(t,p)\}$ is Borel and has no empty sections.

From the selection theorem (see [7]) there exists a universally measurable mapping $q^*(t,p,da)$ from $[0,T]\times R^m_+$ in $\mathcal{P}(A)$ such that

$$\lambda^*(t,p) = \overline{\lambda}(t,p,q^*(t,p)) = \int_A \overline{\lambda}(t,p,a)q^*(t,p,da) \quad \forall(t,p)\notin B.$$

The proof is complete. ■

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