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A Maximal Volume Cone Algorithm for Linear Programming Problems

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Abstract. In this paper, we introduce an efficient algorithm for linear programming problems which is based on the concept of a cone of maximal volume. From each vertex of a polytope X we construct a cone of maximal volume. The cone can be found after not more than $m - n$ times of changing a consequent inequality.

1. Introduction

Since its appearance in 1947 (see [3]), the simplex method has become the main tool for solving linear programming problems. The popularity of this method is explained by various reasons, among which is its effectiveness and its simple understanding in both algebraic and geometric formulations. Only after Khachian's polynomial time ellipsoid method was published [8] did it become more clear that there may be algorithms which are better than the simplex method. This is significant, stimulating the research for new methods. Considerable progress has been made since 1984 when Karmarkar [7] suggested another polynomial time algorithm, which claimed to be fast in practice. Karmarkar's project algorithm solves a linear programming problem after $O(nL)$ steps with $O(n^3)$ operations per step. After the Karmarkar method was published, a number of authors such as Anstreicher and Bosch [1], De Ghellinck and Vial [4], Gay [5], Gonzaga [6], Renegar [9], Todd [10], Ye and Kojima [11] proposed a variant of Karmarkar's algorithm with a reduced complexity. Nevertheless, it is seemly for us that a variant of Karmarkar's algorithm has reached its limited complexity. On the other hand, we think that the systems of linear inequalities could give better properties for finding an optimal solution to the linear programming problem.

Assume that $n < m$ are positive integer numbers. Let X be a nonempty polyhedral compact set (polytope) of R^n given by a system of linear inequalities

$$(a_i, x) \leq b_i, \ i = 1, ..., m. \tag{1}$$

Consider the following linear programming problem (P):

$$\max\{cx \mid x \in X\}.$$

Here, c is a row vector in R^n.

In this paper we present a new method in solving problem (P). In Sec. 2 we introduce the conception of cones of maximal volume and the way to construct them as a key for solving problem (P). In fact, we describe the relation between the cone of maximal volume and the subsystem of linear inequalities of X. From each vertex of X we can construct a cone of maximal volume by not more than $m-n$ steps of changing consequent inequalities. In Sec. 3 we present the algorithm and some computational results are illustrated in Sec. 4.

2. Cones of Maximal Volume and Their Construction

There is no loss of generality if we suppose that rank $(a_i,\ i = 1, \ldots, m) = n$ and it does not contain a redundant inequality, and that a_i, $i = 1, \ldots, n$ is n linearly independent vectors.

Let us consider the system

$$(a_i, x) \leq b_i,\ i = 1, \ldots, n, \tag{2}$$

and its corresponding homogeneous system

$$(a_i, x) \leq 0,\ i = 1, \ldots, n. \tag{3}$$

Let K be the set of solutions of the system (3), then K is a polyhedral cone.

Definition 1. *A linear inequality $(a_k, x) \leq 0$ is called a consequent inequality of the system (3) if $(a_k, x) \leq 0$ for all $x \in K$.*

Definition 2. *The cone K is called cone of maximal volume if $(a_k, x) \leq 0$ is not a consequent inequality of the system (3) for all $k \in \{n+1, \ldots, m\}$.*

From [2] we have the following property of a consequent inequality.

Theorem 1. *The linear inequality $(a_k, x) \leq 0$ is a consequent inequality of the system (3) if and only if the vector a_k is a nonnegative linear representation of vectors a_i of the system (3).*

The following theorem provides us with a condition to check whether a cone K is of maximal volume.

Theorem 2. *Let x^0 be the solution of the system (2) such that*

$$(a_i, x^0) = b_i,\ i = 1, \ldots, n. \tag{4}$$

If x^0 is also a solution of the system (1), then the cone K of the system (3) is of maximal volume.

Proof. Suppose x^0 is a solution of the system (1). We will show that the cone K is of maximal volume. If this is not true, there exists a consequent inequality $(a_k, x) \le 0$ of the system (3), meaning that

$$a_k^T = \sum_{i=1}^{n} \lambda_{ik} a_i^T, \ \lambda_{ik} \ge 0, \ i = 1, ..., n,$$

where a_i^T is a transformation of a_i. There exists at least one $\lambda_{sk} > 0$. Consider the system

$$\begin{cases} (a_i, x) = b_i, & i = 1, ..., n, \ i \ne s, \\ (a_k, x) = b_k . \end{cases} \tag{5}$$

Denote by x^k the solution of the system (4). It is obvious that $x^k \ne x^0$ and

$$(a_s, x^k) < b_s . \tag{6}$$

Notice that

$$(a_k, x^k) = \sum_{i=1, i \ne s}^{n} \lambda_{ik} b_i + \lambda_{sk}(a_s, x^k) = b_k , \tag{7}$$

$$(a_k, x^0) = \sum_{i=1, i \ne s}^{n} \lambda_{ik} b_i + \lambda_{sk}(a_s, x^0). \tag{8}$$

