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Short Communication

On the Length of Generalized Fractions of Modules Having Polynomial Type ≤ 2

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1. Introduction

It is well known that the function defined by the difference between the length and the multiplicity with respect to a system of parameters in a local ring gives a lot of information about the structure of the ring. In this note we shall study a similar problem. Instead of the length of a system of parameters, we take the length of the generalized fraction with respect to a system of parameters which was first introduced in [9]. Then for a system of parameters $\underline{x} = (x_1, \ldots, x_d)$ of a finitely generated A-module M, we consider the function

$$J_M(\underline{n}, \underline{x}) = n_1 \ldots n_d . e(x_1, \ldots, x_d; M) - \ell\Big(M\big(1/(x_1^{n_1}, \ldots, x_d^{n_d}, 1)\big)\Big)$$

as a function in $\underline{n}$. Now the question is: What is the structure of M when $J_M(\underline{n}; \underline{x})$ is bounded above by a constant or $J_M(\underline{n}; \underline{x}) = 0$?

The purpose of this note is to answer the above question under the condition that M has the small polynomial type. Recall that the polynomial type of a module M is an invariant of M and it may be used to study the non-Cohen–Macaulay modules (see [1–3] for more details). All results of this short communication are found in [6] with complete proofs.

2. The Polynomial Type of Fractions

From now on, A is a Noetherian local ring with the maximal ideal m and M a finitely generated A-module, $\dim M = d$. Let $\underline{x} = (x_1, \ldots, x_d)$ be a system of parameters (s.o.p for short) of M and $\underline{n} = (n_1, \ldots, n_d)$ a d-tuple of positive integers.

The first author showed in [1] that the least degree of all polynomials bounding above the function

$$I_M(\underline{n}, \underline{x}) = \ell(M/(x_1^{n_1}, ..., x_d^{n_d})M) - n_1 \cdots n_d \cdot e(\underline{x}; M)$$

is independent of the choice of $\underline{x}$. This invariant of M is called the *polynomial type* of M and is denoted by $p(M)$. The polynomial type was first introduced in [1] and many basic properties as well as some applications of this invariant have been given in [1–5]. Also in [5] we examined first the following function

$$J_M(\underline{n}; \underline{x}) = n_1 \cdots n_d \cdot e(\underline{x}; M) - \ell(M(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))),$$

where $M(1/(x_1^{n_1}, ..., x_d^{n_d}, 1))$ is the submodule of the module of generalized fractions $U(M)_{d+1}^{-d-1}M$ defined in [10]. One of the main results of this note is the following theorem which allows us to define a new invariant on M.

Theorem 1. *The least degree of all polynomials in $\underline{n}$ bounding above the function $J_M(\underline{n}; \underline{x})$ is independent of the choice of $\underline{x}$.*

This new invariant of M is called *the polynomial type of fractions of M* and is denoted by $pf(M)$.

The next result is to give a relation between these invariants $p(M)$ and $pf(M)$ as follows.

Proposition 2. *Let M be a finitely generated A-module. Then*
(i) $pf(M) \leq p(M)$.
(ii) *If M admits a dualizing complex and depth $M > p(M)$, then $pf(M) = p(M)$.*

3. Local Cohomology Modules

A relationship between the top cohomology module $H_m^d(M)$ and the module of generalized fractions $U(M)_{d+1}^{-d-1}M$ was given in [11]. In the following result we shall use the invariant $pf(M)$ to give a sufficient condition for local cohomology modules of M to be zero or to have finite length. (Here we stipulate that the degree of the zero-polynomial is equal to $-\infty$.)

Proposition 3. *Let M be a finitely generated A-module. Then*
(i) *if $pf(M) \leq 0$, then $\ell(H_m^i(M)) < +\infty$ for $i = p(M) + 1, ..., d - 1$.*
(ii) *if $pf(M) = -\infty$, then $H_m^i(M) = 0$ for $i = p(M) + 1, ..., d - 1$.*

From (i) of Propositions 2 and 3, we easily obtain again the basic results on local cohomology modules of Cohen–Macaulay and generalized Cohen-Macaulay modules. Furthermore, Proposition 3 shows that the polynomial type plays an important role in the study of modules which are not generalized Cohen–Macaulay.

Denote by $\widehat{A}$ and $\widehat{M}$ the m-adic completion of A and M, respectively. Then, as an interesting corollary of Proposition 3, the following proposition gives a new characterization of Cohen–Macaulay and generalized Cohen–Macaulay modules in terms of the invariant $pf(M)$.