Since $(a_s, x^0) = b_s$, it follows from (6), (7), and (8) that

$$(a_k, x^0) > (a_k, x^k) = b_k .$$

This contradicts the assumption that x^0 is a solution to the system (1). Therefore, $(a_k, x) \le 0$ cannot be a consequent inequality of the system (3), which implies that K is a cone of maximal volume. ■

Definition 3. *Including a consequent inequality $(a_t, x) \le 0$ into the system (3) instead of an inequality $(a_i, x) \le 0$ is called replacing of consequent inequality if the rank of the new system equals the rank of the system (3).*

Theorem 3. *Let x^0 be a solution to the system (4) and the cone K not of maximal volume. Then a cone of maximal volume K^t can be found after not more than $m - n$ replacings of consequent inequalities.*

Proof. Suppose $a_t^T = \sum_{i=1}^{n} \lambda_{it} a_i^T, \lambda_{it} \ge 0, i = 1, ..., n$. This means that $(a_t, x) \le 0$ is a consequent inequality. Note that there exists at least one positive coefficient λ_{st}. If we replace $(a_s, x) \le 0$ by $(a_t, x) \le 0$, then the new system also has rank n. Let us replace $(a_s, x) \le 0$ by $(a_t, x) \le 0$ and denote by K^t the new cone. Obviously $K \subset K^t$. We will show that $(a_s, x) \le 0$ never becomes a consequent inequality of K^t. Let $r_j, \ j = 1, ..., n$ be the vectors generating K. Determine

$$J_1 = \{j \mid (a_s, r_j) = 0, \ j = 1, ..., n\},$$
$$J_2 = \{j \mid (a_s, r_j) < 0, \ j = 1, ..., n\}.$$

It is plain that $|J_1| = n-1, |J_2| = 1$. Note that

$$(a_t, r_j) < 0 \text{ for some } j \in J_1 .$$

This implies that $K \subset K^t$ and some part P_s of the hyperplane $(a_s, x) = 0$ belongs to the interior of K^t. At step $h > t$ we obtain

$$\begin{aligned} K \subset K^t \subset \cdots \subset K^h, \\ P_s \subset \operatorname{int} K^t \subset \cdots \subset \operatorname{int} K^h. \end{aligned} \tag{9}$$

If at step $h+1$ the inequality $(a_s, x) \leq 0$ becomes a consequent inequality and we replace $(a_i, x) \leq 0$ by $(a_s, x) \leq 0$, then the hyperplane $(a_s, x) = 0$ defines a face of the cone K^{h+1}. This contradicts (7). Therefore, after not more than $m-n$ replacings of consequent inequalities, we obtain a cone of maximal volume. The theorem is proved.■

Theorem 4. *Let c be a nonnegative linear combination of the vectors a_i of the system (3) and suppose the cone K is not of maximal volume. Then there exists replacing of some inequality $(a_s, x) \leq 0$ from the system (3) by a consequent inequality $(a_t, x) \leq 0$ such that the new system has a cone of bigger volume and $(c, x) \leq 0$ is also a consequent inequality of the new system.*

Proof. Let $(a_s, x) \leq 0$ be a consequent inequality of the system (3) and

$$c^T = \sum_{i=1}^{n} \lambda_{ic} a_i^T, \lambda_{ic} \geq 0, \ i = 1, \ldots, n,$$

$$a_t^T = \sum_{i=1}^{n} \lambda_{it} a_i^T, \ \lambda_{it} \geq 0, \ i = 1, \ldots, n.$$

Suppose we replace $(a_s, x) \leq 0$ with positive λ_{st} by $(a_t, x) \leq 0$. Theorem 3 implies that the cone K will be contained in the new cone K^t. We have to show that $(c, x) \leq 0$ is a consequent inequality of the cone K^t. Consider the representation of vector c by vector $a_t, \ a_i, \ i \neq s, \ i = 1, \ldots, n$. Let

$$c^T = \lambda'_{tc} a_t^T + \sum_{i=1, i \neq s}^{n} \lambda'_{ic} a_i^T$$

Note that

$$\lambda'_{tc} = \frac{\lambda_{sc}}{\lambda_{st}},$$

$$\lambda'_{ic} = \lambda_{ic} - \frac{\lambda_{sc}}{\lambda_{st}} \lambda_{it}, \ i \neq s, \ i = 1, \ldots, n.$$

The inequality $(c, x) \leq 0$ will be a consequent inequality of K^t if

$$\begin{aligned} \lambda'_{tc} = \frac{\lambda_{sc}}{\lambda_{st}} \geq 0, \\ \lambda'_{ic} = \lambda_{ic} - \frac{\lambda_{sc}}{\lambda_{st}} \lambda_{it} \geq 0, \ i \neq s, \ i = 1, \ldots, n. \end{aligned} \tag{10}$$

It is easy to verify that if

$$\frac{\lambda_{sc}}{\lambda_{st}} = \min_{k, \lambda_{kt} > 0} \frac{\lambda_{kc}}{\lambda_{kt}},$$

then condition (9) will be satisfied. This concludes the proof of Theorem 4. ■

Remark 1. If we replace the inequality $(a_s, x) \leq 0$ by the consequent inequality $(a_t, x) \leq 0$, then the new representation λ'_{ij} of the vector a_j by vectors a_t, a_i, $i \neq s$, $i = 1, \ldots, n$ are defined as follows:

$$\lambda'_{tc} = \frac{\lambda_{sj}}{\lambda_{st}},$$

$$\lambda'_{ij} = \lambda_{ij} - \frac{\lambda_{sj}}{\lambda_{st}}\lambda_{it}, \; i \neq s, \; i = 1, \ldots, n.$$

Remark 2. The cone K^0, such that the inequality $(c, x) \leq 0$, is its consequent inequality and from K^0 we begin the process of finding a cone of maximal volume which will be called an initial cone. The initial cone K^0 can be found by solving the following problem (Q).

$$\max \sum_{j=1}^{n} -y_j,$$

subject to

$$\sum_{i=1}^{m} a_{ij}\lambda_i + e_j y_j = c_j, \; j = 1, \ldots, n,$$

$$\lambda_i \geq 0, \; i = 1, \ldots, m, \; y_j \geq 0, \; j = 1, \ldots, n,$$

where e_j is a vector such that

$$e_{kj} = \begin{cases} 0, & \forall k \neq j, \\ 1, & \text{if } k = j, \; c_j \geq 0, \\ -1, & \text{if } k = j, \; c_i < 0. \end{cases}$$

This requires not more than n simplex iterations.

Now let us consider the question of how to choose the consequent inequality $(a_t, x) \leq 0$. Our aim is to make the cone K^t as big as possible. We do this in the following way.

Denote by T the index set of consequent inequalities and let $|T| \geq 2$. Determine

$$\gamma = -\frac{1}{n}\sum_{i=1}^{n} a_i,$$

$$T_1 = \{k \mid \gamma_k > 0, \; (a_k, \gamma_k a_k) = 0, \; (c, \gamma + \gamma_k a_k) \leq 0, \; k \in T\}.$$

Find

$$\gamma_1 = \max\{\gamma_k \mid k \in T_1\}. \tag{11}$$

The consequent inequality $(a_t, x) \leq 0$ corresponding to γ_1 in (11) will be chosen for inclusion into the new system.

Theorem 5. *Let the vector c be a nonnegative linear combination of vectors a_i, $i = 1, \ldots, n$ of the system (3) and x^0 a solution to the system (4). If the cone K of the system (3) is of maximal volume and x^0 is a solution to the system (1), then x^0 is an optimal solution to the problem (P).*

Proof. Since $(c, x) \leq 0$ is a consequent inequality of the system (4), it implies

$$(c, x) \leq (c, x^0), \ \forall x \in X.$$

On the other hand, Theorem 2 implies that $x^0 \in X$. Therefore

$$(c, x^0) = \max\{(c, x^0) \mid x \in X\},$$

which concludes the proof of Theorem 5. ■

3. The Algorithm

The idea of this algorithm is that from a vertex of X which corresponds to a cone, we construct a cone with a bigger volume by gradually replacing a consequent inequality until we get a cone of maximal volume (the procedure contains not more than $m - n$ replacement steps) and a corresponding solution x^0. If x^0 satisfies all constraints of problem (P), then x^0 is an optimal solution of (P). Otherwise, we choose a new vertex of X and repeat the procedure.

Initial step. Solve the problem (Q) for finding the representation $\lambda_{ij}, \lambda_{ic}, \lambda_{ib}$ $(\lambda_{ib} = 0)$ of vectors a_i defining the initial K^0. Go to Step 1.

Step 1. Define the set T of consequent inequalities of cone K^0. If $T = 0$, then go to Step 2. Otherwise, define the index s, t and replace by a_t. Go to Step 1.

Step 2. If x^0 is a solution to the system (1), then stop; x^0 is an optimal solution to the problem (P). Otherwise, go to Step 3.

Step 3. Choose a cone K^* such that (c, x) is a consequent inequality of K^* and $(c, x^*) \leq (c, x^0)$, where x^* is a solution to the system (2) corresponding to the cone K^*. Let $K^0 = K^*$, $x^0 = x^*$ and go to Step 1.

4. Computational Experience

The above algorithm has been written using PASCAL. With some examples, we have shown the effectiveness of the algorithm. The number of iterated steps is less than the one of the dual simplex method.

The following table is the result of the testing examples.

(1)	(2)	(3)	(4)	(5)
100	34.57	11.98	15.46	12.25

Note.
(1) The number of the testing examples.
(2) The average number of variables n.
(3) The average number of constraints m.
(4) The average number of iterated steps of the dual simplex method.
(5) The average number of iterated steps of the maximal volume cone algorithm.

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