Proposition 4. *The following statements are true:*

(i) M *is generalized Cohen–Macaulay if and only if* $\widehat{M}/H^0_m(\widehat{M})$ *holds the Serre's condition* (S_1) *and* $pf(M) \leq 0$.

(ii) M *is Cohen–Macaulay if and only if* $\widehat{M}$ *holds the Serre's condition* (S_1) *and* $pf(M) = -\infty$.

4. Length of Generalized Fractions

The aim of this section is to study the structure of M using the local cohomology modules of M when $pf(M) \leq 0$ and M has the small polynomial type. First, in the case $p(M) \leq 1$, we have the following theorem.

Theorem 5. *Suppose* $p(M) \leq 1$. *Then the following conditions are equivalent:*

(i) $pf(M) \leq 0$.

(ii) *There exists a s.o.p* $\underline{x}$ *of* M *and a constant* $K(\underline{x})$ *which depends on* $\underline{x}$ *such that* $J_M(\underline{n}; \underline{x}) \leq K(\underline{x})$ *for all* $\underline{n}$.

(iii) *There exists a constant* K *such that* $J_M(\underline{n}; \underline{x}) \leq K$ *for every s.o.p* $\underline{x}$ *and all* $\underline{n}$.

(iv) $\ell(H^i_m(M)) < +\infty$ *for* $i = p(M) + 1, ..., d-1$.

(v) *For any s.o.p* $\underline{x}$ *of* M, *it holds*

$$J_M(\underline{n}; \underline{x}) = R\ell(H^1_m(M)) + \sum_{i=2}^{d-1} \binom{d-1}{i-1} \ell(H^i_m(M))$$

for $\underline{n} \gg 0$, *where* $R\ell(.)$ *is the residuum length of an Artinian module in the sense of Sharp.*

Remark. (i) When M is a generalized Cohen–Macaulay module, i.e., $p(M) \leq 0$, we again obtain from Theorem 5, one of the main results of [9, Theorem 3.7], on the lengths of generalized fractions.

(ii) If $p(M) = 1$ and $H^i_m(M)$ is not finite for some $i \in \{2, \ldots, d_1\}$, then we can choose a s.o.p $\underline{x}$ of M so that $J_M(\underline{n}; \underline{x})$ is a polynomial of degree 1 (see [5, Corollary 4.7]).

As a consequence of Theorem 5, we get the following result for the case $pf(M) = -\infty$.

Corollary 6. *Suppose* $p(M) \leq 1$. *Then the following conditions are equivalent:*

(i) $pf(M) = -\infty$.

(ii) *There exists a s.o.p* $\underline{x}$ *of* M *such that*

$$\ell(M(1/(x_1, ..., x_d, 1))) = e(\underline{x}; M).$$

(iii) *For every s.o.p* $\underline{x}$ *of* M, *we have*

$$\ell(M(1/(x_1, ..., x_d, 1))) = e(\underline{x}; M).$$

(iv) $H^i_m(M) = 0$ *for* $i = p(M) + 1, ..., d-1$ *and* $R\ell(H^1_m(M)) = 0$.

In the case $p(M) \leq 2$, the problem is more difficult. In order to have $pf(M) = -\infty$ or $pf(M) \leq 0$, we have to use the Matlis dual modules of local cohomology modules to find some necessary conditions and sufficient conditions.

For $0 \leq i \leq d$, we set

$$K_i = \mathrm{Hom}_{\widehat{A}}(H^i_m(M), E(k)),$$

where $k = A/m$ and $E(k)$ is the injective envelope of k. Note that K_i is the Noetherian module for all $i \leq d$.

Theorem 7. *Suppose that A admits a dualizing complex and $p(M) \leq 2$. Then $pf(M) \leq 0$ if and only if $\ell(H^i_m(M)) < +\infty$ for $i = p(M)+1, ..., d-1$ and K_2 is a generalized Cohen–Macaulay module.*

We can derive from this theorem some sufficient and necessary conditions for $pf(M) = -\infty$ as follows.

Corollary 8. *Suppose A admits a dualizing complex and $p(M) \leq 2$. If $pf(M) = -\infty$, then $H^i_m(M) = 0$ for $i = p(M)+1, ..., d-1$ and K_2 is a Cohen-Macaulay module.*

Corollary 9. *Suppose that A admits a dualizing complex and $p(M) \leq 2$. If $H^i_m(M) = 0$ for $i = p(M)+1, ..., d-1$ and K_i is a Cohen-Macaulay module with* $\dim K_i = i$, *for $i = 1, 2$, then $pf(M) = -\infty$.*

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